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ON THE VARIATION OF BEAM BOUNDARIES IN LINEAR BEAM TRANSPORT ELEMENTS AND THE COMPUTATION OF SPACE-CHARGE EFFECTS

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Abstract.

First the properties of monochromatic relativistic beams in linear beam transport elements are considered, neglecting space-charge effects. Under the assumptions that the initial beam occupies an elliptical region in phase space, equations are set up for the variation of the parameters of the ellipse along the beam transport system. The equations are solved for uniform long lenses, point lenses, and drift spaces. It is shown how it is possible to obtain an ellipse of minimum area in phase-space, which completely circumscribes the region defined by two pairs of apertures. Formulae are developed to determine where such apertures should be placed, in order to define an ellipse in the optimum way.

Equations are set up for the perturbation of elliptic boundaries in phase-space by non-linear fields. Although Liouville's theorem must be satisfied, so that phase-space area is conserved, the region occupied by the beam may have an awkward shape which cannot be used in practice. Formulae are derived for estimating the rate of growth of useable phase-space area due to non-linear fields. These formulae are applied to the calculation of the perturbation of the boundaries in phase-space due to space-charge in sheet and axially-symmetric beams. The increase of effective area in phase space due to space-charge is also estimated. Finally, the results are applied to the estimation of the area in phase space which will be occupied by an axially symmetric beam coming out of a linear accelerator such as a Van de Graaff generator. The agreement with the experimental results is shown to be good.

Introduction

The properties of linear focusing elements have been studied by many authors. In most cases e.g. Sturrock (1955), a matrix formulation has been used, and individual particles are followed through the system. In beam transport systems, however, the single particle treatment may be unnecessarily cumbersome. In the matrix formulation, a 2 x 2 matrix is required which has unit determinant. It is usually only the boundary of the beam in transverse phase space which is required. This causes a further redundancy in the data used in the matrix formulation. For these reasons Hereward (1959) and Walsh (1958) have adopted different notations in which only the beam boundary is considered.

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In an important class of practical problems the boundary is an ellipse in phase space. Linear focusing elements have the property that they transform ellipses into other ellipses. Since areas are preserved in phase-space, the properties of these ellipses can be specified by two numbers. Hereward (1959) showed how these two numbers transform in point lenses and drift spaces. In part II of this paper we extend Hereward's treatment to derive equations for the variation of two parameters related to Hereward's. We solve those equations for uniform lenses, point lenses, and drift spaces.

It is another property of linear elements that parallel straight lines transform into parallel straight lines. The usual method of determining ellipses in practice is by the use of parallel apertures. In previous analyses, it has been usual to treat problems connected with apertures by the matrix technique. This is annoying - particularly if most of the analysis is made by techniques analogous to Hereward's. We show here that pairs of parallel lines can also be characterized by a pair of numbers. Moreover, it is usual to define an ellipse by two pairs of parallel apertures. We derive expressions for a one to one correspondence between the parameters of pairs of apertures and those of the ellipse of minimum area which circumscribes the region defined by the apertures.

In part II of this paper we deal only with linear elements. Unfortunately, not all elements of interest have this property. In some cases these elements are non-linear by accident , in other cases, for instance with space-charge forces, the non-linearity is intrinsic. In part III of this paper we investigate the effects of non-linear fields. First the general case of a small non-linear perturbation is considered and its effect on the boundary in phase space of the beam is computed. One effect will be that the ellipt - ical shape will no longer be preserved. Since most beam transport elements are linear, the perturbation from the elliptic shape will cause striations in the beam, and thus increase the useable area in phase space. For this reason we consider the ellipse of smallest area which will enclose the perturbed boundary, and define this as the effective area in phase space. Equations are derived for estimating this increase in effective area.

Since we wish to consider space-charge-effects, we recall the familiar space-charge-spreading equations derived by many authors, e.g. Harrison (1958). In sheet and exidlly symmetric beams, the equations for the edge of the beam do not depend on the charge distribution in phase space. These equations are fundamental to the later development. When the space-charge in every plane is distributed uniformly across the beam (with zero transverse temperature or area in phase-space), the effect of space-charge is that of a linear defocusing lens, and the strength of this lens is given.

The approximation of zero area in phase space is bad for many applications. A far more realistic one is the assumption that the beam occupies uniformly an elliptical area in phase space. Assuming that the beam originally is so distributed , the perturbation due to space-charge is computed using the earlier results for non-linear fields. Comparison is made between the predictions of this theory and the zero-temperature one. The increase of effective phase-space area due to space-charge is estimated for sheet and axially symmetric beams.

