ОБЪЕДИНЕННЫЙ ИНСТИТУТ ЯДЕРНЫХ ИССЛЕДОВАНИЙ

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ON THE EFFECTS OF THERMAL VELOCITIES IN TWO DIMENSIONAL AND AXIALLY SYMMETRIC BEAMS

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Abstract

The effects of thermal velocity in two dimensional and axially symmetric beams are evaluated, assuming a Maxwellian Distribution of velocities at the cathode. The linear treatment is similar to that of Cutler and Hines, but the approach is somewhat different. Analytic expressions and curves are given for the current density in terms of the error and modified Bessel Functions; the arguments of these functions are the solutions of the usual paraxial equations, neglecting thermal effects. Our predictions agree with those of Cutler and Hines for the case they considered.

An approximation is given for the non-linear effects of the electromagnetic fields; these non-linear effects may be due to aberrations in the system, or to changes in the space-charge fields due to the linear effects of thermal velocities. As an example, the problem of a beam with a finite transverse temperature in a field-free drift space is considered. Analytic expressions are given for the current density in such a beam.

1. Introduction

The estimation of the effects of thermal velocities in particle beams has been important since the early work on the cathode ray tubes. In the application of ion beams, where acceleration-deceleration systems are being considered, thermal effects become even more important, and cannot be considered independently of space-charge effects. Pierce [1] has given an excellent review of the work up to 1954. However, his treatment is very general, and does not give results or methods of approach for most of the problems encountered in practice. Cutler and Hines [2] first gave a systematic discussion of thermal velocities in electrostatic round beams with a constant current density at the cathode. Their method can be extended to a wide range of problems, namely to those problems where the field normal to the trajectory of the beam varies linearly with distance from the beam centre. All beams for which the paraxial assumptions are valid [3,4,5,6] fall into this category. Thermal effects change the current distribution in the beam, and this affects the space-charge forces. Danielson et al. [7] extended and formalised Cutler and Hines' treatment, and allowed for the change in space-charge forces by considering their effect on a particular "typical" particle.

In this paper the treatments of Cutler and Hines and Danielson et al. are extended in several directions. First, we relate their method to the paraxial theories developed in Refs 3-6. This method is valid even with transverse magnetic fields, but the effect of the spread in longitudinal velocities is neglected. Both axially symmetric and two dimensional problems are treated. Expressions are derived for the current density and total current as a function of the transverse position in the beam. These expressions are compared with those of Cutler and Hines [2], and are found to be identical for constant cathode current density. In sheet beams with deflection focusing, or in crossed-field guns, the current density varies linearly at the cathode. For this reason the case of sheet beams with a linear variation of cathode current density is treated in detail. Analytic expressions and curves are presented.

A higher order theory is given which gives a correction for the effects of non-paraxial fields, which result from two sources. First, aberrations in the applied fields will produce non-paraxial forces. Secondly, finite temperature effects will, even to the paraxial approximation, change the current distribution; this change in current distribution will lead to a non-linear component of the space-charge field. The theory gives an approximation to the actual current density distribution. This is different to the approach of Danielson et al. [7], who considered a specific "average" particle, and assumed that the corrections to this particle were a good approximation to the behaviour of the beam as a whole. Usually our higher order theory can be only applied numerically. However one particular analytic example is given to illustrate the method. This example is the computation of the current distribution in a two dimensional beam in a field free region, with initially a constant current density and a Maxwellian distribution of transverse velocities. The method may of course be applied to more complex paraxial conditions and non-linear applied fields. Throughout this paper the MKS system of units is used. Our results are valid throughout for relativistic beams.

II. Paraxial Theory

1. Introduction and Notation

In this section we will develop a theory for the spatial distribution of current density in axially symmetric and two dimensional beams. The following assumptions are made:

i) The current density at the cathode is uniform in one direction in two dimensional beams, and is axially symmetric in axially symmetric beams.

ii) The current density at the cathode has a Maxwellian distribution of velocity, and any direction of emission is equally possible.

iii) The effects of longitudinal velocity spread may be neglected.

iv) All electromagnetic transverse forces are linearly proportional to distance from some central trajectory C_o.

Assumption (iv) is almost, but not quite, the assumption which leads to the paraxial theory. In the paraxial theory assumption (iv) is made <u>in the beam.</u> In the absence of thermal effects this covers all paraxial particles. With finite transverse temperature, some particles will stray far from the central trajectory of the paraxial theory. We will assume that assumption (iv) holds for <u>all</u> particles. Later, in the treatment of higher order effects, the corrections which may be made for the error in assumption (iv) are considered.

The coordinate system is illustrated in Fig. 1 for round beams, and Fig. 2 for sheet beams. The theory could equally well be applied to hollow beams with curvilirear central trajectories C_0, but these will not be considered in this section. In the round beam case the central trajectory is the z - axis, and r, θ are the usual cylindrical polar coordinates. All physical parameters of the flow are assumed independent of θ. The cathode is then assumed to be z = constant. The theory would hold equally well in spherical coordinates (r, θ, ϕ). In this case if the substitutions

$$z \to r, \quad r \to r\theta, \quad \theta \to \phi$$

are made, the results would apply to spherical cathodes. In the sheet beam case of Fig. 2, the beam is infinite in the y -direction. The central trajectory of the system is C_0 and the cathode is the plane z=constant. This coordinate system is the same as those of Refs 3,4. In the rest of this section we will discuss the two dimensional case, but the results hold for the round beam by replacing x by r.

From assumption (iv) it is seen that the electromagnetic force has the form $xh(z)$. The relation between $h(z)$ and the variation of fields on the actual trajectory C_0 has been discussed in Refs. 3,4,5,6 and does not concern us here. In this case the relativistic Lorentz Force Law can be written in the form

$$dp_x/dt = e\,h(z)\,x, \qquad dx/dt = p_x/m\,, \tag{1}$$

where

$$\beta = (dz/dt)/c = v_z/c\,, \qquad \gamma = 1/(1-\beta^2)^{1/2}\,, \quad m = m_0\gamma. \tag{2}$$

In Eq. (1), p_x is the momentum in the x-direction, m,e, the mass and charge of the particle; in Eq. (2), v_z is the axial velocity, c the velocity of light, and m_0 the rest mass of the particle. Defining new parameters

$$s = p_x/(m_0 c) , \quad g(z) = (\gamma^2 - 1)^{-1/2} , \quad f(z) = e\,h(z)/(m_0 c^2 \beta) . \tag{3}$$

Eq. (1) becomes

$$dx/dz = g(z)\,s , \quad ds/dz = f(z)\,x . \tag{4}$$

It is to be noted that in Eq. (3) g,f are stated to be functions only of z. This follows from assumptions (iii) and (iv), so that it is assumed that all particles at the same z have the same axial velocity. It is easily verified that if s_1, x_1 and s_2, x_2 are two independent solutions of Eq. (4), then the cross-products

$$s_1\,dx_2/dz - s_2\,dx_1/dz = x_1\,ds_2/dz - x_2\,ds_1/dz = 0. \tag{5}$$

so that

$$s_1 x_2 - x_1 s_2 = \text{constant} . \tag{6}$$

We may therefore define, in the same way as Sturrock[3] the two solutions

$$x = a_{11}(z) , \quad s = a_{12}(z) \quad \text{and} \quad x = a_{21}(z) , \quad s = a_{22}(z), \tag{7}$$

where the pairs of x, s have the values, at z = 0,

$$a_{11} = 1 = a_{22} , \quad a_{12} = 0 = a_{21} . \tag{8}$$

Any solution of Eq. (3) can then be written in the form

$$\begin{pmatrix} x \\ s \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} X \\ S \end{pmatrix} , \tag{9}$$

where X, S are the values of (x,s) at $z = 0$, and, from Eqs (6)-(8), the determinant of a_{ij} is unity.

