ОБЪЕДИНЕННЫЙ ИНСТИТУТ ЯДЕРНЫХ ИССЛЕДОВАНИЙ

Лаборатория теоретической физики

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ON ALGEBRAIC PROBLEMS IN THE THEORY OF COMPLEX ANGULAR MOMENTUM

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Abstract

A method is given for reducing 'algebraic' operations with complex angular momenta (addition, recoupling etc) to ordinary angular momentum algebra, by expanding the matrix elements $\mathfrak{D}^{j}_{mm'}(R)$ for complex j into an orthogonal series. The generalization of vector coupling coefficients and their sum rule for complex total angular momenta are treated as examples.

Г. Домокош

АЛГЕБРАИЧЕСКИЕ ПРОБЛЕМЫ В ТЕОРИИ КОМПЛЕКСНОГО УГЛОВОГО МОМЕНТА

Аннотация

Дается метод сведения "алгебраических" операций с комплексными угловыми моментами /сложение, перестройка и т.д./ к обычной алгебре углового момента путем разложения матричных элементов $\mathfrak{D}^{j}_{mm'}(R)$ для комплексных j в ортогональный ряд. В качестве примеров рассматривается обобщение коэффициентов векторного сложения и их правила суммы для комплексных полных угловых моментов.

Работа издается только на английском языке.

1. Introduction

The theory of complex angular momentum has played a considerable role in later time in predicting asymptotic behaviour of scattering amplitudes, their connection with the exchange of bound and quasistationary systems etc*.

From a certain point of view the main idea of the theory can be stated as a suitable re-ordering of an ordinary Legendre series so as to allow analytic continuation outside its convergence ellipse.

The idea could be applied immediately to field theory with spinless particles in the two-particle approximation, and indeed, it has been shown that the partial wave amplitudes are meromorphic functions of the angular momentum variable/1,2/.

There has been, however, certain confusion, as soon as one wanted to treat many-particle intermediate states. The main difficulty consists in the fact that one has to deal with several angular momenta, coupled together some (or all) of which has to be 'Reggeized'. It is not clear from the beginning, what one has to do in this case with Clebsch-Gordan coefficients, which occur e.g. in the unitarity condition etc.

In the present work we propose a solution to this problem by reducing it to the algebra of 'ordinary' angular momenta. The main point in our argumentation is that the geometrical factors (vector coupling coefficients etc) with complex angular momenta can be expanded into a convergent series according to 'ordinary' quantities. Thus addition, reccupling etc. of complex angular momenta can be performed by operating with 'well-behaved' entities only. In what follows, it will be sufficient to deal with the continuation in j of single valued representations of the rotation group, as usually orbital momenta are 'Reggeized'.

2. Expansion of states with complex angular momentum

All the geometrical problems we have to treat can be reduced to continuation in the total angular momentum quantum number of matrix elements of the irreducible representations of the rotation group. The latter can be written as a function of the Euler angles:

$$\mathfrak{D}^{j}_{mm'}(\alpha\beta\gamma) = e^{im\alpha}\, d^{j}_{mm'}(\beta)\, e^{im'\gamma} \tag{1}$$

where

$$\begin{aligned} d^{j}_{mm'}(\beta) &= C\,(\sin\beta/2)^{p}\,(\cos\beta/2)^{q}\, \times \\ &\times F\left(\tfrac{1}{2}(p+q)+j+1,\quad \tfrac{1}{2}(p+q)-j;\ q+1;\ \sin^2\beta/2\right). \end{aligned} \tag{2}$$

Here C is a normalization coefficient (being a simple algebraic function of j).

$$p = |\,m' + m\,|, \qquad q = |\,m' + m\,|$$

$F(a, b; c; z)$ is a hypergeometric function. If $j=$ integer of half-integer, F in eq. (2) reduces to a polynomial**.

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* The main results are summarized in the corresponding reports of the International Conference on High Energy Physics, Geneva, 1962. We refer the reader to the Prooeedings of this Conference for a general survey.

** In what follows, we make use of standard expressions; the reader may consult any textbook on the so called quantum mechanics of angular momentum or on the theory of group representations for reference.

If we continue eq. (2) in j, then in order to mainfain the single-valuedness of $\mathfrak{D}$ in the Euler angles α and γ, we still must have $m, m' =$ integer, however, without the restriction $|m| \leq j$, $|m'| \leq j$. In general, the hypergeometric function will be singular at $\beta = \pi$ but it will be expandable into an orthogonal series.