All the work up to this point has assumed a constant energy beam. This has allowed the usual (x, x') coordinates to be used for the phase plane coordinates. The assumption is not essential to most of this work, but it greatly simplifies the algebra, and is true in many applications. If there are also axial fields, then conjugate coordinates, (x, p_x) must be used. The space-charge perturbation calculations are extended to the consideration of the increase in effective area of phase space in an accelerating field. Estimates are given for the emittance (i.e. area in phase space $/\pi$), due to space-charge in several numerical examples. The agreement between the theory and the measured emittance of a particular beam is shown to be very close. In the example chosen, that of a pulsed 1.5 Amp. 1.5 MeV electron beam from a Van der Graaf generator, the calculated emittance was more than ten times the thermal value. However, the measured emittance differed by less than 10% from the calculated value.

Throughout this paper, space-charge neutralisation effects are neglected. In dc beams, or pulsed ones with long pulse lengths, these effects may become important. It is also assumed that the beam uniformly occupies a region of phase-space; hence finite transverse temperatures effects are only approximated. The methods can be applied, however, to other phase-space configurations; the extension of the method to beams with Maxwellian distribution, will be considered in a later paper.

Throughout this paper, MKS units are used unless stated otherwise.

II. Linear Bear Transport Systems

1. The Phase Space Concept and the Matrix Notation

In this section we will define a coordinates system to describe the transverse oscillations of particles in a beam transport system. We will show how, under certain conditions, the transverse motion of the particles may be represented by two-component vectors, one component representing distance from an axis, the other direction of motion of the particle.

A beam transport system will usually consist of a series of lenses, drift lengths, and deflectors. If the trajectory s_0 of one particle of momentum p_0 through the transport system is known, then this trajectory may define, at each value of the longitudinal coordinate z, an origin of coordinates. A plane may be constructed perpendicular to the trajectory, for each z and two perpendicular directions, x and y, may be taken in this plane. In static electromagnetic fields, the trajectory of any other particle is then completely described by finding the values of the x and y coordinates as a function of z. The notation is illustrated in Fig.I. In order to characterize the motion of the particle, it is necessary to know its transverse position coordinates x, y, its transverse momenta in each plane p_x, p_y and the longitudinal momentum p_z. When one is dealing with axially symmetric systems, it is useful to use (r,ϕ) as in Fig. 2, for the transverse coordinates. In this case the variations of r, ϕ, p, p_ϕ, and p_z are required as a function of z. In most of this paper we will treat the x, y case; however many of the results apply in the (r, ϕ) system, particularly if p_ϕ is always zero. When the electromagnetic fields do not couple the oscillation in the x -plane with those in the y -plane the differential equations of motion for the particles take a particularly simple form. This situation will arise with appropriately oriented quadrupoles, electrostatic deflectors, simple bending magnets, drift spaces, and axial fields. In axially symmetric systems, the motion in the r -plane will couple with that in the ϕ -plane only if there is a p_ϕ. If all the fields are axially symmetric, and if there is no p_ϕ, then the motion in the r -plane obeys very similar equations to that in the x-, y -planes mentioned above. In the problems to be considered in this paper, it will always be assumed that the motions in the two planes are uncoupled. We will restrict ourselves to small transverse oscillations, in which the transverse momenta p_x, p_y, are assumed much smaller than the longitudinal momentum.

In most of this paper, we will restrict ourselves to focusing elements without appreciable axial fields, so that the total momentum of the particle will remain essentially constant. This restriction is not necessary, but it often applies in any case to the kind of beam transport system we have in mind and simplifies the equations. When the energy of the particles varies, it is necessary to use x, y, p_x, p_y as the dependent variables; when the energy of the particles is constant, one may use, even for relativistic beams, x, y, x', y', where

$$x' \equiv dx/dz \, , \quad y' \equiv dy/dz \, . \tag{2.1}$$

The deviation (x, y) of the particles from the axis of coordinates S_0 is given by the equations

$$\left. \begin{aligned} d^2x/dz^2 + f(x,z) = 0 \\ d^2g/dz^2 + g(y,z) = 0 \end{aligned} \right\} , \tag{2.2}$$

where f and g depend on the details of the beam transport system and the momentum of the particle. We will assume that all particles are monochromatic, so that f, g will depend only on the transverse and axial position of the particle. The monochromatic assumption is not necessary /e.g. see Kirstein (1962)/, but will be assumed, for simplicity, in this report. Equations (2.1) and (2.2) follow directly from the Lorentz Force Law, and the assumption that the x - and y plane motions are not coupled. It is possible to consider the motion of particles in only one plane; the motion in the other plane will obey essentially the same equations. Equations (2.2) and (2.3) have beam derived by many authors and extensively studied /e.g. Sturrock (1955)/.