We will not concern ourselves with detailed expressions for a_{ij}. They result from the normal solution of the paraxial equation with the boundary conditions of Eq. (8). The initial plane $z = 0$, may be taken as the cathode, or any other plane where the initial x, s are known or assumed. The conclusions which will be drawn are independent of the actual expressions for the a_{ij}.

2. Boundary Conditions at the Initial Plane

It is experimentally observed that from an emitting cathode or plasma sheath, particles are emitted with equal probability in all directions according to a Maxwellian distribution of velocities. This experimental fact, which was assumption (iii), leads to the conclusion that the transverse velocity distribution is also Maxwellian. In particular in the two dimensional beams of Fig.2, the probability of having an initial transverse momentum in the x-direction between p_x and $p_x + dp_x$, is $P(p_x)\, dx$ where

$$P(p_x) = (2\pi m_0 k T)^{-1/2} \exp -\left[p_x^2/(2 m_0 k T)\right] , \qquad (10)$$

where k is Boltzman's constant, and T is the temperature in ^oK of the source. If we define the parameter λ by the relation

$$\lambda = \left[m_0 c^2/(2 k T)\right]^{1/2} , \qquad (11)$$

and evaluate the probability of having transverse coordinate s, as defined by Eq. (2), between s and $s+ds$ this probability is $P(s)ds$ where

$$P(s) = (\lambda/\sqrt{\pi}) \exp(-\lambda^2 s^2) \quad . \tag{12}$$

In an axially symmetric system, Eq. (12) holds identically with $P(s_x)\, P(s_y)\, ds_x\, ds_y$, replacing $P(s)ds$.

If the current density at the cathode varied linearly with x, then the current between x and $x + dx$ is $i_c(x/a)dx$ where

$$i_c(x/a) = [I_c/(2a)]\,[1 + q\,(x/a)] \quad , \tag{13}$$

the boundaries of the two dimensional beam are $\pm a$, I_c is the total current per unit length in the beam, and q is the factor of proportionality. Equation (12) is often satisfied in practice, and is always satisfied in paraxial beams from a planar cathode. If $j(x,s)\,dx\,ds$ is the current density in x,s phase space, then from Eqs (12) and (13) the value of $j(x,s)$ at the cathode is $j_c(x,s)$, where

$$\left.\begin{aligned} j_c(x,s) &= [I_c/(2a)]\,[1 + q(x/a)]\,(\lambda/\sqrt{\pi}) \exp(-\lambda^2 s^2) \;, & -a \leqslant x \leqslant a \\ &= 0 & \text{otherwise} \end{aligned}\right\} . \tag{14}$$

At the cathode of axially symmetric beams with current density variation $i_c(r/a)$, the current density at the point (x,y,s_x,s_y) in four-dimensional phase space is given by

$$\left.\begin{aligned} j_c(x,y,s_x,s_y) &= i_c(r/a)(\lambda^2/\pi) \exp[-\lambda^2(s_x{}^2 + s_y{}^2)] \;, & 0 \leqslant r \leqslant a \\ &= 0 & \text{otherwise} \end{aligned}\right\} , \tag{15}$$

where now s_x, s_y are the x and y components of s, a is the radius of the beam, and r,s are related to x, y and s_x,s_y by the formulae

$$r^2 = x^2 + y^2 \quad , \qquad s^2 = s_x^{\,2} + s_y^{\,2} \quad . \tag{16}$$

We will usually consider the case of constant current density, where i_0 is constant, but will derive expressions for the general case.

3. The Current Distribution in Round Solid Beams

We are now in a position to find the current density in (x,s) or (r,s) space - and therefore in real space - in two dimensional and axially symmetric beams. Let us first consider round beams.

In Cartesian coordinates with axially symmetric fields, it may be verified that the x,y coordinates obey identical equations, namely Eq. (1), so that their transformation laws are given by Eq. (9). Hence the point (x,y,s_x,s_y) comes from (X,Y,S_X,S_Y) if, from Eq. (9),

$$a_{11}X + a_{12}S_x = x \quad , \qquad a_{11}Y + a_{12}S_y = y \quad . \tag{17}$$

The line elements ds_x, ds_y, for constant x,y are related to dS_x dS_y by the relations

$$ds_x = a_{21}\,dX + a_{22}\,dS_x = dS_x/a_{11} \quad , \qquad ds_y = dS_y/a_{11} \quad , \tag{18}$$

so that the current density $i(x,y,z)$ at (x,y) is given by

$$i(x,y,z) = \int_{s_x}\int_{s_y} j(x,y,s_x,s_y)\, ds_x\, ds_y$$

$$= \int_{S_x}\int_{S_y} j_o(X,Y,S_x,S_y)\, dS_x\, dS_y$$

$$= \frac{1}{a_{11}{}^2}\int_{S_x}\int_{S_y} \left[j_o\ (x-a_{12}S_x)/a_{11},\ (y-a_{12}S_y)/a_{11},\ S_x,\ S_y\right] dS_x\, dS_y \ . \quad (19)$$

Clearly $i(x,y,z)$ depends on z through the a_{ij}. To obtain the current density in an axially symmetric beam, we may transform into axially symmetric coordinates (r,θ) in real space. With the transformation

$$\left.\begin{array}{ll} x = r\cos\theta\ , & y = r\sin\theta\ , \\ a_{12}S_x = r\cos\theta - \zeta\cos\phi\ , & a_{12}S_y = r\sin\theta - \zeta\sin\phi \end{array}\right\}, \quad (20)$$

the differential line elements become

$$dx\ dy\ dS_x\ dS_y = (r\zeta/a_{12}{}^2)\ d\theta\ d\phi\ dr\ d\zeta \ . \quad (21)$$

Substituting Eqs.(15), (20) and (21) into Eq. (19), and integrating over θ to find the total current $i_1(r,z)dr$ between r and $r+dr$, we find that

$$i_1(r,z) = \int_\theta r\ i(x,y,z)\, d\theta$$

$$= \int_\theta d\theta \int_\phi d\phi \int_\zeta \left[r\zeta/(a^2a_{11}{}^2)\right] j_c\left[(\zeta/a_{11})\cos\theta, (\zeta/a_{11})\sin\theta, S_x, S_y\right] d\zeta$$

$$= \frac{(\lambda/a_{12})^2}{\pi\, a_{11}{}^2}\int_0^{2\pi} d\theta \int_0^{2\pi} d\phi \int_0^{a_{11}a} r\zeta\ i_o\left[\frac{\zeta}{a_{11}a}\right] \exp-\left[\frac{r^2+\zeta^2-2r\zeta\cos(\theta-\phi)}{(a_{12}/\lambda)^2}\ d\zeta\right]. \quad (22)$$

Defining now the convenient parameters

$$\chi = a\, a_{11}\lambda/a_{12}\ ,\quad \psi = r\lambda/a_{12}\ ,\quad \xi = \lambda\zeta/a_{12}\ , \tag{23}$$

and using the relation [2]

$$\int_0^{2\pi} \exp(a \cos u)\, du \;=\; 2\pi\, I_0(a)\ , \tag{24}$$

where $I_0(a)$ is the modified Bessel function of zero order, Eq. (22) becomes

$$i_1(r,z) = (4\pi/\chi^2)\int_0^{\chi} i_0(\xi/\chi)\, \xi \exp-(\xi^2 + \psi^2)\, I_0(2\xi\psi)\, d\xi\ , \tag{25}$$

where i_0 was defined in Eq. (15).

