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P.T.Kirstein

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ON THE ENERGY SPREAD OF A STACKED BEAM CAUSED BY THE MULTIPLE TRAVERSALS OF RF BUCKETS

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* On leave from the Centre of European Nuclear Research, Geneve, Switzerland.

A b s t r a c t

Approximate expressions are derived for calculating the increase of energy spread of a stacked beam due to the multiple traversal of rf buckets. These expressions are evaluated for two initial energy distributions of the stacked beam; in one of these cases the results are compared with exact numerical calculations performed by Swenson, and the agreement is shown to be fairly close.

1. Introduction

In fixed field accelerators, it is possible to stack a large number of pulses of current. The injected pulses of current are captured by an rf field; then either the frequency of the rf[1,2] is modulated, or energy is given to the bunch by a betatron field[2,3], and the energy of the injected beam is increased to a final stacking energy. This procedure is repeated as often as desired with identical rf programmes. It is of interest to investigate the resulting energy spread of the stacked beam. This problem has been treated theoretically by several authors[1,4,6]. The investigation leads to the solution of a differential equation for the energy of the particle as a function of its original parameters. This differential equation may be solved as a function of the initial energy and phase, with respect to the rf of the particle. The repetition rate of the rf cycles is usually many orders of magnitude greater than the frequency of the rf; hence any phase coherence between the starting frequency of the rf and the circulation frequency of the beam will be lost due to the inherent spread of revolution frequency. For this reason, the quantity of interest is not the distribution function of particles, but the value of this function averaged with respect to all possible initial phases. We may therefore define a function $P_n(E)$ which is the number of particles having energy in the range E, $E + dE$ after the n^{th} cycle of the rf. Lebedev[4] has shown that $P_n(E)$ satisfies the integral equation

$$P_n(E) = \exp(-\alpha) \int_{-\infty}^{\infty} F(\Delta, E)\, P_{n-1}(E-\Delta)\, d\Delta + \Psi(E) \quad , \quad (1)$$

and that under certain assumptions, Eq. (1) can be solved explicitly. In Eq. (1), $\Psi(E)$ is the energy spread of the beam brought up by each cycle of the rf; $F(\Delta,E)$ is the probability that a particle originally in the energy range $\Delta, \Delta+dE$ will have final energy in the range $E, E+dE$ after one cycle of the rf, and $\exp(-\alpha)$ is the loss of particles in one rf cycle due to gas scattering. Lebedev solved Eq.(1) for two limiting cases, when a steady state was achieved, and when n was very large. We will make no attempt to discuss further the derivation of Eq.(1), but will refer to Ref. [4] to justify the equation.

The actual form of $F(\Delta,E)$ depends on the detailed parameters of the machine and the rf programme, while Ψ depends on these parameters and on the energy spectrum of the injected beam. In the particular case that the rf passes through the region of the beam and is switched off far away, F may be independent of E in Eq.(1). This case will be considered in this paper. The results therefore will have limited validity, through they may serve as a qualitative approximation to the behaviour when F also depends on E in Eq.(1). The solution of this simplified problem is of interest in the compensation of the effects of radiation in a stacked beam.

We will find a general solution of Eq.(1) by a method essentially the same as that used by Lebedev[4]. We will then make an empirical assumption for the form of F and solve for P_n explicitly. Finally, we will compare our results with exact calculations made by Swenson[5] on a digital computer. Our results will be shown to be a good approximation to the exact solution for all n for one particular case.

2. The General Solution

In this section we will solve Eq.(1) exactly for the case that F does not depend on E. Our procedure will be to use Fourier transform methods, in a way similar to Lebedev[4]. The exact forms of F, Ψ and the original distribution are immaterial for the formal solution - though the explicit functions are required for evaluating the resulting expressions. Several limiting forms of the solution will be given.