We choose the matrix elements of the finite-dimensional, irreducible representations of the ordinary rotation group as our basis system, i.e. we write:

$$\mathfrak{D}^{j}_{mm'}(\alpha\beta\gamma) = \sum_{\ell\mu\mu'} \langle jm|\ell\mu\rangle \, \mathfrak{D}^{\ell}_{\mu\mu'}(\alpha\beta\gamma) \langle \ell\mu'|jm'\rangle . \tag{3}$$

(Here and in what follows, letters j -denote complex, ℓ -ordinary, integer of half-integer angular momenta). Eq. (3) is the Clebsch-Gordan series for $\mathfrak{D}^{j}_{mm'}(\alpha\beta\gamma)$.

The expansion coefficients are of course given by the integral:

$$\langle jm|\ell\mu\rangle\langle\ell\mu'|jm'\rangle = \frac{2\ell+1}{8\pi^2} \int dR\, \mathfrak{D}^{j}_{mm'}(R)\, \mathfrak{D}^{\ell}_{\mu\mu'}(R) \tag{4}$$

where R is an element of the rotation group

$$dR = d\alpha \sin\beta \; d\beta d\gamma .$$

Taking into account eq. (1), we immediately see that the right hand side of (4) will be proportional to $\delta_{m\mu}\ \delta_{m'\mu'}$ and making use of Dougall's expansion[3] for $P^{m}_{j}(\cos\beta)$, we find:

$$\langle jm|\ell\mu\rangle\langle\ell\mu'|jm'\rangle = \delta_{m\mu}\ \delta_{m'\mu'}\ \frac{\sin j\pi}{\pi} \times \frac{(-1)^{\ell}\,(2\ell+1)}{(j-\ell)(j+\ell+1)} \tag{5}$$

Eq. (5) determines $\langle jm|\ell\mu\rangle$ up to a phase, which, however, is unimportant*.

The reader can immediately verify that for $j \to n$ ($n \geq 0$, integer)

$$\langle nm|\ell\mu\rangle\langle\ell\mu'|nm\rangle = \delta_{m\mu}\ \delta_{m'\mu'}\delta_{n\ell} ,$$

as it should be.

3. Continuation of Clebsch-Gordan coefficients in the angular momentum

As an application of the expansion found in the previous section, we show, how one continues a vector coupling coefficient in the total angular momentum quantum number.

Problems of this kind occur in treating many-particle intermediate states in crossed channels of a scattering amplitude[4].

We start with the definition of vector coupling coefficients:

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* Eq. (5) can be found either directly by calculating the integral (4) or, alternatively, by making use of the fact that the expansion coefficient is not only diagonal in m, but independent of it. So it can be found for same special value of m, e.g. $m = 0$. This is the way we followed.

$$< \ell_1 \mu_1 \ell_2 \mu_2 | j\mu > < j\mu' | \ell_1 \mu_1' \ell_2 \mu_2' > =$$

$$= \frac{2j+1}{8\pi} \int dR\, \mathfrak{D}^{j*}_{\mu\mu'}(R)\, \mathfrak{D}^{\ell}_{\mu_1\mu_1'}(R)\, \mathfrak{D}^{\ell}_{\mu_2\mu_2'}(R) \qquad (6)$$

If here one of the angular momenta (j , say) is compelx, we insert the expansion (4)-(5) for the corresponding matrix element; and analogously, if we have to deal with several complex momenta we have to insert the expansion for every $\mathfrak{D}$ with a complex angular momentum. Let us illustrate the procedure for the case, when only j is complex. We have:

$$< \ell_1 \mu_1 \ell_2 \mu_2 | j\mu > < j\mu | \ell_1 \mu_1 \ell_2 \mu_2 > =$$

$$= \frac{2j+1}{8\pi^2} \sum_{\ell} < j\mu | \ell\mu > < \ell\mu | j\mu > \times$$

$$\times \int dR\, \mathfrak{D}^{\ell*}_{\mu\mu}(R)\, \mathfrak{D}^{\ell_1}_{\mu_1\mu_1}(R)\, \mathfrak{D}^{\ell_2}_{\mu_2\mu_2}(R)$$

