ОБЪЕДИНЕННЫЙ ИНСТИТУТ ЯДЕРНЫХ ИССЛЕДОВАНИЙ

Лаборатория теоретической физики

A.A.Logunov, A.N.Tavkhelidze

E - 1145

QUASIOPTICAL APPROACH IN QUANTUM FIELD THEORY

Дубна 1962 г.

A.A.Logunov, A.N.Tavkhelidze

E - 1145

QUASIOPTICAL APPROACH IN QUANTUM FIELD THEORY

Submitted to Nuovo Cimento.

Дубна 1962 г.

Abstract

A quasioptical approach in the quantum field theory is developed.

А.А.Логунов и А.Н.Тавхелидзе

"КВАЗИОПТИЧЕСКИЙ ПОДХОД В КВАНТОВОЙ ТЕОРИИ ПОЛЯ"

Аннотация

В работе разработан квазиоптический подход в квантовой теории поля.

Работа печатается только на английском языке.

In order to study the nucleon interaction with nuclei an optical nuclear model was advanced according to which the nucleon scattering by nuclei looks like the scattering of light by the semi-transparent optical medium. In such an approach the problem of scattering is considered not as the many-body problem, but as a problem about the nucleon motion in the field described by the complex potential. The imaginary part of the potential describes the inelastic scattering processes. In the light of presently available data one can definitely say that the optical nuclear model for nucleon scattering gives a true picture of the reality. It is quite evident that this model is of a phenomenological nature, and its success in each concrete case depends, in the main, upon a fortunate choice of the complex potential.

The aim of this paper is to develop the quasioptical approach to the problem of elementary particle scattering according to the concepts of the quantum field theory. We shall see that the principles of the quantum theory make it possible to construct the generalized complex potential dependent on velocities. The imaginary part of the generalized potential characterizes possible inelastic processes. Such an approach is general enough and allows to consider both the problems of scattering and those of the bound states.

The introduction of the potential into the quantum field theory was discussed earlier. For instance, in Refs./1/ the potential was introduced as a subtraction constant in the dispersion relations. According to such a definition the potential depends only upon the momentum transfer and the particle masses.

In Ref./2/, the potential was defined as a difference between the imaginary part of the amplitude and a certain integral of the double spectral function of elastic scattering. In this definition the potential is a function of the variables s , t and of particle masses.

In Ref./1/, the problem of introducing the potential dependent only upon the momentum transfer t and the particle masses was also discussed from another point of view. The potential in this approach was chosen so that the Schrödinger equation should lead us to the relativistic S matrix. It is quite obvious that such a potential has a limited meaning (as is pointed out by the authors of /1/ themselves). The problems of elementary particle interaction according to the concept of the nuclear optical model were treated in a paper by D.I.Blokhintsev et al/9/.

Before we proceed to obtaining the Schrödinger equation we extend in § 1 the Lippmann-Schwinger method to the problems with the complex potential.

In § 2 we will discuss the problems dealing with the 4-time Green functions.

In § 3 the properties of the 2-time Green functions and of the generalized complex potential will be studied.

In § 4 a method for constructing the Schrödinger equation with the complex potential will be worked out.

In § 5 an approximate method of summing the perturbation theory graphs will be developed.

1. Lippmann-Schwinger's Method in Problems with Complex Potential

Let the system be described by the Schrödinger equation

$$i\frac{\partial\psi(t)}{\partial t} = (H_0 + H_1)\,\psi(t)$$
$$H = H_0 + H_1, \quad H_1 = U - i\Gamma, \quad \Gamma \geq 0. \qquad /1.1/$$

Using the unitary transformation

$$\psi(t) = e^{-iH_0 t}\,\phi(t) \qquad /1.2/$$

we pass to the interaction representation

$$i \frac{\partial \phi(t)}{\partial t} = H_I(t)\, \phi(t)$$

where

$$H_I(t) = e^{iH_0 t} H_I e^{-iH_0 t} = U(t) - i\Gamma(t), \qquad \Gamma(t) \geq 0 . \qquad /1.3/$$

With the aid of Eq. /1.3/ , we have

$$\frac{\partial < \phi^*(t) \quad \phi(t) >}{\partial t} = -2 < \phi^*(t)\, \Gamma(t)\;\; \phi(t) > \lessapprox 0 \qquad /1.4/$$

Now we see that the imaginary part of the potential characterizes inelastic processes. Introduce the operators S_t and S

$$\phi\;(t) = S_t \phi(-\infty), \qquad \phi(\infty) = S\phi(-\infty) \qquad /1.5/$$

According to /1.4/ and /1.5/ we get

$$\overset{+}{S}_t\; S_t \leqq 1, \qquad \overset{+}{S}\; S \leqq 1, \qquad /1.6/$$

or

$$S = 1 + iT, \qquad i(T - \overset{+}{T}) + \overset{+}{T}\; T \leqq 0 . \qquad /1.7/$$

The presence of inelastic processes in the system leads to the violation of unitarity of the S -matrix.

Let ϕ_a be the eigenfunctions of the operator H_o

$$H_o\, \phi_a = E_a \phi_a \qquad /1.8/$$

If it is assumed that the interaction H_I is switched adiabatically on and off at $t=-\infty$, $t=\infty$, respectively, what is easily accomplished by introducing the factor $\exp(-\epsilon|t|)$, then the eigenfunctions ϕ_a at $t=\pm\infty$ will be the eigenfunctions of the total operator H as well. In the final formulas we have to go over to the limit $\epsilon \to 0$ With account of these remarks, we have

$$\phi(t) = \phi_\beta - i \int_{-\infty}^{t} H_I(t)\;\; \phi(t)\, dt \qquad /1.9/$$

$$iT\phi_\beta = -i \int_{-\infty}^{+\infty} e^{-\epsilon|t|} H_I(t)\, \phi(t)\, dt$$

Hence

$$i\, T_{a\beta} = (\phi_a^* T \phi_\beta) = -i \int_{-\infty}^{+\infty} e^{-\epsilon|t|} \quad (\overset{*}{\phi}_a e^{iH_o t} H_I e^{-iH_o t} \quad S_t\, \phi_\beta)\, dt =$$

$$= -i \int_{-\infty}^{\infty} (\overset{*}{\phi}_a H_I\, e^{i(E_a - H_o)t - \epsilon|t|} \quad S_t \phi_\beta)\, dt \qquad /1.10/$$