If F does not depend on E in Eq.(1), the equation may be written

$$P_n(E) = \exp(-\alpha) \int_{-\infty}^{\infty} F(\Delta)\, P_{n-1}(E-\Delta)\, d\Delta + \Psi(E) \quad . \qquad (2)$$

If we now take Fourier transforms, using the relation[7] between a function x and its Fourier transform

$$\left.\begin{aligned} x(\lambda) &= 1/\sqrt{2\pi} \int_{-\infty}^{\infty} \exp(i\lambda E)\ X(E)\ dE \\ X(E) &= 1/\sqrt{2\pi} \int_{-\infty}^{\infty} \exp(-i\lambda E)\ x(\lambda)\ d\lambda \end{aligned}\right\} \quad , \qquad (3)$$

we obtain

$$\begin{aligned} p_n(\lambda) &= 1/\sqrt{2\pi} \int_{-\infty}^{\infty} \exp(i\lambda E)\ P_n(E)\ dE \\ &= e^{-\alpha}/\sqrt{2\pi} \int_{-\infty}^{\infty} F(\Delta)\ d\Delta\ e^{i\lambda\Delta} \int_{-\infty}^{\infty} P_{n-1}(E-\Delta)\ e^{i\lambda(E-\Delta)} dE + 1/\sqrt{2\pi} \int_{-\infty}^{\infty} \Psi(E)\ e^{i\lambda E} dE \\ &= \sqrt{2\pi} \exp(-\alpha)\ f(\lambda)\ p_{n-1}(\lambda) + \psi(\lambda) \quad . \end{aligned} \qquad (4)$$

$$F(\Delta) = (1/D)\sqrt{2/\pi}\ \exp\left[-(\Delta-\bar{\Delta})^2/(2D^2)\right] \quad , \qquad (11)$$

where $\bar{\Delta}$ is the average width of the rf bucket near the stack and D is its root-mean-square deviation. Most of the $F(\Delta)$ which occur in practice have approximately this form; however our main justification is the close fit between exactly computed energy spreads, and ones derived from Eq.(11). Since $F(\Delta)$ is a probability, it must be such that its integral from $-\infty$ to ∞ is unity; Eq.(11) satisfies this condition. If $F(\Delta)$ has the form of Eq.(11), the g of Eq. (5) is given by

$$g(\lambda) = \exp(-\alpha)\exp(-\lambda^2D^2/2 + i\lambda\bar{\Delta}) \quad . \qquad (12)$$

For large E, the P_n resulting from this $g(\lambda)$ may be obtained by the WKBG approximation irrespective of F. Let us now consider a beam of N particles which originally uniformly occupies the region shown in Fig.1. Then P_o is given by the expression

$$\left.\begin{aligned} P_o(E) &= N/(2c) \quad , -c \leq E \leq c \\ &= 0 \quad , \text{otherwise} \end{aligned}\right\} \quad . \qquad (13)$$

We wish to find the energy distributions after the passage of n empty buckets. In this case $p_o(\lambda)$ is given from Eqs (3) and (13) by the relation

$$p_o(\lambda) = N/\sqrt{2\pi}\quad (\sin c\lambda)/(c\lambda) \quad . \qquad (14)$$

In this case no additional rf buckets are brought up, so that Ψ is zero, and Eq.(7) becomes

$$p_n(\lambda) = (N/\sqrt{2\pi})\exp(-n\alpha - n\lambda^2D^2/2 + in\bar{\Delta})\sin c\lambda/(c\lambda) \quad . \qquad (15)$$

Now the Fourier transform of dP_n/dE is $-i\lambda$ times that of P_n, hence

$$\frac{dP_n}{dE} = \frac{N}{\sqrt{2\pi}c}\int_{-\infty}^{\infty} \exp(-n\alpha - i\lambda E - nD^2\lambda^2/2 + i\lambda n\bar{\Delta})\left[e^{-i\lambda c} - e^{i\lambda c}\right]/(2i)\, d\lambda$$

$$= \frac{N \exp(-n\alpha)}{D\sqrt{2n\pi}}\left[\exp-\left\{(E + c + n\bar{\Delta})^2/(2nD^2)\right\} - \exp-\left\{(E - c - n\bar{\Delta})^2/(2nD^2)\right\}\right]. \quad (16)$$