(We made use of the reality of the expansion coefficients (5)). Remembering (6), the latter equation can be written as follows:

$$< \ell_1 \mu_1 \ell_2 \mu_2 | j\mu > < j\mu | \ell_1 \mu_1 \ell_2 \mu_2 > =$$

$$= (2j+1) \sum_{\ell} < j\mu | \ell\mu > < \ell\mu | j\mu > (2\ell+1)^{-1} \times$$

$$\times < \ell_1 \mu_1 \ell_2 \mu_2 | \ell\mu > < \ell\mu | \ell_1 \mu_1 \ell_2 \mu_2 >$$

or, finally, with eq. (5) and making use of the reality of the vector coupling coefficients for integer ℓ :

$$< \ell_1 \mu_1 \ell_2 \mu_2 | j\mu > < j\mu | \ell_1 \mu_1 \ell_2 \mu_2 > =$$

$$= \frac{(2j+1)\sin j\pi}{\pi} \sum \frac{(-1)^{\ell}}{(j-\ell)(j+\ell+1)} (< \ell_1 \mu_1 \ell_2 \mu_2 | \ell\mu >)^2 . \qquad (7)$$

Eq. (7) of course determines the continuation of the vector coupling coefficient up to an arbitrary phase. The latter can be chosen to correspond to that at integer values of j , in most expressions, however, we have to deal with the modulus squared of the vector coupling coefficients, so that the arbitrary phase drops out (of. ref. /4/).

Let us remark that the sum of the right hand side of eq. (7) is finite, as the ordinary Clebsch-Gordan coefficients vanish, unless the angular momenta ℓ_1 , ℓ_2 , ℓ satisfy the traingle inequality:

$$|\ell_1 - \ell_2| \leq \ell \leq \ell_1 + \ell_2 .$$

At this point we can generalize the so-called sum rule for the vector coupling coefficients.

As is well-known, the latter states for integer (or half-integer) ℓ_1 , ℓ_2 , ℓ :

$$\sum_m (< \ell_1 0 \ell_2 m | \ell m >)^2 = \frac{2\ell+1}{2\ell_1+1} . \qquad (8)$$

Combining this expression with eq. (7) we immediately find:

$$\sum_{\mu} \langle \ell_1 0 \; \ell_2 \mu \mid j \mu \rangle \langle j \mu \mid \ell_1 0 \; \ell_2 \mu \rangle = \frac{2j+1}{2\ell_1+1} \; \frac{\sin j\pi}{\pi} \sum_{\ell=|\ell_1-\ell_2|}^{\ell_1+\ell_2} \frac{(-1)^{\ell}(2\ell+1)}{(j-\ell)(j+\ell+1)} \tag{9}$$

We remark that the sum of the right hand side of eq. (9) can be expressed with the help of polygammafunctions (cf. ref./3/ Ch. I).

In fact, we have:

$$\sum_{\ell=|\ell_1-\ell_2|}^{\ell_1+\ell_2} \frac{(-1)^{\ell}(2\ell+1)}{(j-\ell)(j+\ell+1)} = \sum_{\ell} (-1)^{\ell} \left(\frac{1}{j-\ell} - \frac{1}{j+\ell+1} \right) =$$

$$= (-1)^{\ell_1-\ell_2} \sum_{k=0}^{2N} \frac{(-1)^k}{j-|\ell_1-\ell_2|-k} + (j \to -j-1)$$

where

$$2N = \ell_1 + \ell_2 - |\ell_1 - \ell_2| .$$

Hence, with the help of eqs. 1.7 (32) and 1.8 (1) of ref./3/ we have:

$$\sum_{\mu} \langle \ell_1 0 \; \ell_2 \mu \mid j\mu \rangle \langle j\mu \mid \ell_1 0 \; \ell_2 \mu \rangle =$$

$$= \frac{2j+1}{2\ell_1+1} \; \frac{\sin j\pi}{\pi} \; (-1)^{\ell_1-\ell_2} \left\{ \frac{1}{j-|\ell_1-\ell_2|} + G(\ell_1+\ell_2+1-j) - \right. \tag{10}$$

$$\left. - G(|\ell_1-\ell_2|+1-j) + (j \to -j-1) \right\}$$

Expressions like eq. (10) turn out to be useful when one estimates sums over products of vecrtor coupling coefficients with several complex angular momenta.

4. Discussion

One sees from the foregoing sections, how geometrical entities are to be continued in angular momenta. The same procedure can be applied to other quantities as well, like Racah coefficients or to transformation coefficients between two different angular momentum representations (e.g. to the transformation coefficient between a helicity representation and an (LS) representation) . The unicity of the continuation can be proved by the application of standard unicity theorems from the theory of integer functions, as it has been done in the case of the continuation of partial wave amplitudes. (Cf. ref./5/).

If one has several complex angular momenta, then some of the resulting series may not converge for the whole domain necessary. In such cases a further analytic continuation procedure, like that leading to eq. (10), combined with a contour integral representation of the series in question, should be applied.

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