Denote

$$\Psi_\beta^{(+)}(E) = \int_{-\infty}^{\infty} dt\; e^{i(E - H_o)t - \epsilon|t|} \quad S_t\;\phi_\beta \qquad /1.11/$$

then

$$T_{\alpha\beta} = -(\overset{*}{\phi}_\alpha \; H_I \; \Psi^{(+)}_\beta(E_\alpha)) \qquad /1.12/$$

Since

$$S_t \; \phi_\beta = \phi_\beta - i \int_{-\infty}^{t} e^{-\epsilon|t|} \; H_I(\tau) \; S_\tau \; \phi_\beta d\tau$$

then

$$\Psi^{(+)}_\beta (E) = \int_{-\infty}^{\infty} dt \, e^{i(E-H_o)t - \epsilon|t|} \; \phi_\beta \; -$$

$$- i \int_{-\infty}^{\infty} dt \; e^{i(E-H_0)t - \epsilon|t|} \int_{-\infty}^{t} e^{-\epsilon|\tau|} \; H_I(\tau) \; S_\tau \; \phi_\beta d\tau =$$

$$= 2\pi \; \delta \; (E - E_\beta) \, \phi_\beta - i \int_{-\infty}^{\infty} d\tau \; \{ \int_{\tau}^{\infty} dt \; e^{i(E-H_o)t - \epsilon(t-\tau)} \; \} \times$$

$$\times \exp(-\epsilon |\tau|) \; e^{iH_o\tau} \; H_I \; e^{-iH_o\tau} \; S_\tau \; \phi_\beta$$

It follows from here that

$$\Psi^{(+)}_\beta (E) = 2\pi \, \delta \, (E - E_\beta) \; \phi_\beta + \frac{1}{E - H_0 + i\epsilon} \int_{-\infty}^{\infty} H_I \; e^{-\epsilon|\tau| + i(E-H_0)\tau} \; S_\tau \; \phi_\beta d\tau \qquad /1.13/$$

or, taking into consideration /1.11/, we get

$$\Psi^{(+)}_\beta \; (E) = 2\pi\delta \, (E - E_\beta) \, \phi_\beta + \frac{1}{E + i\epsilon - H_0} \; H_I \, \Psi^{(+)}_\beta \; (E) \qquad /1.14/$$

Supposing that

$$\Psi^{(+)}_\beta (E) = 2\pi\delta \, (E - E_\beta) \, \Psi_\beta \qquad /1.15/$$

we obtain the integral equation for Ψ_β

$$\Psi_\beta = \phi_\beta + \frac{1}{E_\beta + i\epsilon - H_0} \; H_I \, \Psi_\beta \qquad /1.16/$$

Solving this integral equation we find $\Psi^{(+)}_\beta$ and are able, with the aid of /1.12/, to calculate the scattering matrix $T_{\alpha\beta}$

The integral equation /1.16/ may be regarded as a formal solution of the differential equation

$$(E_\beta - H + i\epsilon) \, \psi = 0 \qquad /1.17/$$

which corresponds to outgoing waves.

Solving Eq. /1.16/ by iterations, we get the expansion

$$\Psi_\beta = \left(1 + \frac{1}{E_\beta + i\epsilon - H_0} H_I + \frac{1}{E_\beta + i\epsilon - H_0} H_I \frac{1}{E_\beta + i\epsilon - H_0} H_I + \dots \right) \qquad /1.18/$$

which can be written as

$$\Psi_\beta = \left(1 + \frac{1}{E_\beta + i\epsilon - H} H_I \right) \phi_\beta \qquad /1.19/$$

From here, taking account of /1.12/, we have

$$T_{\alpha\beta} = -2\pi \phi_\alpha^{+} \left(H_I + H_I \frac{1}{E_\beta + i\epsilon - H} H_I \right) \phi_\beta \, \delta (E_\alpha - E_\beta) \qquad /1.20/$$

If the interaction H_I is a function of energy, and

$$H_I (E + i\epsilon) - H_I (E - i\epsilon) = - i\Gamma (E) \qquad /1.21/$$

where $\Gamma (E) \geqq 0$

then, by repeating the above arguments, we get

$$\Psi_\beta = \phi_\beta + \frac{1}{E_\beta + i\epsilon - H_0} H_I (E_\beta + i\epsilon) \Psi_\beta \qquad /1.22/$$

In the relativistic problems we shall be concerned with the equations of the form

$$(E^2 - F^2 (p) - V (E)) \psi = 0 ,$$

where $V (E) = u (E) + i\Gamma (E), \;\; \Gamma (E) \geq 0$

$$E > 0 , \quad F^2 (p) = m^2 + \vec{p}^{\,2}. \qquad /1.23/$$

In this case, as can be easily seen, we arrive at the integral equation of the form

$$\Psi_\beta = \phi_\beta + \frac{1}{(E_\beta + i\epsilon)^2 - F^2 (p)} V (E_\beta + i\epsilon) \Psi_\beta \qquad /1.24/$$

here the amplitude $T_{\alpha\beta}$ is equal to

$$T_{\alpha\beta} = - \frac{1}{2 E_\beta} \delta (E_\alpha - E_\beta) 2\pi < \phi_\alpha^{*} (V + V \frac{1}{(E_\beta + i\epsilon)^2 - F^2} V) \phi_\beta > \qquad /1.25/$$

In virtue of the invariance under rotation group, the total momentum is conserved. Hence, the scattering matrix can be reduced to a diagonal form. The partial amplitudes in this case may be put as

$$S_\ell (E) = e^{\eta_\ell (E) + 2i \delta_\ell (E)}$$

where $\eta_\ell (E)$ characterize inelastic scattering processes.

2 The 4-time Green Functions and Bethe-Salpeter Equation

For the 4-time Green functions, in case of identical particles, Bethe and Salpeter derived the following equation

$$G'(1,2;1',2') = G'_0(1,2;1',2') +$$
$$+ \int G_0(1,2;3,4)\, K(3,4;5,6)\, G'(5,6;1',2')\, dx_3\, dx_4\, dx_5\, dx_6 \qquad /2.1/$$

the expression for the kernel K being obtained by expanding in a power series according to the coupling constant. For simplicity we are concerned here with the scalar particles only.