If i_0 is constant, the i_1 of Eq. (25) is the same as that obtained by Cutler and Hines [2] in Eq. (31) of their paper. To compare their results with ours, one must use the substitution

$$\sigma = a_{12}/(\sqrt{2}\lambda)\ ,\quad r_e = a_{11}r\ ,\ \chi = r_e/(\sqrt{2}\sigma)\ ,\quad \psi = r/(\sqrt{2}\sigma)\ . \tag{26}$$

For the case i_0 constant, curves are given in Ref. 2. We see that the effect of the current density variation at the cathode puts in the extra term $i_0(\xi\ /\chi)$ in the integral. The $i_1(r,z)$ from Eq. (25) have been evaluated numerically in Refs 2 and 7, for constant i_0. We present curves, however, for the constant current density case in a slightly different way from the curves of Ref. 2. In Fig. 3, the quantity $i(r,z)/i_0$,

where $i_0(r,z)$ is the value of i in the beam in the absence of thermal effects, is plotted versus $\psi\,/\,\chi$, which is $(r/\ a\, a_{11})$. For highly convergent beams, with χ less than unity, these curves are difficult to use. Hence in Fig. 4 we plot $i(r,z)/(\chi^2 i_0)$ versus ψ. For hollow beam applications, the integral is taken from χ_1 to χ_2 instead of from 0 to χ where χ_1 and χ_2 are expressions analogous to χ with the inner and outer cathode radii replacing a.

Finally it is important in part III to have expressions for the total current $I(r,z)$ inside a radius r. Using the definitions of Eq. (25) we see that

$$\begin{aligned} I(r,z) &= \int_0^r i_1(r,z)\, dr \\ &= (4\pi a^2/\chi^2) \int_0^{\psi} \psi e^{-\psi^2} d\psi \int_0^{\chi} i_c(\xi/\chi)\, \xi e^{-\xi^2} I_0(2\xi\psi)\, d\xi \, . \end{aligned} \tag{27}$$

Equation (27) has been integrated for the constant current density case in Ref. 2.

We will not evaluate any particular electromagnetic structure as an example of this theory. Suffice to say that it is valid, to the linear approximation, for any beam or gun for which the paraxial equation is valid. For the higher-order corrections, it is necessary to use the methods of part III. The theory of this section can easily be applied to hollow beams with a linear variation of current density at the cathode. In this case the same parameters can be used, but the limits of integration, and the functions $i(r,z)$ will be slightly changed. It is still possible however, to express these functions analytically in terms of the error functions.

4. The Current Density in Two Dimensional Beams

The methods of the last section can be applied equally well to the evaluation of the current density and total current in strip beams, infinite in one direction. Assuming a linear variation of current density according to Eq. (13) at the initial plane, which will usually be taken as the cathode, the current density in phase space at this plane is given by Eq. (14). Assuming a current density $i_c(x/a)$ at $z = 0$, we may make a computation similar to that of Eqs. (19)-(26), with x replacing r. The current density $i(x,z)$ at x may be derived by similar transformations, namely

$$\begin{aligned} i(x,z) &= \int_{-\infty}^{\infty} j(x,s)\, ds = \int_{-\infty}^{\infty} j_c(X,S)\, dS/a_{11} \\ &= \lambda/(\sqrt{\pi}\, a_{11}) \int_{-\infty}^{\infty} i_c\left[(x-a_{12}S)/(a_{11}a)\right] \exp-(\lambda S)^2\, dS \\ &= \left[1/(\sqrt{\pi}\, a_{11})\right] \int_{-\chi}^{\chi} i_c(\xi/\chi) \quad \exp-(\xi-\psi)^2 \quad d\xi \quad , \end{aligned} \tag{28}$$

where now a is the half-width of the beam at the cathode, and ψ, χ are given by Eq. (23) with x replacing r. If we replaced the exponential term by the Dirac Delta Function, Eq. (28) would give the current density to be expected in the absence of thermal effects. For the particular case of a linear variation of current density at the cathode, $i_c(x/a)$ takes the form of Eq. (13), and the current density is given by the expression

$$i(x,z) = I/(2\sqrt{\pi}\, a_{11}a) \int_{-\chi}^{\chi} (1 + q\xi/\chi)\, \exp-(\psi-\xi)^2\, d\xi$$

$$= \frac{I_0}{2\, a_{11}a}\left\{\left[1 + \frac{q\psi}{\chi}\right]\left[\frac{1}{2}\, E(\psi+\chi) - \frac{1}{2}\, E(\psi-\chi)\right] + \frac{q/\chi}{2\sqrt{\pi}}\left[e^{-(\psi+\chi)^2} - e^{-(\psi-\chi)^2}\right]\right\}, \quad (29)$$

where $E(x)$ is the usual error function $(2/\sqrt{\pi}) \int_0^x \exp(-x^2)dx$. In the low temperature limit, $|\psi|, |\chi| \to \infty$, the exponential terms tend to zero, while the bracket with the error functions gives unity inside the original beam, where $|\psi| \leq \chi$ and zero outside it. This is the current density in the absence of thermal effects. The expression for $i(x,z)$ takes a particularly simple form for constant current density, where $q = 0$.

Curves are given in Figs 5 and 6 for $i(x,z)/\chi$. For highly convergent beams, i.e. for small χ less than unity, the curves of Fig. 6 are more useful. The different curves refer to values $q = 0$, $q = 1$ of the parameter q, which determines the initial current density variation at the cathode. Since $i(x,z)$ is linear in q, the values of $i(x,z)$ for different q may be found by linear interpolation between the values on the $q = 0$ curve, and those on the $q = 1$ curve.

A quantity which is of value later in part III, is the total current $I(x,z)$, between the nominal centre of the beam and the plane x, at the axial position z. This current is given by the relations

$$I(x,z) = \int_0^x i(x,z)\,dx = a\,a_{11}\int_0^{\psi} i(x,z)\,d\psi/\chi$$

$$= \left[I_c/(2\chi)\right]\left[\tfrac{1}{2}(1-q)\,F_2(\psi+\chi) - \tfrac{1}{2}(1+q)\,F_2(\psi-\chi) + q\,F_2(\chi) + \right.$$
$$\left. + \{q/(2\chi)\}\{F_1(\psi+\chi) - F_1(\psi-\chi) - 2\,F_1(\chi)\}\right] \qquad (30)$$

where F_1 and F_2 are given by the relations

$$\left.\begin{aligned} F_1(x) &= \tfrac{1}{2}(x^2 + \tfrac{1}{2})\,E(x) + \left[x/(2\sqrt{\pi})\right]\exp(-x^2) \\ F_2(x) &= x\,E(x) + (1/\sqrt{\pi})\,\exp(-x^2) \end{aligned}\right\} . \qquad (31)$$

In Fig. 7, curves are given for $I(x,z)/I_c$ as a function of ψ for different values of the parameters χ, q. The function $I(x,z)/I_c$, is the percentage of the total current contained between x = 0 and x = x at the plane z.

An illustration of the use of this theory is given at the end of part III.

III. HIGHER ORDER CORRECTIONS

1. Introduction

In the last part of the paper, we discussed the effects of linear applied fields; i.e. paraxial fields, on particles which originally had a Gaussian distribution in phase space. The effects discussed in Part II will be called linear effects. In this part will be discussed the perturbation introduced by slightly non- linear fields. These non- linear fields may come from two factors. First, aberrations may cause the applied fields to become non- linear. Secondly the linear thermal effects will redistribute the space- charge, causing non- linear space - charge fields. In Section 2 we develop the general non- linear theory of the motion of particles in phase space. In this theory, the linear behaviour is assumed known. In Section 3 the theory is applied to round and strip beams to evaluate the space - charge distribution to the non- linear fields. Finally we consider in Section 4 a particular example, the spreading of a sheet beam in the absence of applied fields, but allowing for non- linear space charge effects due to an initially Maxwellian distribution of transverse velocities.