Hereafter we will use the notation $'$ to denote differentials with respect to z ; in particular the slope of a trajectory, dx/dz , will be denoted by x' . If we consider three neighbouring particles starting, at $z=0$, from (x_0 , x'_0), (x_0 , $x'_0 + \Delta x'_0$) ($x_0 + \Delta x_0$, x'_0), then it is well known they will have positions and slope at the plane z of (x , x') ($x + \Delta x_1, \bar{x}'_1 + \Delta x'_1$), ($x + \Delta x_2\, x' + \Delta x'_2$) where

$$(\Delta x_1\, \Delta x'_2 - \Delta x_2 \Delta x'_1) = \Delta x_0 \Delta x'_0 \; . \qquad (2.3)$$

Equation (2.4) has been derived from Eq. (2.2) by many authors /e.g. Sturrock (19?5)/.

If we consider a steady beam of particles passing along the beam transport system, then at any particular z , e.g. z_0 , we may consider the transverse motion of every particle in the beam. Each particle will have, at z_0 , an x , x' , y , y' . If a dot is put for the (x , x') of each particle, at z_0 , the beam may be represented as occupying a region such as that shown in Fig. 3. Since the number of particles in the beams we will consider is very large, the discrete points may be replaced by a continuous distribution in the (x , x') plane as in Fig. 4. It is often implicitly assumed that the density of particles is constant over some part of the (x , x') space and zero over the rest. This assumption is made in the space-charge calculations of part III. The diagrams of Figs 3 and 4 are called <u>phase plots</u>, and the region occupied by the beam is called the region of <u>phase space</u> occupied by the beam. It is to be noted that such phase plots exist in both the x- and y- directions. The effect of Eq. (2.2) may be regarded as that of transforming the phase plot at z_0 into some other phase plot at z_1 . As a result of Eq. (2.4) , such a transformation is area conserving. Hence if a distribution of particles is initially uniform inside one region, bounded by a curve Σ and zero outside, it will always have such a distribution, but the boundary Σ may be transformed, For the study of the properties of the beam, it is sufficient to study the motion of the boundary Σ in phase space. A useful concept in studying the behaviour of beams is the <u>emittance</u> in phase space. This is the area of the space in the phase plane occupied by the beam divided, for later convenience, by π . The emittance is, from Eq. (2.4), an invariant of the beam under transformations such as Eq. (2.2). However, after a complicated system of lenses, an initial phase plot such as Fig. 5a may be transformed into that of Fig. 5b, in which the emittance is the same as that of Fig. 5a. The complicated nature of the phase plot may mean that the effective emittance is considerably increased namely to the area inside Σ_1 of Fig. 5c, for the purpose of injecting into an accelerator, for example. This notion of effective emittance will be discussed further in part III.

In a very important class of problems, those with linear focusing elements, the electro-magnetic forces are such that they produced transverse forcing terms $f(z)x$, so that Eq. (2.2) becomes

$$d^2x/dz^2 + f(z)x = 0. \tag{2.4}$$

Equation (2.4) has been derived by many authors /e.g. Courant and Snyder (1958)/, and the matrix notation based on its solution is usually employed in beam transport system calculations. Equation (2.5) with always apply, if the particles already satisfy Eq. (2.2), for sufficiently small oscillations. In some focusing elements, as for instance quadrupole lengths, bending magnets, and drift lengths, it is valid even for the relatively large oscillations of interest.

We now define two functions a_1 and a_2 satisfying Eq.(2.5) and the initial conditions

$$a_1 = 1 \ , \quad a_1' = 0 \qquad \text{and} \qquad a_2 = 0 \ , \quad a_2' = 1 \ , \ at\ z = 0 \ . \tag{2.5}$$

where as usual ′ denotes differentiation with respect to z. For Eq. (2.4) the invariant of Eq. (2.3) takes the form

$$x_1 x_2' - x_2 x_1' = const. , \tag{2.6}$$

where x_1 and x_2 are any solutions of Eq. (2.4). Hence it follows that

$$a_1 a_2' - a_2 a_1' = 1 \tag{2.7}$$

From the linear form of Eq. (2.5), it may be verified that any solution may be written

$$x = x_0 a_1 + x_0' a_2 \qquad and \qquad x = x_0 a_1' + x_0' a_2' \tag{2.8}$$

Equation (2.8) may be written in the convenient matrix form

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ a_1' & a_2' \end{pmatrix} \begin{pmatrix} x_0 \\ x_0' \end{pmatrix} = \vec{A} \begin{pmatrix} x_0 \\ x_0' \end{pmatrix} , \tag{2.9}$$

where, from Eq. (2.7), the determinant of A is unity.