Representing the function G' as

$$G'(1,2;1',2') = (1+P_{12})\, G(1,2;1',2') \qquad /2.2/$$

where P_{12} is the commutation operator, and

$$G_0(1,2;1',2') = D(1,1')\, D(2,2')$$

and taking into account the symmetry of a kernel

$$K(1,2;1',2') = K(2,1;2',1') \qquad /2.3/$$

we get the following equation for the G-function

$$G(1,2;1',2') = D(1,1')\, D(2,2') +$$
$$+ \int D(1,3)\, D(2,4)\, K(3,4;5,6)\, G(5,6;1',2')\, dx_3\, dx_4\, dx_5\, dx_6 \qquad /2.4/$$

Hence, the equation for the four-dimensional wave function takes on the form

$$\Psi(1,2) = \int D(1,3)\, D(2,4)\, K(3,4;5,6)\, \Psi(5,6)\, dx_3\, dx_4\, dx_5\, dx_6 \qquad /2.5/$$

This equation is applicable both to scattering problems and to bound states.

Eq. /2.4/ in the momentum representation reads

$$G(p_1, p_2; q_1, q_2) = (2\pi)^{-8}\, D(p_1)\, D(p_2)\, \delta(p_1 - q_1)\, \delta(p_2 - q_2) +$$
$$+ (2\pi)^{-8}\, D(p_1)\, D(p_2) \int dq'_1\, dq'_2\, K(p_1, p_2; q'_1, q'_2)\, G(q'_1, q'_2; q_1, q_2) \qquad /2.6/$$

It can be rewritten in the following symbolic form

$$G = G_0 + G_0 K G \qquad /2.7/$$

One can easily get from here the expansion

$$G = G_0 (1 + K G_0 + K G_0 K G_0 + \dots) \qquad /2.8/$$

In the next paragraph we shall study the properties of the 2-time Green functions $G(t, \vec{x}_1; t, \vec{x}_2; \tau \vec{x}'_1; \tau \vec{x}'_2)$

Let us establish the relationship between the Fourier-transform of the 2-time function $G(\vec{p}, p_0; \vec{q}, p_0; \vec{p}', p_0'; \vec{q}', p_0')$ and that of the 4-time function $G(p, q, p', q')$

$$G(x_0, \vec{x}; x_0, \vec{y}; x_0', \vec{x}'; x_0', \vec{y}') = \int G(x, y; x', y')\, \delta(x_0 - y_0)\, \delta(x_0' - y_0')\, dy_0\, dy_0'$$

Substituting into this expression

$$G(x, y; x', y') = \int e^{ipx + iqy - ip'x' - iq'y'}\, G(p, q; p', q')\, dp\, dq\, dp'\, dq'$$

and having in mind, that

$$G(x_0 \vec{x}; x_0 \vec{y}; x_0' \vec{x}'; x_0' \vec{y}') =$$

$$= \int e^{-i\vec{p}\vec{x} - i\vec{q}\vec{y} + i\vec{p}'\vec{x}' + i\vec{q}'\vec{y}' + ip_0 x_0 - ip_0' x_0'}\, \tilde{G}(\vec{p}, p_0; \vec{q}, p_0; \vec{p}', p_0'; \vec{q}', p_0')\, d\vec{p}\, d\vec{p}'\, d\vec{q}\, d\vec{q}'\, dp_0\, dp_0'$$

we get

$$\tilde{G}(\vec{p}, p_0; \vec{q}, p_0; \vec{p}', p_0'; \vec{q}', p_0') = \int d\epsilon\, d\epsilon'\, G(\vec{p}, p_0 - \epsilon; \vec{q}, \epsilon; \vec{p}', p_0' - \epsilon'; \vec{q}', \epsilon') \qquad /2.9/$$

The momenta entering this expression satisfy the conservation law.

$$\vec{p} + \vec{q} = \vec{p}' + \vec{q}', \quad p_0 = p_0'.$$

In the centre- of- mass system of $\vec{p} + \vec{q} = 0$ the Fourier transform of the 2-time function depends both on the momenta $\vec{p}$, $\vec{p}'$ and on the energy of the system E. So, it can be put as $\tilde{G}(\vec{p}, \vec{p}'; E)$. Integrating both sides of the expansion /2.8/ and using /2.9/, we get the following expansion for the Fourier-transform of the 2-time Green function

$$\tilde{G} = \tilde{G}_0 + \widetilde{G_0 K G_0} + \widetilde{G_0 K G_0 K G_0} + \dots \qquad /2.10/$$

It follows from here for the $\tilde{G}^{-1}$ function that

$$\tilde{G}^{-1} = \tilde{G}_0^{-1} - \tilde{G}_0^{-1} \cdot \widetilde{G_0 K G_0} \cdot \tilde{G}_0^{-1} -$$
$$- \tilde{G}_0^{-1} \cdot \widetilde{G_0 K G_0 K G_0} \cdot \tilde{G}_0^{-1} + \tilde{G}_0^{-1} \cdot \widetilde{G_0 K G_0} \cdot \tilde{G}_0^{-1} \cdot \widetilde{G_0 K G_0} \cdot \tilde{G}_0^{-1} \qquad /2.11/$$

3. 2-Time Green Functions and Their Properties

Here we shall be concerned with a study of the spectral properties of the 2-time Green functions. It is not so difficult to establish these properties, as far as we are dealing in this case with one variable, and the problem is actually equivalent to the study of the analytic properties of the amplitudes with a fixed source/3/. The 2-time Green functions are also used in the problems of statistical physics/4/.

Let A and B be some operators which are, generally speaking, a product of the field operators.

