ОБЪЕДИНЕННЫЙ ИНСТИТУТ ЯДЕРНЫХ ИССЛЕДОВАНИЙ

Лаборатория теоретической физики

B.A. Arbusov, A.A. Logunov, A.N. Tavkhelidze, R.N. Faustov

E 1030

THE ASYMPTOTIC BEHAVIOUR OF THE SCATTERING AMPLITUDES AND THE RENORMALIZATION GROUP METHOD

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In recent years in connection with the Regge's papers[1], the problems of the asymptotic behaviour of the scattering amplitudes at high energies[2] have been widely discussed. If it is assumed that the asymptotic behaviour of the scattering amplitudes is dominated by the pole in the plane of the complex angular momentum with the largest real part, the amplitude at high energies s takes on the following form:

$$f(s,t) \approx \beta(t) s^{\alpha(t)} \tag{1}$$

where t is the momentum transfer, $\alpha(t)$ is the position of the pole of the t -channel partial wave amplitude in the ℓ- plane. Such an approach seems rather attractive and arouses enthusiasm of some physicists. However, the description of the scattering processes with the aid of the Regge poles has so far a phenomenological character, as long as it does not yet take into account the interaction dynamics.

Not long ago a number of papers appeared[3], in which attempts were made to obtain the asymptotic behaviour of type (1), starting from particular interaction models.

Note, that in some papers the problem of the asymptotic behaviour of different Green functions has been already investigated by summing the "principal" logarithmic diagrams[4,5]. However, this method took into account, as a matter of fact, only the two- particle states in the s -channel, what, of course, is not justified at high energies.

Here, to investigate the asymptotic behaviour of the scattering amplitude we resort to the renormalization group method[6]. It is essential in our approach that in order to find the asymptotic behaviour of the scattering amplitude at large s, we make use of the differential equations of the renormalization group over the variable t (the momentum transfer).

Such an approach is consistent with the concept of "strip" approximation[7]. The application of this method can be illustrated by two models of the quantum field theory.

Consider first the case of the scalar neutral mesons with the interaction $g\phi^3(x)$. We represent the meson-meson scattering amplitude as

$$T(t, s, m^2, g^2) = \frac{g^2}{m^2 - s} M(t, s, m^2, g^2) \tag{2}$$

Employing usual procedure[6], for M we get the following equation of the renormalization group

$$\frac{\partial}{\partial x} \ln M(x, y, \mu, g^2) = \frac{1}{x} \left[\frac{\partial}{\partial \xi} \ln M\left(\xi, \frac{y}{x}, \frac{\mu}{x}, g^2\psi(x,\mu,g^2)\right)\right]_{\xi = 1} \tag{3}$$

where $x = \frac{t}{\lambda}$, $y = \frac{s}{\lambda}$, $\mu = \frac{m^2}{\lambda}$, λ is the normalization momentum, $g^2\psi(x,\mu,g^2)$ is the invariant charge described by the corresponding differential equation[5]. In accord with the renormalization group method, as an initial approximation for the righthand side of (3), we make use of the following diagrams

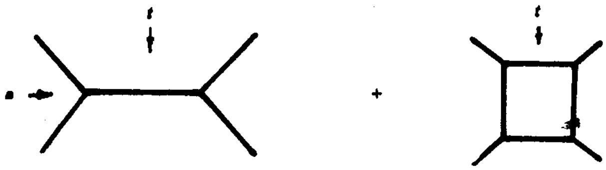


Fig. 1.

For large s and t , not very close to $4m^2$, this yields

$$M_0(x, y, \mu, g^2) = 1 + \frac{g^2}{8\pi^2 m^2} \Phi\left(\frac{x}{\mu}\right) \ln\left(-\frac{y}{\mu}\right) \tag{4}$$

where

$$\Phi(x) = \int_4^\infty \sqrt{\frac{x'-4}{x'}} \frac{dx'}{(x'-4)(x'-x)}$$

Substituting (3) into (4) and solving the differntial equation for large s, we have

$$T(t, s, m^2, g^2) = \frac{g^2}{m^2} \left(-\frac{s}{m^2}\right)^{a(t)} \tag{5}$$

where

$$a(t) = -1 + \frac{g^2}{8\pi^2 m^2} \int_{-\infty}^{t} \Phi'\left(\frac{t'}{m^2}\right) \psi\left(\frac{t'}{m^2}, 1, g^2\right) dt' \tag{6}$$

in the approximation when $\psi\left(\frac{t}{m^2}, 1, g^2\right) = 1$

$$a(t) = -1 + \frac{g^2}{8\pi^2} \int_{4m^2}^{\infty} \sqrt{\frac{t'-4m^2}{t'}} \frac{dt'}{(t'-4m^2)(t'-t)} \tag{7}$$