Integrating Eq. (16), we obtain

$$P_n = \frac{N e^{-n\alpha}}{2}\left[\mathrm{Erf}\left\{(E + c - n\bar{\Delta})/(nD\sqrt{2})\right\} - \mathrm{Erf}\left\{(E - c - n\bar{\Delta})/(nD\sqrt{2})\right\}\right] , \quad (17)$$

where Erf(x) is the usual error function of x, $(2/\sqrt{\pi}) \int_0^x \exp(-x^2)\, dx$
We may easily verify that the maximum of P_n occurs at $E = n\bar{\Delta}$, and that the value of P_n there is $N \exp(-n\alpha)\ \mathrm{Erf}\ c/(nD\sqrt{2})$. Since P_n is symmetric about this point, the average energy also changes by $n\bar{\Delta}$.

Finally, we may easily verify that the mean-square-deviation D_n is given by

$$D_n = \left[\int_{-\infty}^{\infty} E^2 P_n(E)\, dE \Big/ \int_{-\infty}^{\infty} P_n(E)\, dE\right] - \left[\int_{-\infty}^{\infty} E\, P_n(E)\, dE \Big/ \int_{-\infty}^{\infty} P_n(E)\, dE\right]^2$$

$$= c^2/3 + nD^2 . \quad (18)$$

Thus in this model the mean-square-energy spread increases by D^2 at each repetition cycle of the rf. Swenson[5] has calculated numerically a particular example of this case for a particular voltage programme, with no losses due to vacuum. The comparison of Swenson's exact calculations and our analytic ones is given in Fig.2. Good agreement is obtained for all n, where Lebedev only claimed his results valid for large n.

Using the formulae developed, with the F of Eq.(11), and thus the g of Eq.(12), it is possible to investigate the effects of the passing buckets on any initial distribution. If the bucket stops near or in the stack, F will depend on Δ and E, and the theory will no longer be valid. However for large distances from the place where the bucket stops, the theory will be again approximately true, and we may use non-zero Ψ to represent the particles brought up.

As another easily calculable example, let us consider the case when

$$\Psi = 0, \qquad P_0(E) = (N/d)\sqrt{2/\pi}\ \exp-\left[E^2/(2d^2)\right] \quad . \tag{19}$$

Then in a similar way to the previous example,

$$p_0 = N/(2d\sqrt{\pi})\ \exp-(\lambda^2 d^2/2) \quad , \tag{20}$$

so that

$$P_n = \left\{N\ e^{-n\alpha}/\left[2nD^2 + d^2\right]\right\} \exp-\left[(E-n\bar{\Delta})^2/(2nD^2 + 2d^2)\right] . \tag{21}$$

The mean E again shifts by $n\bar{\Delta}$ and the mean-square-deviation is now $(d^2 + nD^2)$. This method may be extended to any other initial distribution; for large n the final distribution is almost independent of the initial one.

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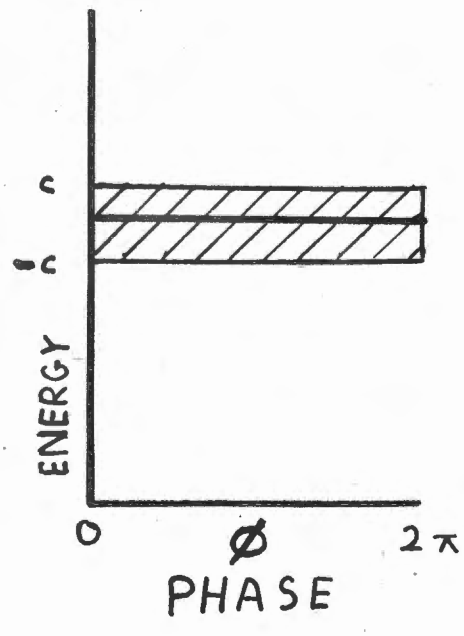

Fig. I. Initial distribution of particles in Phase Space in the stacked beam.