Consider the Green function of the form

$$K_C(x,x') = < 0 \,/\, T\,(\, A\,(x)\; B(x')\,)\,/\,0> \qquad /\,3.1/$$

Before proceeding to the study of the spectral properties of the function /3.1/, we investigate the analytic structure of the retarded and advanced functions

$$K_r\,(\,x,x') = \theta\,(\,t-t')\,<0\,|\,A\,(x)\;B\,(x')\,|\,0>$$

$$K_a(\,x,\,x') = -\,\theta(\,t'-t)\,<0\,/\,A(x)\;B(x')\;/\;0> \qquad /3.2/$$

Here by $\vec{x}$ and $\vec{x}'$ we mean the groups of variables

$$\vec{x} = (\,\vec{x}_1,\,\vec{x}_2,\,...\,),\;\; \vec{x}' = (\,\vec{x}_1',\,\vec{x}_2',\,...\,) \qquad /3.3/$$

Let us find the Fourier-transforms of the functions /3.2/

$$K_r\,(\,\vec{x},\,\vec{x}';\,E\,) = \int e^{iE\tau}\;\theta(\tau)<0\,/\,A(x)\;B(x')\,/\,0>\,d\tau$$

$$K_a\,(\,\vec{x},\,\vec{x}';\,E) = -\int e^{iE\tau}\;\theta\,(-\tau)<0/\,A\,(x)\;B(x')\,/\;\;0>\,d\tau \qquad /3.4/$$

where

$$\tau = t - t'\,.$$

It can be seen from /3. 3/, that the function K_r is analytical in the upper half-plane E, and the K_a - in the lower half-plane. Let us calculate the difference between the functions $K_r - K_a$

$$I_{ab}(E) = \tfrac{1}{2}\,(\,K_r - K_a) = \tfrac{1}{2}\sum_n \delta\,(E-E_n) < 0\,/\,A\,(0,\vec{x})\,/\,n><n\,/\,B\,(0,\vec{x}')/0> \qquad /3.5/$$

Since $E_n \geq 0$, then the functions K_r and K_a coincide for the real values of $E<0$. Thus, there exists a single function $K\,(E;\vec{x},\vec{x}')$ holomorphic in the complex plane E cut along the real axis $E \geq 0$. For the Fourier-transform of this function with respect of the variables $\vec{x} = (\vec{x}_1, \vec{x}_2, ...)$ and $\vec{x}' = (\vec{x}_1', \vec{x}_2', ...)$ we get the following spectral representation

$$K_r\,(\,E;\,\lambda,\,\lambda'\,) = \frac{1}{\pi\, i}\int_0^\infty dE'\,\frac{I_{ab}\,(E')}{E' - E - i\epsilon} \qquad /3.6/$$

$$K_a(\,E;\,\lambda,\lambda'\,) = \frac{1}{\pi\, i}\int_0^\infty dE'\,\frac{I_{ab}(E')}{E' - E + i\,\epsilon}$$

Repeating the above arguments for the functions

$$K_r'\,(x,x') = -\,\theta\,(\,t-t'\,)<0\,/\,B\,(x')\;\;A(x)\,/\,0>$$

$$K_a'(\;x,x') = \theta\,(\,t'-t\,)<0\,/\;B(x')\;A(x)\;/0> \qquad /3.7/$$

we obtain

$$K_{r,a}'\;(E;\,\lambda,\lambda') = -\,\frac{1}{\pi\;\;i}\int_{-\infty}^{0} dE'\,\frac{I_{ba}(E')}{E' - E \mp i\epsilon} \qquad /3.8/$$

where

$$I_{ba}(E) = \tfrac{1}{2}\sum_n \delta\,(\,E+E_n)\;<0\,/\,B\,(0,\vec{x}')\,/n><n\,/\,A\,(0,\vec{x})/0> \qquad /3.9/$$

With the aid of the representations /3.6/ and /3.8/, we get the following spectral representation for the Green function K_C

$$K_c(E;\lambda,\lambda') = \frac{1}{\pi i}\int_0^\infty dE' \left[\frac{I_{ab}(E')}{E'-E-i\epsilon} + \frac{I_{ba}(-E')}{E'+E-i\epsilon} \right] \qquad /3.10/$$

In the case of the 2-time Green function we are interested in

$$G(t,\vec{x}_1;t,\vec{x}_2;t',\vec{x}'_1;t',\vec{x}'_2) = <0/T(\phi(t,\vec{x}_1)\overset{+}{\phi}(t,\vec{x}_2)\phi(t',\vec{x}'_2)\overset{+}{\phi}(t',\vec{x}'_1)/0> \qquad /3.11/$$

the operators A and B are equal

$$A(x) = \phi(t,\vec{x}_1)\overset{+}{\phi}(t,\vec{x}_2),\ B(x') = \phi(t',\vec{x}'_2)\overset{+}{\phi}(t',\vec{x}'_1) \qquad /3.12/$$

and, hence,

$$I_{ab}(E;\vec{x}_1,\vec{x}_2;\vec{x}'_1,\vec{x}'_2) = \tfrac{1}{2}\sum_n \delta(E-E_n)\, I^{(n)}_{ab}(\vec{x}_1,\vec{x}_2;\vec{x}'_1,\vec{x}'_2) \qquad /3.13/$$

$$I_{ba}(E;\vec{x}_1,\vec{x}_2;\vec{x}'_1,\vec{x}'_2) = \tfrac{1}{2}\sum_n \delta(E+E_n)\, \overset{*}{I}{}^{(n)}_{ab}(\vec{x}_2,\vec{x}_1;\vec{x}'_2,\vec{x}'_1)$$

where

$$I^{(n)}_{ab}(\vec{x}_1,\vec{x}_2;\vec{x}'_1,\vec{x}'_2) = <0/\phi(0,\vec{x}_1)\overset{+}{\phi}(0,\vec{x}_2)/n><0/\phi(0,\vec{x}'_1)\overset{+}{\phi}(0,\vec{x}'_2)/n>^*$$

or for the Fourier transforms

$$I_{ab}(E;\lambda,\lambda') = \tfrac{1}{2}\sum_n \delta(E-E_n)\, I^{(n)}_{ab}(\vec{p}_1,\vec{p}_2;\vec{p}'_1,\vec{p}'_2) \qquad /3.14/$$

$$I_{ba}(E;\lambda,\lambda') = \tfrac{1}{2}\sum_n \delta(E+E_n)\, \overset{*}{I}{}^{(n)}_{ab}(-\vec{p}_2,-\vec{p}_1;-\vec{p}'_2,-\vec{p}'_1)$$

but in the centre-of-mass system

$$I^{(n)}_{ab}(\vec{p}_1,\vec{p}_2;\vec{p}'_1,\vec{p}'_2) = I^{(n)}_{ab}(-\vec{p}_2,-\vec{p}_1;-\vec{p}'_2,-\vec{p}'_1) \qquad /3.15/$$