2. The Formalism with Slightly Non-linear Fields.

In part II the effects of linear fields were discussed. In this section we discuss the perturbations which arise in slightly non- linear fields. Let it be assumed that there is a non- linear transverse field $\mathcal{E}(x,z)$ in the x-direction in addition to the linear $h(z)$. In the axially symmetric case the extra field would be $\mathcal{E}(r,z)$ in the r-direction . In the x,s coordinate system of Eqs (3) and (4), the equations of motion become, for a two dimensional beam

$$dx/dz = g(z)\, s \;, \qquad ds/dz = f(z)\, x + F(x,z), \tag{32}$$

where f,g,s were defined in Eq.(3) and F is related to $\mathcal{E}$ by the expression

$$F(x,z) = e\mathcal{E}(x,z) \,/\, (m_0 \beta c^2). \tag{33}$$

If we write

$$x = x_0 + \xi \;, \qquad s = s_0 + \zeta \;, \tag{34}$$

where x_0, s_0 are the solution of Eq.(32) with F=0, then substitution into Eq.(32) yields the relation

$$d\xi/dz = g(z)\,\zeta \;, \qquad d\zeta/dz = f(z)\,\xi + F(x_0 + \xi, z). \tag{35}$$

If the perturbation is small, then to first order in ξ we may write

$$F(x_0+\xi, z) \simeq F(x_0,z) + \xi\, F_x(x_0,z) \;, \tag{36}$$

where $F_x(x_0,z)$ is the derivative of $F(x_0,z)$ with respect to x. By succesive iteration, we find that to second order the solution of Eq.(35) is

$$\left.\begin{aligned} \xi &= \xi_0 + \int_0^z \zeta_0\, g(u)\, du + \int_0^z g(u)\, du \int_0^u \left[f(v)\xi_0 + F(x_0,v)\right] dv \\ \zeta &= \zeta_0 + \int_0^z \left[\xi_0 f(u) + F(x_0,u)\right] du + \int_0^z f(u)\, du \int_0^u \zeta_0\, g(v)\, dv + \int_0^z \xi_0 F_x(x_0,u)\, du \end{aligned}\right\}. \tag{37}$$

In Eq.(37) ξ_0 and ζ_0 are the original value of ξ and ζ. If we assume that the initial position of a particle in phase space is given by the linear theory of part II, so that ξ_0, ζ_0 are zero, then Eq.(37) takes the simple form

$$\xi = \int_0^z g(u)\, du \int_0^u F(x_0,v)\, dv, \quad \zeta = \int_0^z F(x_0,u)\, du. \tag{38}$$

Equation (38) shows that while the linear theory transforms the region of the beam from Fig.8a to 8b, the non- linear theory to this approximation transforms the region into that of Fig.8c. How - ever, since the perturbation term is only dependent on the position of the particle in the absence of non- linear effects x_0 , the actual current density at the point corresponding to x_0 is unaltered. However, the point corresponding to x_0 is shifted by the amount ξ of Eq. (38). The total current between x and (x+dx) remains $\int i(x_0,z)\, dx_0$, while the element of length is $(dx_0 + d\xi)$. Therefore the current density $i(x,z)$ at the physical point x_0 is given by

$$\begin{aligned} i(x_0,z) &= i_0(x_0 - \xi, z)/(1 + d\xi/dx_0) \simeq i_0(x_0,z) - \xi\, di_0/dx_0 - i_0\, d\xi/dx_0 \\ &= i_0(x_0,z) - d[i_0(x_0,z)\xi]/dx_0 \;. \end{aligned} \tag{39}$$

where i_0 is the i of the linear theory. The formulae for $I(x,z)$ are unchanged, though it must be remembered that they refer to the current inside $(x_0 + \xi)$. The parameter ψ of Eq.(23) is given by

$$\psi = x_0 \lambda / a_{12} \;. \tag{40}$$

In axially symmetric systems all the results of this section are applicable with r, r_0 replacing x, x_0.
The spread of transverse momenta is not changed at x by the non-linear perturbation, however, the s of each particle is given the extra increment ζ of Eq.(38). Thus the non-linear term is equivalent to a long lens of strength $\left(\int_0^z F(x_0,u)\, du\right)$ at the point corresponding to x_0. We note that the method adopted here is quite different from that used by Danielson et al [7]. They considered the non-linear forces experienced by an average particle, and estimated that these forces were a good approximation to the forces experienced by the beam as a whole. We compute, to the second approximation in z , the forces experienced at each point in space. Clearly at no part of this section is it material whether the problem is two dimensional or axially symmetric. Such considerations only enter into the computation of F.

3. The Computation of the Space- Charge Forces.

In the previous section a theory was developed based on a non-linear forcing term $F(x,z)$. It was stated that this non- linear term was made of two terms; the first term is due to aberrations of the system, the second is due to space- charge. The first term can be computed by the methods developed by Sturrock [3]. This term will not be discussed in this paper, We will however give an estimate of the space- charge term F_s.

a) Round Beam. Let us assume an axially symmetric beam, uniform in the z-direction. It can be shown from Green's functions considerations, that the electrostatic radial field $\mathcal{E}_s$ due to a current distribution $i(r,z)$ can be given by the expression

$$\mathcal{E}_s(r,z) = \int_0^r i(r,z)/(2\pi r \varepsilon_0 v_z)\, dr \;, \tag{41}$$

where the axial velocity v_z is assumed constant across the beam, $i(r,z)$ is assumed a slowly varying function of z, and ε_0 is the dielectric constant of free space. In the relativistic case the self- magnetic field reduces the force due to the field by γ^2 so that the space-charge force $F_s(r,z)$ is given using Eqs (25) and (33) by the expression

$$F_s(r,z) = (\gamma^2-1)^{\frac{1}{2}}\Lambda\, I(r,z)/I_c \quad , \tag{42}$$

where Λ has the value

$$\Lambda = \lambda I_0(\gamma^2-1)^{-3/2}/(\pi a) \quad , \tag{43}$$

and I(r,z) given by Eq.(25). In Eq.(43) , λ is a constant which has the value in MKS units from Ref.9

$$\left.\begin{aligned} \lambda &= 3.82 \times 10^{-4} && \text{for electrons} \\ &= 2.08 \times 10^{-7} && \text{for protons} \end{aligned}\right\} \quad . \tag{44}$$

In the non- relativistic case, if the beam has voltage V, it is shown in Ref.9 that

$$\Lambda = [\lambda'/(\pi a)][I_0/V^{3/2}] \quad , \tag{45}$$

where

$$\left.\begin{aligned} \lambda' &= 4.78 \times 10^{4} && \text{for electrons} \\ &= 2.05 \times 10^{6} && \text{for protons} \end{aligned}\right\} \quad . \tag{46}$$

If the rate of change of axial velocity with distance is fast, as at a space- charge- limited cathode, then Eq.(41) is not strictly valid.

b) <u>Two- Dimensional Beam</u>. If the beam is two- dimensional, and it is assumed again that the axial velocity does not vary too rapidly in the z - direction, the expression analogous to Eq.(41) for the transverse electric field is

$$\mathcal{E}_s(x,z) = \left[\int_{-\infty}^{x} i(x,z)\,dx - \int_{x}^{\infty} i(x,z)\,dx\right]/(2\varepsilon_0 v_z). \tag{47}$$

Using the definitions of I(x,z) and F(x,z) of Eqs.(30) and (33), it can be shown that the space- charge force F_s (x,z) is given by [9]

$$\begin{aligned} F_s(x,z) &= [2I(x,z) - I(-\infty,z) - I(\infty,z)][e/(2m_0\beta c^2\varepsilon_0 v_z)] \\ &= (\Lambda/I_0)\sqrt{\gamma^2-1}\,[2I(x,z) - I(-\infty,z) - I(\infty,z)] \;, \end{aligned} \tag{48}$$

where Λ is given by the expression

$$\Lambda = \lambda I_c(\gamma^2-1)^{-3/2} \;, \tag{49}$$

and λ is given by Eq.(44), while I_c is the current per unit width of beam. In the non- relativistic case,

$$\Lambda = \lambda'(I_0/V^{3/2}), \tag{50}$$

where λ' is given by Eq.(46). The higher order correction due to space- charge may now be evaluated by the methods of section 2.