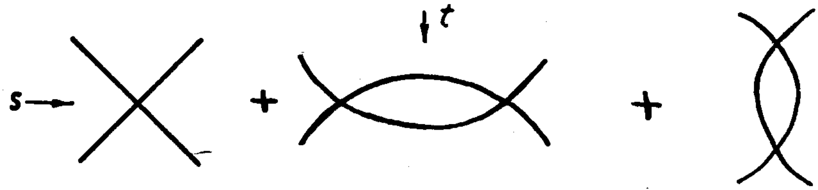
Let us now turn to the model of pseudo-scalar neutral particles with the interaction Lagrangian $h\phi^4(x)$.In this case the renormalization group equations for the scattering amplitudes are of the form[5]

$$\frac{\partial}{\partial x}\ln\Box(x,y,\mu,h)=\frac{1}{x}\left[\frac{\partial}{\partial\xi}\ln\Box\left(\xi,\frac{y}{x},\frac{\mu}{x},h\phi(x,\mu,h)\right)\right]_{\xi=1} \tag{8}$$

where the invariant charge $h\phi(x,\mu,h)$ satisfies the equation

$$\frac{\partial}{\partial x}h\phi(x,\mu,h)=\frac{h\phi(x,\mu,h)}{x}\left[\frac{\partial}{\partial\xi}\phi\left(\xi,\frac{\mu}{x},h\phi\right)\right]_{\xi=1} \tag{9}$$

Choosing, as an initial approximation, the expression obtained by calculating the diagrams drawn in Fig. 2.,



F i g. 2.

for large y, we find

$$\Box_0(x,y\mu,h)=1+hF(x,\mu)-\frac{h}{8\pi^2}\ln(-y) \tag{10}$$

where

$$F(x,\mu)=\frac{x-1}{8\pi^2\mu}\int_4^\infty\sqrt{\frac{x'-4}{x'}}\,\frac{dx'}{(x'-\frac{1}{\mu})(x'-\frac{x}{\mu})}$$

Substituting (10) into (8) and taking into account the terms of the first order in $h\ln(-y)$, we arrive at a solution

$$\frac{\Box(t,s,m^2,h)}{\Box(t_0,s,m^2,h)}=\gamma(t,t_0)\left(-\frac{s}{m^2}\right)^{\tilde{a}(t,t_0)} \tag{11}$$

where t_0 is an arbitrary constant

$$\gamma(t,t_0)=\exp\left[h\int_{t_0}^{t}dt'\,\phi(t',m^2,h)\,F'(t',m^2)\right]$$
$$\tilde{a}(t,t_0)=\frac{h^2}{8\pi^2}\int_{t_0}^{t}dt'\,\phi^2(t',m^2,h)\,F'(t',m^2) \tag{12}$$

To find the asymptotic behaviour in s of the scattering amplitude $\Box(t,s,m^2,h)$ at the finite momentum transfer, it is necessary to impose the boundary condition. As such, one can choose the well-known experimental fact that the cross sections at high energies are constant. This, in its turn, leads one to a conclusion, that the imaginary part of the scattering amplitude $\Box(0,s,m^2,h)$ at large s behaves like $C\cdot s$ (C is some well-known constant), while the real part grows more slowly, what can be easily seen if use is made of the dispersion relation for the forward scattering. Employing the boundary condition formulated above, we obtain

$$\Box(t, s, m^2, h) = iCm^2\gamma(t,0)\left(-\frac{s}{m^2}\right)^{a(t)} \qquad (13)$$

where

$$a(t) = 1 + \tilde{a}(t,0)$$

In the previous model we could also apply a similar boundary condition. However, it seemed more convenient there to impose the boundary condition from the requirement of the consistence with the perturbation theory.*

To investigate the analytical properties of the exponent $a(t)$, it is necessary to study the analytical structure of the invariant charge $h\phi(t, m^2, h)$ (or $g^2\psi(t, m^2, g^2)$ respectively). The invariant charge $h\phi(t, m^2, h)$ equals the value of the function

$$h\Box(t, s, p_1^2, p_2^2, p_3^2, p_4^2, m^2, h)\, d^2(t, m^2, h) = h\tilde{\Box}(t, m^2, h)\, d(t, m^2, h)$$
$$s + t + u = \sum_{i=1}^{4} p_i^2$$

at the symmetrical point

$$t = s = u\,, \quad p_1^2 = p_2^2 = p_3^2 = p_4^2 = p^2$$

where $d(t, m^2, h)$ is the meson Green function.