Following the notation of Sturrock (1955), the general expression for $\vec{A}$ may be written in the form

$$\vec{A} = \begin{pmatrix} \cos(\theta + \psi)/\cos\psi & \rho \sin\theta/\cos\psi \\ -\sin\theta/(\rho\cos\psi) & \cos(\theta-\psi)/\cos\psi \end{pmatrix} \quad (2.10)$$

All the familiar beam transport elements have matrices where θ , ρ , ψ take special values. In any symmetric element, for example, ψ is zero so that $\vec{A}$ becomes

$$\vec{A} = \begin{pmatrix} \cos\theta & \rho\sin\theta \\ -(1/\rho)\sin\theta & \cos\theta \end{pmatrix} . \quad (2.11)$$

The most familiar element is the long lens, of strength f per metre. For this element

$$\rho = 1/\sqrt{f} \quad , \quad \theta = \sqrt{f}\, z \quad . \quad (2.12)$$

Drift lengths are obtained by putting

$$\rho = 1 \ , \quad \theta = 0 \ , \quad (2.13)$$

and point lenses of strength C by letting

$$\theta \to 0 \ , \quad \theta/\rho \to C \ . \quad (2.14)$$

Actually the matrices of Eqs. (2.13) and (2.14) may be derived from that of Eq. (2.12); the first by letting $z \to 0$, the second by letting $z \to 0$, $fz \to C$. These matrices have been obtained by several authors, /e.g. Regenstreif (1960)/. If the energy of the particles were allowed to vary slightly, the f of Eq. (2.5) would become energy dependent. It can be shown /c.f. Kirstein (1962) for example/, that the effect of this energy variation is, to first order, merely to shift the position of the phase plot without changing the shape. It is necessary to add an energy dependent vector, independent of (x , x'),to Eq. (2.9). Those problems will not be treated further in this paper, and only monochrom - atic beams will be considered.

The matrix notation of Eq. (2.9) is very useful. It reduces the analysis of a complicated beam transport system to a series of matrix multiplications. However, it gives more information than we need. Four numbers have to be computed at each stage;these numbers are not independent, since Eq. (2.7) must be satisfied. Moreover if we are interested in the whole region of phase space occupied by the beam,not in the motion of individual particles, it is more convenient to follow the transformation of the boundary of the beam, Σ in Fig. 4, than to use the matrix notation for the individual particles. A notation which will characterize the beam boundary for certain important practical cases is given in the next section.

2. The Transformation of Ellipses and Pairs of Parallel Straight Lines

The transformation of points in phase space by Eq. (2.5) is, from Eq. (2.9), a linear one. Hence two curves, ellipses and parallel straight lines, transform into other ellipses and straight lines. These two curves are of considerable practical importance. The phase shapes which come out of most practical devices approximate to ellipses. Moreover, the acceptance regions of circular accelerators usually require the injection of particles occuping an elliptical area in phase space. For this reason, the study of the transformation of ellipses in electromagnetic fields leading to Eq. (2.5) is very important. In practice the phase ellipses are often measured or defined by two pairs of parallel apertures. A pair of parallel apertures is represented by a pair of parallel lines in the phase plane, hence the properties of the transformation of parallel lines are also interesting. In this section we will show how ellipses and parallel straight lines can each be characterized by 2 numbers. Actually it requires 3 numbers to specify an ellipse, however one, the area, is invariant for transformations such as those of Eq. (2.5); for this reason the variation of only two numbers is important. The equations governing the parameters will be set up in general linear fields, and will be solved for the particular cases of uniform long lenses, with their special cases of point lenses and drift lengths. When only point lenses and drift lengths are used, Hereward (1959) has shown that a related set of numbers transform in a particularly simple manner. The connection between Hereward's parameters and ours is given.

a) Pairs of Parallel Straight Lines.