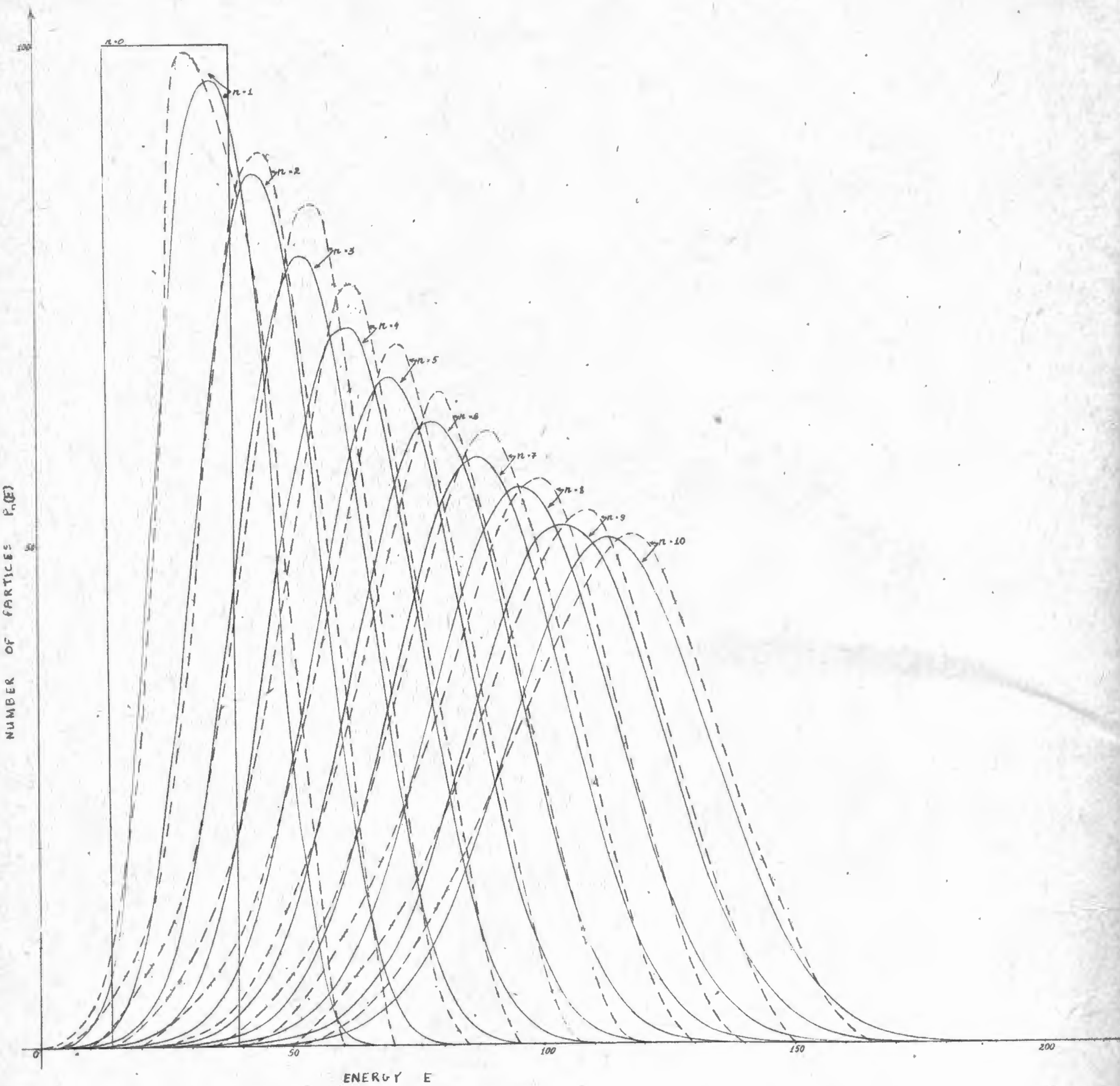

Fig. 2. Comparison of the Energy Spread in the stacked beam as predicted by Eq. (17) with Swenson's exact calculations. The different curves refer to the number n of passages of the rf bucket. The solid curves are predictions of Eq.(17), the broken lines those obtained by Swenson.