Therefore:

$$I_{ba}(-E;\lambda,\lambda') = \overset{*}{I}_{ab}(E;\lambda,\lambda') \qquad /3.16/$$

Using /3.10/ and /3.16/, for the 2-time Green function we get the following spectral representation

$$\tilde{G}(E;\lambda,\lambda') = \frac{1}{\pi i}\int_0^\infty dE' \left[\frac{I_{ab}(E';\lambda,\lambda')}{E'-E-i\epsilon} + \frac{\overset{*}{I}_{ab}(E';\lambda,\lambda')}{E'+E-i\epsilon} \right] \qquad /3.17/$$

The operator $I_{ab}(E;x,x')$ is positive-definite since the form

$$<I_{ab}(E)> = \int \overset{*}{\psi}(\vec{x}_1)\overset{*}{\psi}(\vec{x}_2)\, I_{ab}(E;\vec{x}_1,\vec{x}_2;\vec{x}'_1,\vec{x}'_2)\,\psi(\vec{x}'_1)\psi(\vec{x}'_2)\,d\vec{x}'_1\,d\vec{x}'_2\,d\vec{x}_1\,d\vec{x}_2$$

$$<I_{ab}(E)> = <I_{ab}(E)>^* \qquad /3.18/$$

has a definite sign. Let us establish some properties of the operator I_{ab}. According to /3.13/, we have

$$I^{(n)}_{ab}(\vec{x}_1,\vec{x}_2;\vec{x}'_1,\vec{x}'_2) = \overset{*}{I}{}^{(n)}_{ab}(\vec{x}'_1,\vec{x}'_2;\vec{x}_1,\vec{x}_2) \qquad /3.19/$$

from here

$$I^{(n)}_{ab}(\vec{p}_1,\vec{p}_2;\vec{p}'_1,\vec{p}'_2) = \overset{*}{I}{}^{(n)}_{ab}(\vec{p}'_1,\vec{p}'_2;\vec{p}_1,\vec{p}_2) \qquad /3.20/$$

or in the centre-of-mass system

$$I^{(n)}_{ab}(\vec{p},\vec{p}') = \overset{*}{I}{}^{(n)}_{ab}(\vec{p}',\vec{p}) \qquad /3.21/$$

Therefore, by Eq. /3.14/ we have

$$I_{ba}(-E;\lambda,\lambda') = I_{ab}(E;\lambda',\lambda) \qquad /3.22/$$

or taking into account /3.16/

$$\overset{*}{I}_{ab}(E;\lambda,\lambda') = I_{ab}(E;\lambda',\lambda) \qquad /3.23/$$

On the other hand,

$$I^{(n)}_{ab}(\vec{p}_1,\vec{p}_2;\vec{p}'_1,\vec{p}'_2) = I^{(n)}_{ab}(-\vec{p}_1,-\vec{p}_2;-\vec{p}'_1,-\vec{p}'_2) \qquad /3.24/$$

But, since

$$I^{(n)}_{ab}(\vec{p}_1,\vec{p}_2;\vec{p}'_1,\vec{p}'_2) = I^{(n)}_{ab}(-\vec{p}'_1,-\vec{p}'_2;-\vec{p}_1,-\vec{p}_2) \qquad /3.25/$$

then

$$I^{(n)}_{ab}(\vec{p}_1,\vec{p}_2;\vec{p}'_1,\vec{p}'_2) = I^{(n)}_{ab}(\vec{p}'_1,\vec{p}'_2;\vec{p}_1,\vec{p}_2) \qquad /3.26/$$

It follows from here that

$$I_{ab}(E;\lambda,\lambda') = I_{ab}(E;\lambda',\lambda) \qquad /3.27/$$

According to /3.23/ and /3.27/ the spectral representation for the function $\tilde{G}(E;\lambda,\lambda')$ can be put as

$$\tilde{G}(E;\lambda,\lambda') = \frac{E}{\pi i}\int_0^\infty dE'^2\,\frac{I_{ab}(E'^2;\lambda,\lambda')}{E'^2-E^2} \qquad /3.28/$$

where

$$I_{ab}(E^2;\lambda,\lambda') = \sum_n \delta(E^2-E_n^2)\,\theta(E)\,I^{(n)}_{ab} \qquad /3.29/$$

From (3.18) it is seen that the operator I_{ab} is positive-definite.

Since the imaginary part of the operator $\tilde{G}$ has a definite sign, one can establish that the imaginary part of of the operator $\tilde{G}^{-1}$ will have a definite sign. either.

Indeed, let A and B be two non-commutating operators, and

$$D = A + i(B+\epsilon), \quad \epsilon > 0. \qquad /3.30/$$

$B \geq 0$ (in the sense of /3.18/).

Writting D as

$$A + i(B+\epsilon) = \sqrt{B+\epsilon}\,[\,(B+\epsilon)^{-1/2} A (B+\epsilon)^{-1/2} + i\,]\sqrt{B+\epsilon} \qquad /3.31/$$

we have

$$[A + i(B+\epsilon)]^{-1} = (B+\epsilon)^{-1/2} \frac{(B+\epsilon)^{-1/2} A (B+\epsilon)^{-1/2} - i}{(B+\epsilon)^{-1/2} A (B+\epsilon)^{-1} A (B+\epsilon)^{-1/2} + 1} (B+\epsilon)^{-1/2}$$

From here

$$Im\, D^{-1} = -[\,A(B+\epsilon)^{-1} A + B + \epsilon\,]^{-1} < 0 \qquad /3.32/$$

4. The Equation for a System of Two Particles

The Fourier-transform of the 2-time Green function $\tilde{G}$ satisfies the following equation

$$\int \Sigma(\vec{p}, \vec{q}; E)\, \bar{G}(\vec{q}, \vec{p}'; E)\, d\vec{q} = \delta(\vec{p} - \vec{p}') \qquad /4.1/$$