4. The Space- Charge Spreading of a Sheet Beam in a Drift Region

To illustrate the method, we will make a computation of the space- charge spreading of a sheet beam in a drift region.

This problem has been treated by many authors [2,7,9]. It will be assumed that the beam is in a region of zero applied field, and that originally the current density is uniform across the beam with a Maxwellian distribution of velocities. This means that the beam originally occupies the region of phase space shown in Fig.6 . Unlike previous authors, we will make a non- linear computation which allows for the change of space - charge forces due to the change in current distribution. The linear behaviour assumes no space- charge spreading, the non- linear term will include all the space- charge effects. It is possible to choose, for the linear model, the situation with constant space- charge. In order to solve the problem on a digital computer, this would be the best procedure. We intend to give an analytic solution; this is difficult to do if the a_{11}, a_{12} of the matrix of Eq.(9) are complicated.

Let us first consider the linear behaviour. The assumption is made that the beam originally has thickness 2a, temperature T, current I, and ratio of energy to rest energy γ . Since the equations of motion in the absence of any field are, from Eqs (3) and (4)

$$dx/dz = s/\sqrt{\gamma^2-1} \quad , \qquad ds/dz = 0, \tag{51}$$

the a_{11}, a_{12} of Eqs.(7) and (8) are given by

$$a_{11} = 1 \quad , \qquad a_{12} = z/\sqrt{\gamma^2-1} \quad . \tag{52}$$

Hence according to the linear theory, using Eq.(19) ψ, χ may be defined by the relations

$$\chi = a\alpha/z \quad , \quad \psi = x_0\,\alpha/z\,, \tag{53}$$

where x_0 is the position of the particle according to the linear theory and

$$\alpha = \lambda\sqrt{\gamma^2-1}. \tag{54}$$

Using Eqs (52) and (53) for a_{11} , χ, ψ the current density is obtained directly from Eq.(28). Since it is assumed that the initial current density is constant in real space, q=0 in Eq.(28). The expression for $i_0\,(x_0,z)$ then becomes

$$i_0\,(x_0,z) = \left[I_c/(2a)\right]\left[\tfrac{1}{2}E(\psi+\chi) - \tfrac{1}{2}E\,(\psi-\chi)\right] \quad . \tag{55}$$

Using the expression for $F_s\,(x_0,z)$ of Eq.(48) and for $I(x,z)$ of Eq.(30) with q=0, it is seen that $F_s\,(x_0,z)$ is given by the expression

$$F_s\,(x_0,z) = \Lambda\sqrt{\gamma^2-1}\;\left[F_2(\psi+\chi) - F_2(\psi-\chi)\right] \quad , \tag{56}$$

where ψ, χ are given by Eq.(53) and $F_2\,(x)$ from Eq.(31) by the expression

$$F_2(x) = x\,E(x) - e^{-x^2}/\sqrt{\pi}. \tag{57}$$

Using the F_s of Eq.(56), g (z) of Eq.(3) and ψ, χ of Eq.(53), we obtain the expression for ξ, from Eq.(38),

$$\xi = \Lambda \int_0^z du \int_0^u \left\{ F_2[\alpha(x_0+a)/v] - F_2[\alpha(x_0-a)/v] \right\} dv \quad . \tag{58}$$

The right hand side of Eq.(58) can be integrated analytically in terms of the error function E(x) and the exponential integral Ei(x) defined by

$$Ei(x) = \int_x^\infty e^{-x}/x \; dx \quad . \tag{59}$$

In terms of these functions it may be verified that Eq.(58) yields the relation

$$\xi = [\Lambda z^3/(2a\alpha)] \left\{ F_3[\alpha(x+a)/z] - F_3[\alpha(x-a)/z] \right\} , \tag{60}$$

where $F_3(x)$ has the form

$$F_3(x) = \tfrac{1}{2} x E(x) + 2x^3[\pm 1 - E(x)] + 3x^2/(2\sqrt{\pi})\, Ei(x^2) - 2x^2/\sqrt{\pi}\, e^{-x^2} , \tag{61}$$

and the positive sign holds for positive x, the other for negative. Clearly for general α, x, it would be necessary to evaluate Eq.(60) – and this does not seem worth the trouble. However, for large x, the asymptotic expression may be used

$$F_3(x) \simeq \pm x/2 + 1/(4\sqrt{\pi}\, x^2) \exp(-x^2) \quad , \tag{62}$$

the positive sign refering to positive x. Using this expansion, Eq.(60) becomes

$$\xi(x_0) = \frac{\Lambda z^2}{2a} \left\{ \frac{1}{2}\left[(x_0+a) - \frac{(x_0-a)^2}{|x_0-a|} \right] + \frac{(z/\alpha)^3}{4\sqrt{\pi}\, a} \left[\frac{e^{-[\alpha(x+a)/z]^2}}{(x+a)^2} - \frac{e^{-[\alpha(x-a)/z]^2}}{(x-a)^2} \right] \right\} , \tag{63}$$

although this approximation is not good enough near the edge of the beam if $\alpha(x_0 \pm a) \lessapprox z$. The first term gives the usual linear behaviour in the absence of thermal effects, since the bracket gives x_0 for $x_0 \leqslant a$ and a for $x_0 \geqslant a$, while the second term gives the non- linear term due to the thermal velocities changing the space- charge distribution. The current density $i(x_0, z)$ may be found from Eq.(39) namely

$$i(x_0, z) = i_0(x_0, z) - d(\xi\, i_0)/dx_0 \quad . \tag{64}$$

In Eq.(64), i_0, ξ are given by Eqs.(55) and (60). This problem is of insufficient practical interest to justify calculating i exactly from Eq.(64). However, one may use the asymptotic expansions for ξ of Eq.(63) for large $\alpha(x_0 \pm a)/z$ and a similar one for $i(x_0, z)$ using the asymptotic expansion

$$E(x) \simeq |x|/x - [1 - 1/(2x^2)]\, e^{-x^2}/(\sqrt{\pi}\, x) \quad . \tag{65}$$

Using these expansions, the current density is given by

$$i(x_0,z) = i_0(x_0,z) - \left[I_c \Lambda z^2/(4a^2)\right] F_4(x_0,z) \quad , \tag{66}$$

where $F_4\ (x,z)$ is given by the expression

$$\left.\begin{aligned} F_4(x,z) &= 1 + \left[2\alpha x/(\sqrt{\pi} z)\right]\left\{ e^{-\left[\alpha(x-a)/z\right]^2} - e^{-\left[\alpha(x+a)/z\right]^2}\right\} \quad ,|x|<a \\ &= \left[2\alpha a/(\sqrt{\pi} z)\right]\left\{ e^{-\left[\alpha(x-a)/z\right]^2} - e^{-\left[\alpha(x+a)/z\right]^2}\right\} \quad ,|x|>a \end{aligned}\right\} . \tag{67}$$

The approximation of Eq.(67) is not good for x = a. The $i_0\ (x_0,z)$ in Eq.(65) has already been plotted in Fig.6, the other term is the non- linear perturbation. It seems of little practical purpose to evaluate the ξ and i of Eqs.(63) and (66). This example illustrates the method, which can be used for far more complex problems. As even this simple example shows, the analytic application of the method will usually be impossible.However the method can be applied numerically without difficulty . It is to be noted that unlike Danielsons inclusion of non- linear effects, this application is equally good both inside and outside the beam. Finally, more complicated electromagnetic forces, due to aberrations in the system, can be used for F .