As far as the function $d(t, m^2, h)$ is analytical in the t -plane with the cut along the real axis from $4m^2$ up to ∞, for investigating the analytical properties of the invariant charge it is sufficient to study the analytical structure of the function $\tilde{\Box}(t, m^2, h)$.

Using the analytical properties of the n- th term in the perturbation theory [8], one can easily show that the function $\tilde{\Box}(t, m^2, h)$ is analytical in the complex t -plane with the cut along the positive real axis $t > 4m^2$.

Thus, even the elementary consideration shows that the function $a(t)$ (see formula (12)) is analytical in the complex t -plane with the cut along the positive real axis from $4m^2$ up to ∞ and $\frac{da}{dt} > 0$, $t < 4m^2$

The above investigation points out the sufficient effectiveness of the renormalization group method in the qualitative consideration of the asymptotic behaviour of the scattering amplitude at high energies and establishes interesting relationship between the invariant charge and the Regge exponent.

The results obtained show also that the class of the diagrams we have taken into account makes an essentially greater contribution to the asymptotic behaviour in s

* The account of the crossing-symmetry leads to the appearance of the factor $(1+e^{-i\pi a(t)})$ in formulae (5) and (13).

than those corresponding to the "principal" logarithmic terms in the notation adopted by the authors of [4].

The application of our method to the investigation of the asymptotic properties of the scattering amplitudes in electrodynamics and meson theory will be considered in a more detailed paper.

In conclusion we express our deep gratitude to N.N. Bogolubov and I.T. Todorov for interesting discussions and remarks.

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Note added in proof

The method developed can be applied for investigating the asymptotic properties of the amplitude involving fermions and bo - sons, e.g., for $\pi - N$ scattering where the amplitude may be represented as

$$T(s,t) = M_1(s,t) + \frac{\hat{q}}{s} g^2 M_2(s,t).$$

please, see the next page

The equation of the renormalization group can be written both for M_1 and M_2 :

$$\ell n \frac{M_1(x, y, \mu, g^2)}{M_1(x_0, y_0, \mu, g^2)} = \int_{x_0}^{x} \frac{dx'}{x'} [\frac{\partial}{\partial \xi} \ell n M_1 (\xi, \frac{y}{-x'}, \frac{\mu}{-x'}, g^2 \psi(-x', \mu, g^2))]_{\xi=-1} +$$

$$+ \int_{y_0}^{y} \frac{dy'}{y'} [\frac{\partial}{\partial \eta} \ell n M_1 (\frac{x_0}{y'}, \eta, \frac{\mu}{y'}, g^2 \psi (y', \mu, g^2))]_{\eta=1} .$$

Substituting the expression for the lowest order of the perturbation theory, we obtain the asymptotic behaviour of the Regge type for $\pi - N$ scattering /1/. Analogous results take place when treating the Compton-effect /2/. The application of this method for solving the problems of the potential scattering leads us to the asymptotic behaviour of the Regge type and allows to calculate $a(t)$ for a wide class of potentials (see /3/).

For the sake of simplicity we made use earlier of the equations of the renormalization group in order to investigate the asymptotic behaviour of the scattering amplitudes. However, it turns out possible to get further information if the renormalization group equations are written for the imaginary part of the amplitude. Employing then the dispersion relations, the real part of the scattering amplitude can be reconstructed. This is illustrated by the model with the interaction $g \phi^3(x)$ where the following expression is obtained for the imaginary part

$$Im\, T(s, t) = \frac{g^4}{8\pi} \Phi(t) (\frac{s}{m})^{a(t)} .$$

Hence, by using the dispersion relations for the asymptotic behaviour we get

$$T(s, t) = \frac{g^4}{8\pi} \Phi(t) (\frac{s}{m^2})^{a(t)} \frac{[1 + e^{-i\pi a(t)}]}{\sin \pi a(t)} .$$

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