The expression for the operator Σ may be obtained in the framework of the perturbation theory. In this sense, the situation here is alike that with the Bethe-Salpeter equation /2.1/ where the kernel K is also found from the perturbation theory. The algorithm for constructing the operator Σ is given by /2.11/. The equation for the wave function of a system of two particles is

$$\int \Sigma(\vec{p}, \vec{p}'; E)\, f(\vec{p}')\, d\vec{p}' = 0 \qquad /4.2/$$

Making use of /2.11/ the equation for f may be written as

$$(E^2 - \vec{p}^{\,2} - m^2)\, f(\vec{p}) = \int V(\vec{p}, \vec{p}'; E)\, f(\vec{p}')\, d\vec{p}' \qquad /4.3/$$

We have shown earlier that using the Lippmann-Schwinger method and starting from Eq. /4.3/ it is possible to construct the scattering matrix. It can be easily seen that this scattering matrix has the same form as that usually defined in the quantum field theory. Indeed, since

$$G(t, \vec{p}; t, \vec{q}; \tau, \vec{p}'; \tau, \vec{q}') = \langle 0 / T(\phi(\vec{p},t)\, \phi(\vec{q},t)\, \phi^{+}(\vec{q}',\tau)\, \phi^{+}(\vec{p}',\tau) / 0 \rangle \qquad /4.4/$$

then, taking into account the conventional relationship between the operators and the state vectors in the Heisenberg representation and in the interaction representation, we get

$$G(t,\tau) = \langle 0 \,|\, S(\infty,t)\,\phi(\vec{p},t)\,\overset{+}{\phi}(\vec{q},t)\,S(t,\tau)\,\phi(\vec{q}',\tau)\,\overset{+}{\phi}(\vec{p}',\tau)\,S(\tau,-\infty)\,/\,0\rangle\, S_0^{-1} .$$

Then, with the aid of the adiabatic hypothesis about the switching on and switching off the interaction, we may go over to the limit $\tau \to -\infty$

$$\psi(t) = \lim_{\tau\to-\infty} G(t,\tau)$$

At the same time the function $\psi(t)$ obtained is a solution of the homogeneous equation .

$$\Sigma\psi = 0$$

which at $t\to-\infty$ represents a plane wave, while at $t\to\infty$ -a scattered wave.

So, passing to the limit $t\to\infty$, we arrive at a conventional expression for the matrix element of the scattering amplitude

$$\lim_{t\to\infty}\psi(t) = \lim_{\substack{\tau\to-\infty\\ t\to\infty}} G(t,\tau) = \rho\sqrt{p_0\,p'_0\,q_0\,q'_0}\,\langle 0 \,|\, a(p')\,b(q')\,S\,\overset{+}{a}(p)\,\overset{+}{b}(q)\,|\,0\rangle$$

It follows from here that according to our definition of the potential the scattering amplitude constructed by the Lippmann-Schwinger method leads to a customary expression for the scattering amplitude in the quatum field theory.

The generalized potential is a complex function of the momenta $\vec{p}$, $\vec{p}'$ and of the energy of the system E. The imaginary part of this potential has a definite sign /3.17/ and characterizes the absorption in the system, i.e., the inelastic scattering processes. Since it follows from Eq. /4.1/ that $\Sigma = \tilde{G}^{-1}$, the potential is an analytic function of E, and its spectral representation results directly from the spectral representation /3.12/ for the 2-time Green function.

So, we have shown that in the framework of the concepts of the quantum field theory the system of two particles may be described by an equation of the Schrödinger type with complex potential which can be constructed according to the algorithm developed in the framework of the perturbation theory. The description of the system of two particles with the aid of Bethe-Salpeter equation has some shortcomings.

For instance, it is absolutely obscure what physical condition is necessary to formulate with respect to the relative time of two particles in solving this equation. Nor is it clear how to interpret from the physical point of view the 4-dimensional wave function because its norm is not positive-definite. It is quite obvious that the approach to the description of two particles we developed above is free from these shortcomings. To summarize, several remarks should be made.

Making use of the results of §1, with the aid of the generalized complex potential V one can show that the matrix elements of the scattering amplitudes in any order of the perturbation theory coincide with those obtained in a usual manner.

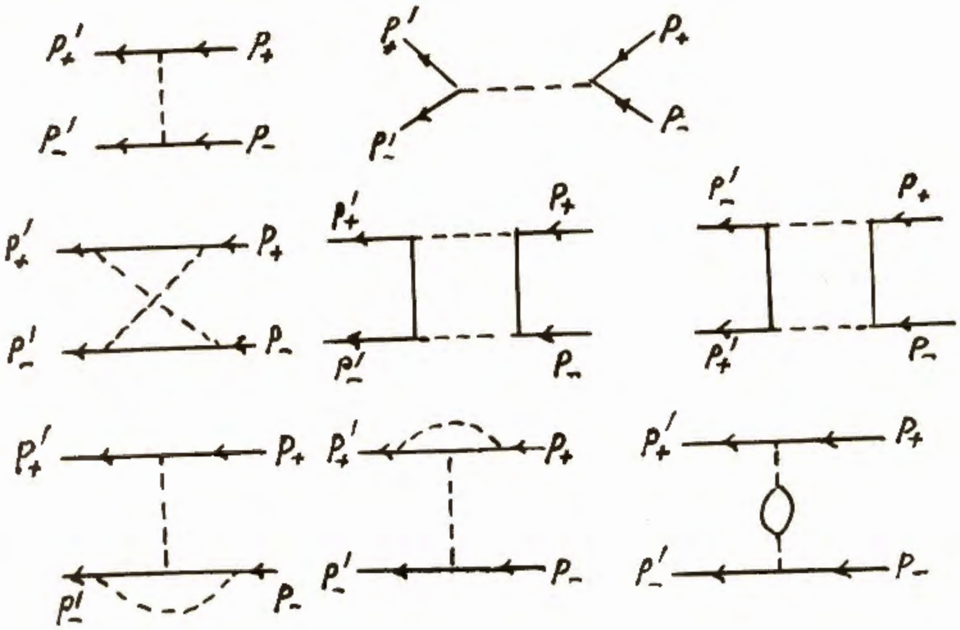
On the other hand, the potential V allows to depart beyond the mass shell and to get the equation for the 2-time Green function /4.1/. The situation with this side is the same as that with the Bethe-Salpeter equation which permits, on the one hand, to describe correctly the matrix elements of the scattering amplitude within the perturbation theory, and, on the other, to go outside the mass shell and to find the 4-time Green function.