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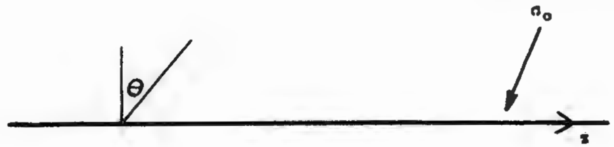
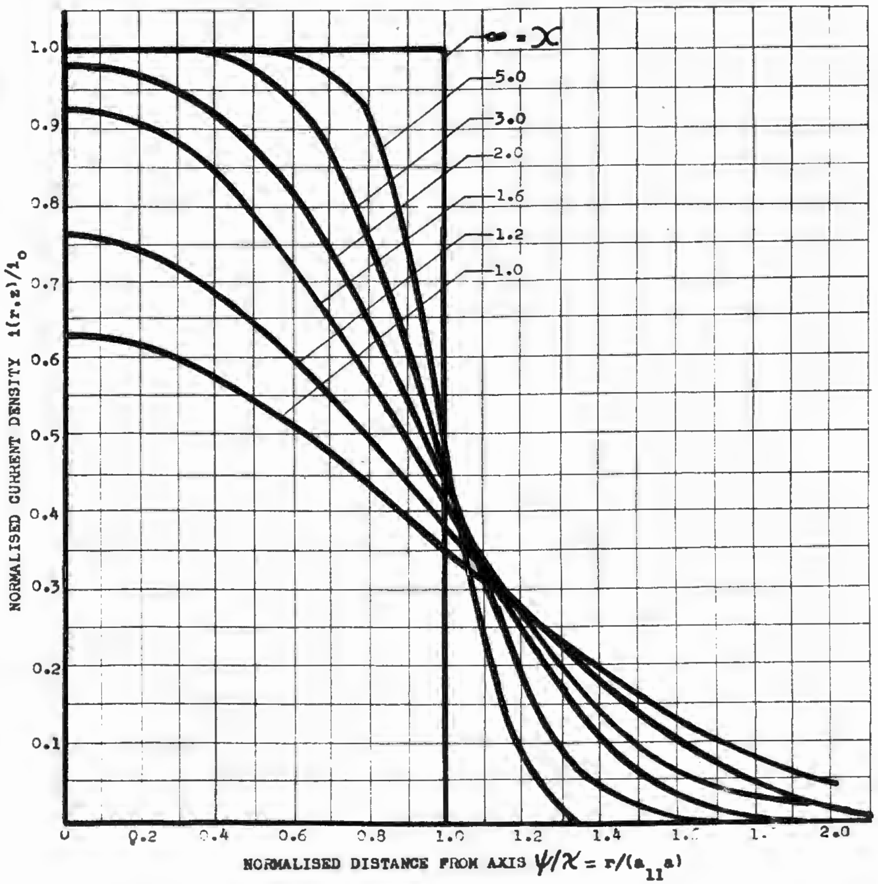


F i g. 1.

The coordinate system for round beams.

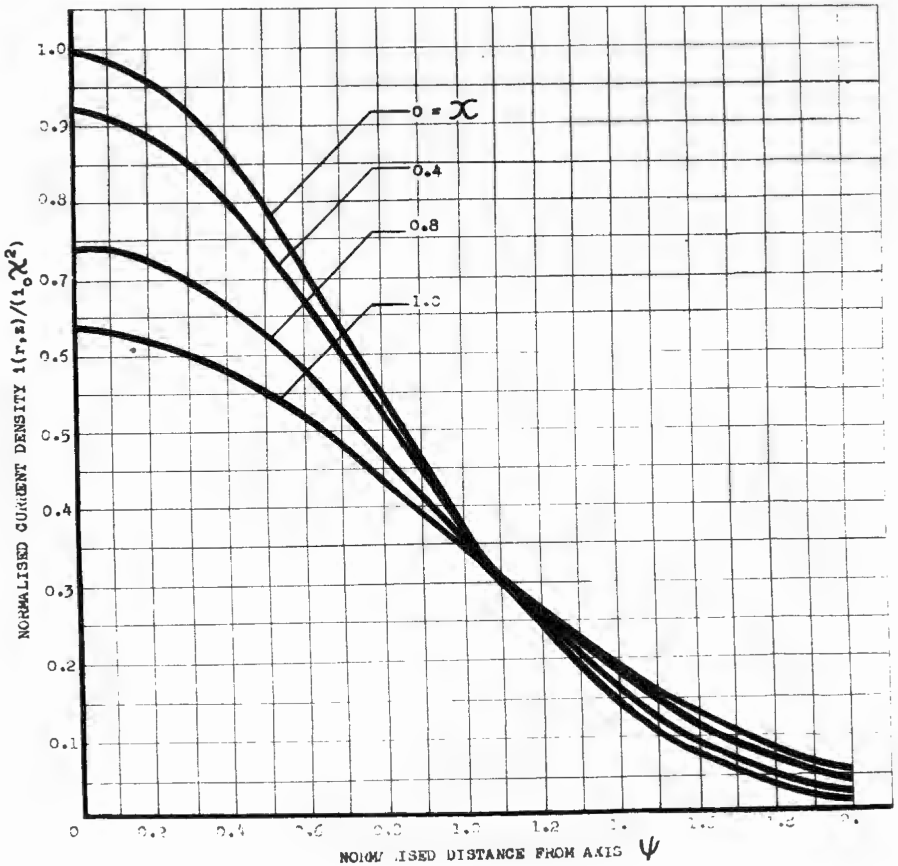
F i g. 2.

The coordinate system for curvilinear sheet beams.



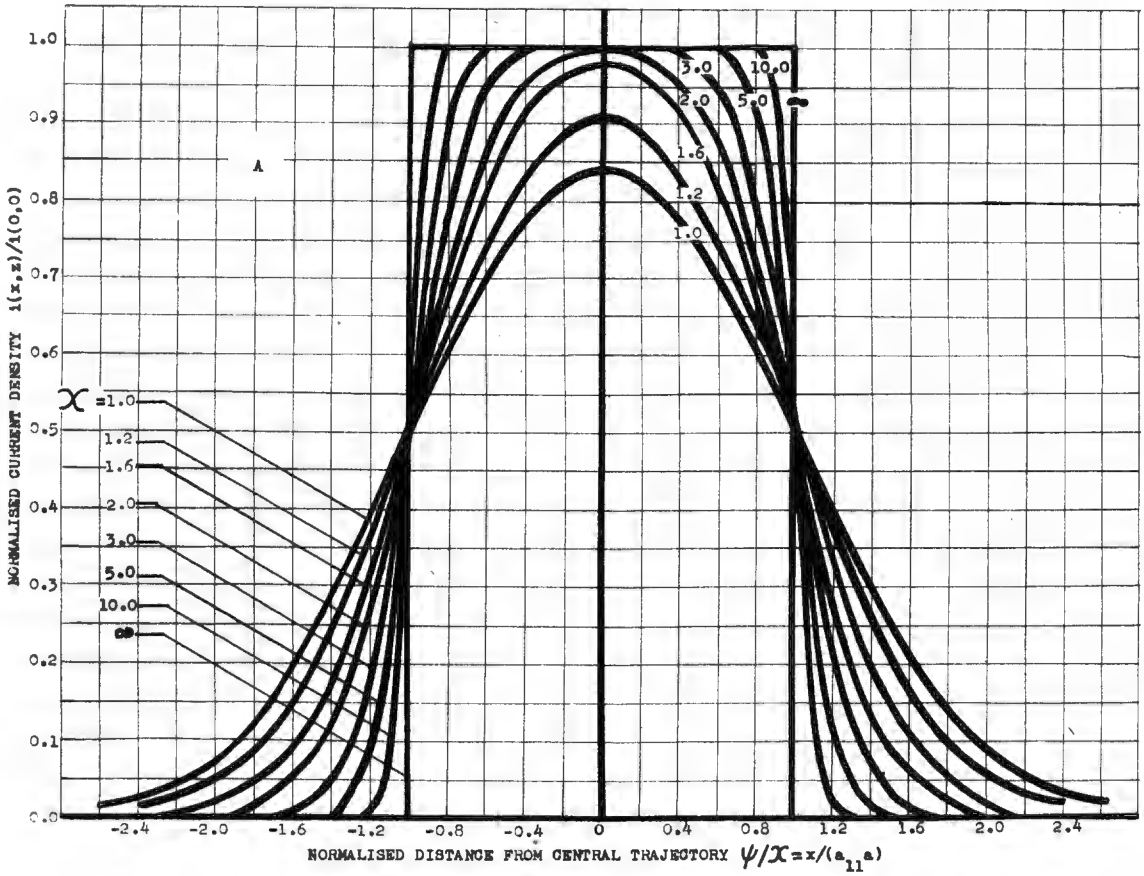
F i g. 3.