Let us illustrate the method developed above by the interaction of the form

$$H_I = g\,\overset{+}{\phi}\,\phi\,A(x)$$

where ϕ is the field operator of scalar particles with the mass M, and A is the field operator of scalar particles with zero masses.

In the lowest orders of the perturbation theory the following irreducible graphs will give the contribution to the kernel:



where p_+ is the initial momentum of a positively charged particle, and p_- is the initial momentum of the negatively charged particle.

It can be seen that the generalized potential has the real and imaginary parts. By /2.11/ and /4.3/ in the lowest order of the perturbation theory, for the real and imaginary parts of the potential we have:

$$Re\ V(\vec{p},\vec{p}\,';E) = e^2 \tilde{G}_0^{-1}(p)\, \tilde{G}_0^{-1}(p') \int\int d\epsilon\, d\epsilon'\, G_0(\epsilon,\vec{p})\, G_0(\epsilon',\vec{p}) \cdot$$

$$\cdot \left[\frac{1}{(\vec{p}-\vec{p}\,')^2 - (\epsilon-\epsilon')^2} - \frac{1}{2E^2} \right]$$

$$Im\ V(\vec{p},\vec{p}\,';E) = e^4 \tilde{G}_0^{-1}(\vec{p})\, \tilde{G}_0^{-1}(\vec{p}\,') \int dk\, \delta(k^2)\, \delta((E-k)^2)\, I_k(\vec{p})\, I_k(\vec{p}\,')$$

where

$$G_0^{-1}(\epsilon,\vec{p}) = [(\tfrac{1}{2}E+\epsilon)^2 - p^2 - m^2]\,[(\tfrac{1}{2}E-\epsilon)^2 - \vec{p}^{\,2} - m^2]$$

$$I_k(\vec{p}) = \int d\epsilon\, G_0(\epsilon,p)\, [\epsilon^2 - (\vec{p}-\vec{k})^2 - m^2]^{-1}$$

The real part of the potential determines the energy levels of the system, the imaginary part characterizes the inelastic processes and on being averaged, coincides with the probability of breaking up the system into two scalar quanta.

5. A Method of Summing the Graphs

As was shown in /5/, the perturbation theory terms contain the information about the bound states of the system of two particles. The problem as to how to get this information from the perturbation theory is far from being trivial. The difficulty is that in the ℓ plane appear the cuts which make the picture essentially complicated/6/. This should be taken into consideration if one wants to single the Regge singularities out of the perturbation theory. The mechanical mixing of the perturbation theory terms may lead to erroneous conclusions.

On the other hand, it should be borne in mind, that the perturbation theory terms which are essential for describing the bound states are proportional not only to the small coupling constant e^2 , but to the parameter $\frac{e^2}{\sqrt{W}}$ as well (W is the binding energy). This parameter is not at all small in this case. Therefore, it seems worthwhile to develop a method of summing such terms and to get the expansion in a small parameter e^2 /7/ . Such a method is, in fact, the description of the bound state with the aid of the Schrödinger equation /4.3/ with the complex potential which, in contrast to the expansion terms of the scattering amplitude, no longer contains the terms of the form $\frac{e^2}{\sqrt{W}}$

This approach is general enough as long as it leads, on the one hand, to the relativistic S-matrix, and, on the other, allows to depart outside the energy shell and to construct the 2-time Green function.

Here we consider a less general problem of constructing the generalized potential which leads to the matrix elements in the quantum field theory. Such a formulation of the problem is ambiguous since there may exist a fairly wide class of potentials so that the equation

$$(E^2-\vec{p}^{\,2}-m^2)\,\psi(p)-\int V(\vec{p}^{\,2},\vec{p}'^{\,2},E^2,(\vec{p}-\vec{p}')^2)\,\psi(p')\,d\vec{p}' \qquad /5.1/$$

will lead to the same scattering matrix. The question arises as to how to choose the simplest potential satisfying this requirement. In what follows we shall develop the method of constructing the complex potential dependent only upon energy and upon relative momentum $(p-p')^2$. In this manner we reduce the problem of summing the graphs to the solution of the Schrödinger equation with the complex potential depending on the variables E and $(p-p')^2$. It is worthwhile nothing that such a reconstruction of the perturbation theory series makes it possible to describe the bound state.

Before proceeding to the construction of the potential, we dwell upon the discussion of some properties of the scattering amplitudes. Following Mandelstam, the amplitude $M(s,t)$ can be represented as

$$M(s,t)=u_1(s)+u_2(s)+u_3(u)+\int_{-\infty}^{\infty}ds'\int_{4M^2}^{\infty}dt'\,\frac{R_1(s',t')}{(s'-s)(t'-t)}+ \qquad /5.2/$$

$$+\int_{-\infty}^{\infty}ds'\int_{4M^2}^{\infty}du'\,\frac{R_2(s',u')}{(s'-s)(u'-u)}$$

the functions R_1 and R_2 are expressed in terms of the Mandelstam spectral functions as

$$R_1(s,t)=\rho_1(s,t)\,\Theta(s-4M^2)\,\theta(t-4M^2)-$$

$$-\rho_3(t,u)\,\theta(u-4M^2)\quad\theta(t-4M^2) \qquad /5.3/$$

$$R_2(s,t)=\rho_2(s,u)\,\theta(s-4M^2)\,\theta(u-4M^2)-\rho_3(t,u)\,\theta(t-4M^2)\,\theta(u-4M^2)$$

Owing to the invariance under the 3-dimensional rotation group one can, instead of the amplitude M, introduce the amplitudes $M^{(+)}$ and $M^{(-)}$ acting correspondingly on the even and odd states. They can be determined in the following manner

$$M\phi_{e}=M^{(+)}\phi_{e}=\psi_{e},\quad M\phi_{n}=M^{(-)}\phi_{n}=\psi_{n}. \qquad /5.4/$$

where ϕ_s denotes the even state

$$\phi_s(\vec{q}) = \phi_s(-\vec{q}) \qquad /5.5/$$

and ϕ_a designates the odd one

$$\phi_a(\vec{q}) = -\phi_a(-\vec{q}) \qquad /5.6/$$

Thus, one can regard the even and odd states independently. Since the operator $M^{(+)}$ acts on the even states, and $M^{(-)}$ on the odd ones, then, by /5.2/, the following spectral representations can be given for them

$$M^{(+)}(s,t) = u_1(s) + u_2(t) + u_3(t) + \int_{-\infty}^{\infty} ds' \int_{4M^2}^{\infty} dt' \frac{R_1(s',t') + R_2(s',t')}{(s'-s)(t'-t)}$$

$$M^{(-)}(s,t) = u_2(t) - u_3(t) + \int_{-\infty}^{\infty} ds' \int_{4M^2}^{\infty} dt' \frac{R_1(s',t') - R_2(s',t')}{(s'-s)(t'-t)} \qquad /5.7/$$

As far as the amplitude is divided into the even and odd parts, we construct two generalized potentials $V_+(s,t)$ and $V_-(s,t)$ correspondingly. The potential can be reconstructed by the given expansion of the scattering amplitude in the perturbation theory, if use is made of the expression for the scattering amplitude (see the formula /1.25/)

$$M = v + v \frac{1}{E[(E+i\epsilon)^2 - F^2] - v} v \qquad /5.8/$$

where

$$T = -\frac{\pi}{E^2} \delta(E_i - E_f) M, \quad V = v/E .$$

Representing M_+ and v_+ as an expansion

$$M^{(+)} = M_0^{(+)} + M_1^{(+)} + \dots$$
$$v^{(+)} = v_0^{(+)} + v_1^{(+)} + \dots \qquad /5.9/$$

and substituting into the expression of form /5.8/, we get

$$v_+(E, p-p') = M_0^{(+)} + M_1^{(+)} - M_0^{(+)} \frac{1}{E[(E+i\epsilon)^2 - F^2(p)]} M_0^{(+)} \qquad /5.10/$$

A similar expression can be obtained for the potential v_- . In the given case the variable E stands in the denominator of the potentials V_+ and V_- which makes the investigation at the point $E=0$ difficult. In order to avoid this, it is convenient in a number of cases to determine the potential in Eq. /5.1/ as follows:

$$(E^2 - \vec{p}^2 - m^2)\psi(p) - \frac{1}{\sqrt{\vec{p}^2 + m^2}} \int W(\vec{p}^2, \vec{p}'^2; E^2, (\vec{p}-\vec{p}')^2)\psi(p')\, d\vec{p}' \qquad /5.11/$$

Then W is related to the scattering amplitude by

$$M = W + W \frac{1}{\sqrt{m^2 + \vec{p}^2}\,[(E+i\epsilon)^2 - F^2(p)] - W} W \qquad /5.12/$$

In a similar manner we find

$$W_+(E, \vec{p}-\vec{p}\,') = M_0^{(+)} + M_1^{(+)} - M_0^{(+)} \frac{1}{\sqrt{\vec{p}^{\,2}+m^2}\,[(E+i\epsilon)^2 - F^2]} M_0^{(+)} \qquad /5.13/$$

Note, that the third term in the expressions/5.10/ and /5.13/ takes off from M_1 the part of the amplitude which is due to the Coulomb scattering.

From /5.10/ and /5.13/ we see, that the potentials in this approach is determined only by the matrix elements of the scattering amplitude. As soon as in constructing the potential, the latter was required to be consistent with the matrix elements of the perturbation theory as well as to be dependent only upon the energy and relative momentum, the departure beyond the mass shell with the aid of the given potential will fail to describe truly the situation in the quantum field theory.

In conclusion we should like to note that, as long as the potential in the quantum field theory is complex and energy dependent, in order to clear up how far Regge's ideas can be applied to the quantum field theory, it seems extremely desirable to investigate the analiticity in the ℓ plane by resorting to the Schrödinger equation with the complex potential having definite(see/4/) analytical properties by the variable E, and in some cases (see /5/) by the variable t, as well/8/.

The authors express their deep gratitude to Academician N.N.Bogolubov whose advice and discussions stimulated the writing of this paper. They are also grateful to B.A.Arbuzov, Nguen Van Hieu, R.N.Faustov, A.T.Filippov, and O.A.Khrustalev for useful discussions.

References

1. T.M.Charap and S.P.Fubini. Nuovo Cim., XIV, N. 3 (1959). Nuovo Cim. XV, N. 1 (1960).
2. G.F.Chew and Frautschi. Phys. Rev. 124, N 1 (1961).
3. A.A.Logunov and A.N.Tavkhelidze. Proceedings of the Georgian Academy of Sciences, XVIII, N 1, N 5 (1957).
4. V.L.Bonch-Bruevich, S.V.Tyablikov, The Green Function Model in Statistical Mechanics, GIF, ML Moscow (1961).
5. V.A.Arbuzov, A.A.Logunov, A.N.Tavkhelidze and R.N.Faustov, Phys. Rev. Lett. 2, 150 (1962).
   B.A.Arbuzov, A.A.Logunov, A.N.Tavkhelidze, R.N.Faustov and A.T.Filippov. Phys. Rev. Lett. 2, 305 (1962).
   A.A.Logunov, Nguen Van Hieu, A.N.Tavkhelidze, O.A.Khrustalev (in print).
6. B.A.Arbuzov, B.M.Barbashev, A.A.Logunov, Nguen Van Hieu, A.N.Tavkhelidze, R.N.Faustov and A.T.Filippov, preprint JINR, E-1095.
7. B.A.Arbuzov, A.A.Logunov, A.N.Tavkhelidze and R.N.Faustov. (in print).
8. B.A.Arbuzov, A.A.Logunov, A.N.Tavkhelidze, R.N.Faustov, and A.T. Filippov, (to be published).
9. D.I.Blokhintsev, V.S.Barashenkov, and B.M.Barbashov. Uspekhi fis. nauk, volume LXVIII is. 3.

Received by Publishing Department
on December 18 , 1962 .