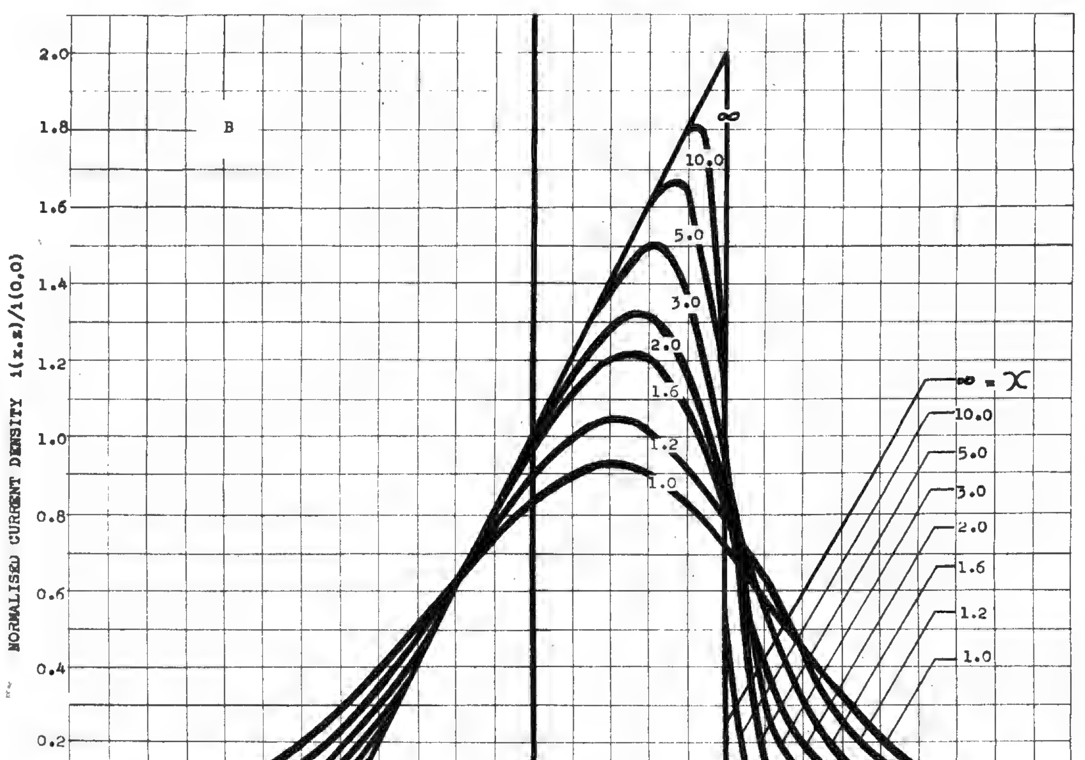
Curves showing the character of the current density variation in a round beam which has been dispersed by thermal velocities.



F i g. 4.

Curves showing the character of the current density variation in a round beam which has been dispersed by thermal velocities. These curves are in more suitable units than those of Fig. 3 for highly convergent beams.

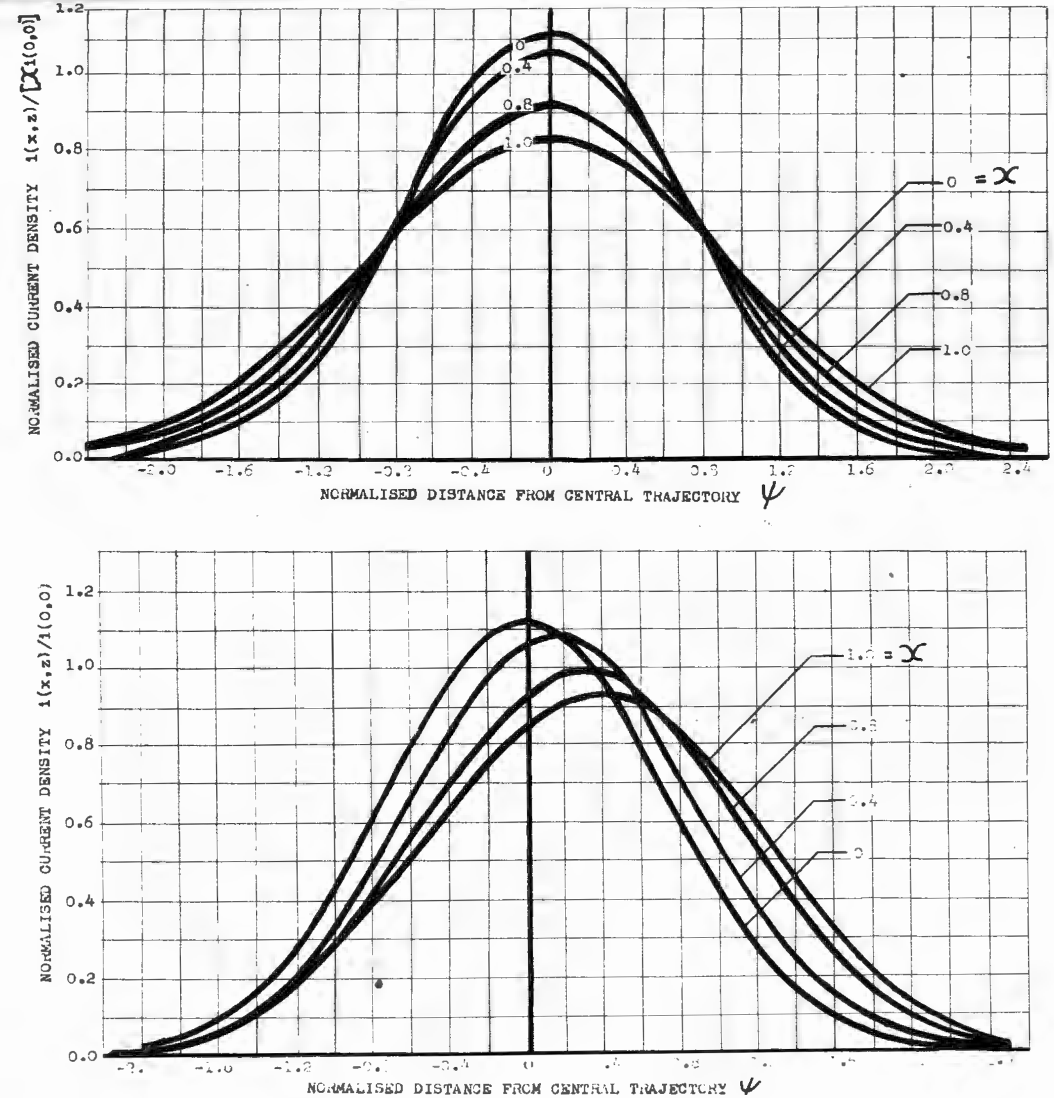




F i g. 5.

Curves showing the character of the current density variation in a sheet beam, with linear current density variation at the cathode, dispersed by thermal velocities. The ratio of maximum to minimum current density at the cathode is $(1+q)/(1-q)$.

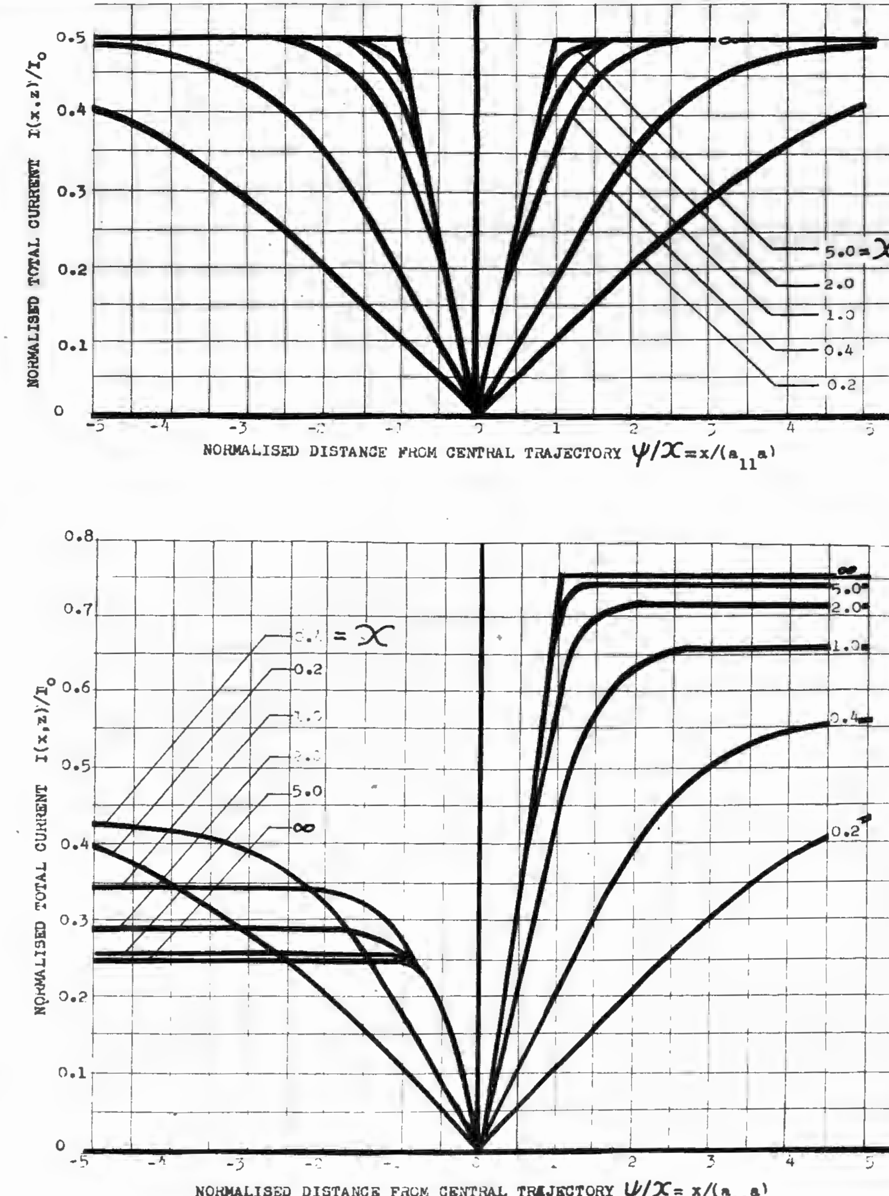
a) Constant current density, $q = 0$.
b) linear current density, $q = 1$.



F i g. 6.

Curves showing the character of the current density variation in a sheet beam dispersed by thermal velocities as in Fig.5. These curves are in more suitable units for highly convergent beams.

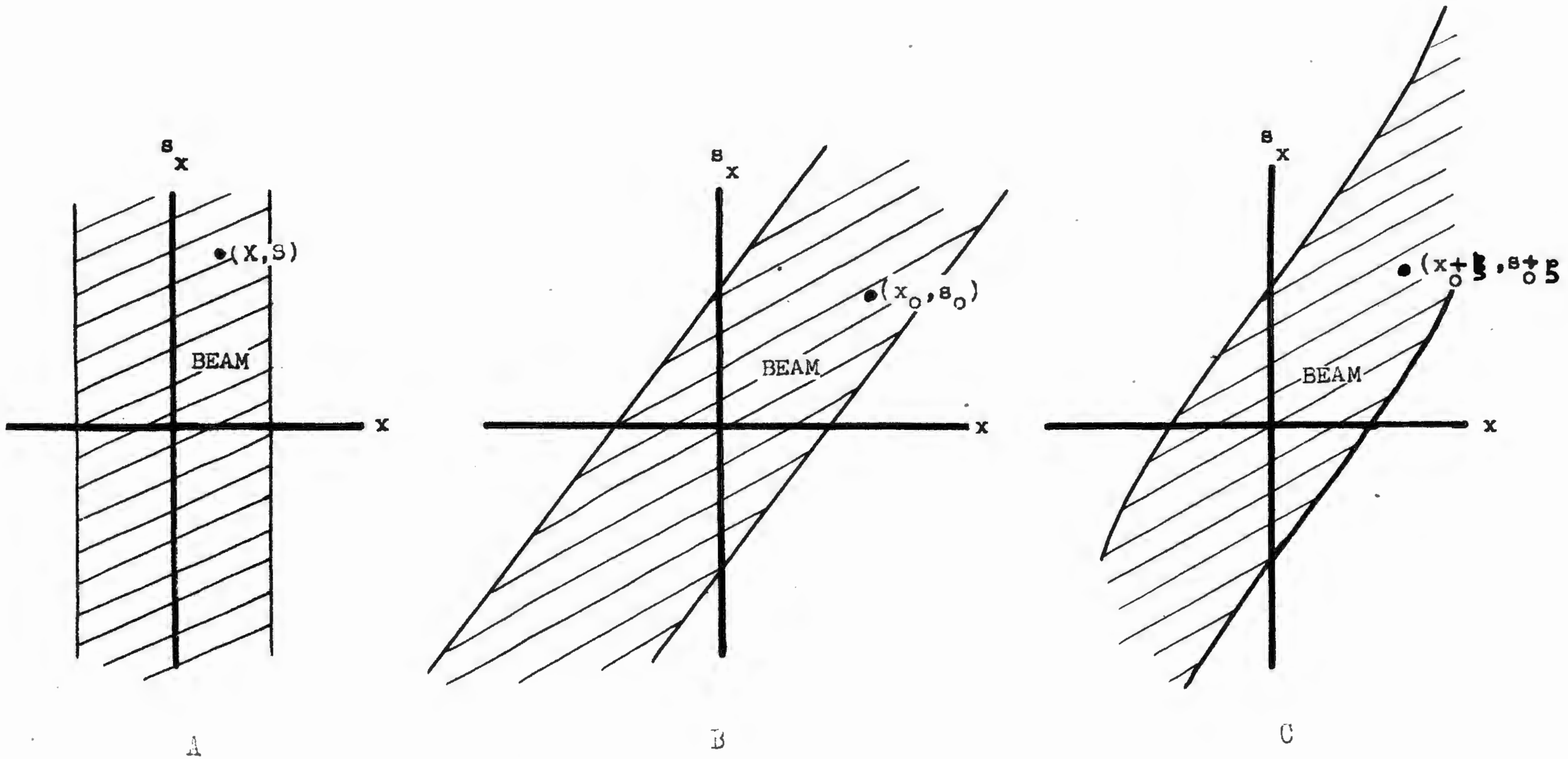
a) constant current density, q = 0.
b) linear current density, q = 1.



F i g. 7.

Curves showing the percent of the total beam current to be found between the planes x = 0 and x = x in a sheet beam dispersed by thermal velocities as in Fig. 5.

a) Constant current density, q = 0.
b) linear current density, q = 1.



F i g. 8:

The phase plots of points in a sheet beam in the (x,s) plane. a) The phase plots at the cathode, b) The phase plot at a different axial position, according to the linear theory.
c) The phase plot as in (b) according to the non-linear theory.