The pair of parallel straight lines of Fig. 6 are clearly determined by their intercept with the axes ($\pm X$, 0), (0 $\pm Y$). For reasons of later convenience, we will characterize them by related numbers m , n so that the intercepts are ($\mp 1/n$, 0),(0, $\pm 1/m$). If one line has originally intercepts ($-1/n_0$, 0), (0, $1/m_0$), then using the matrix notation of Eq. (2.9), these points will transform into

$$\begin{aligned} (-1/n_0, 0) &\rightarrow [-(1/n_0)a_1, -(1/n_0)a'_1] \\ (0, 1/m_0) &\rightarrow [(1/m_0)a_2, (1/m_0)a'_2] \end{aligned} \qquad (2.15)$$

The straight line joining the points ($-a_1/n_0, -a'_1/n_0$) and (a_2/m_0, a'_2/m_0) has intercepts with the axes ($-1/n$, 0),(0, $1/m$) where

$$m = a_1 m_0 + a_2 n_0 , \quad n = a'_1 m_0 + a'_2 n_0 .$$

We may therefore write

$$\begin{pmatrix} m \\ n \end{pmatrix} = \vec{A} \begin{pmatrix} m_0 \\ n_0 \end{pmatrix}, \qquad (2.16)$$

where $\vec{A}$ is the matrix of Eq. (2.9). It should be stressed that there is a difference between Eqs (2.9) and (2.16), in spite of their apparent identity. Equation (2.9) describes the transformation of an arbitary point (x_0 , x_0'), while Eq. (3.16) describes that of a pair of parallel lines characterized by intercepts $(-1/n, 0)$, $(0, 1/m)$ and $(1/n, 0)$ $(0, -1/m)$. The fact that it is possible to characterize a line in this way is later shown to be important in the positioning of apertures to determine a specific ellipse in an optimum manner.

b)Ellipses. The ellipse of Fig.7 has parametric representation

$$x = \sqrt{Ep} \quad \sin\theta \quad , \quad x' = \sqrt{E/p}\,(\cos\theta + q\sin\theta) \; . \tag{2.17}$$

where E , p , q can be related to the dimensions a , b , c , d , e , f ; however these relations do not concern us at this point. Since the area πE of the ellipse is invariant through the beam transport system, the changes in p and q characterize the ellipse. By considering the equation of the ellipse of Eq. (2.15) in the form

$$(x,\ x') \begin{pmatrix} (1+q^2)/p & -q \\ -q & p \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix} = 0 \; . \tag{2.18}$$

and transforming x , x' by the matrix $\vec{A}$ of Eq. (2.10), it is easy to see that (p_0 , q_0) will transform into (p , q) by the transformation laws

$$\left.\begin{aligned} p &= \{ \frac{(1+q_0^2)}{p_0}\,\rho^2 \sin^2\phi + p_0 \cos^2(\phi - \psi) + 2\,q_0 \sin\phi \cos(\phi - \psi) \} \;/ \cos^2\psi \\ q &= \{ \frac{1+q_0^2}{p_0}\,\rho\cos(\phi + \psi)\sin\theta - \frac{p_0}{\rho}\sin\phi\cos(\phi - \psi) + \\ &+ q_0 [\cos(\phi + \psi)\cos(\phi - \psi) - \sin^2\phi] \} \;/ \cos^2\psi \end{aligned}\right\} . \tag{2.19}$$

This transformation is still very complicated. For the uniform lens, where $\rho = 1/\sqrt{f}$, $\phi = \sqrt{f}\,z$, $\psi = 0$, the expressions for p , q are much simpler, namely

$$\left.\begin{aligned} p &= [(\frac{1+q_0^2}{f p_0} + p_0) - \frac{1+q_0^2}{f p_0}\cos(2\sqrt{f}\,z) + \frac{q_0}{\sqrt{f}}\sin(2\sqrt{f}\,z)] \\ q &= [(\frac{1+q_0^2}{\sqrt{f}\,p_0} - \sqrt{f}\,p_0) \sin(2\sqrt{f}\,z) + q_0 \cos(2\sqrt{f}\,z)) \end{aligned}\right\} . \tag{2.20}$$

For the particular case of a point lens of strength C , Eq. (2.20) gives, taking the limit in the usual way of $z \to 0$, $fz \to C$,

$$p = p_0 \quad , \quad q = q_0 - C\,p_0 \quad , \tag{2.21}$$

while for a drift length z ,

$$\left.\begin{array}{l} p = p_0 + 2q_0 z + (1 + q_0^2) z^2/p_0 \\ q = q_0 + (1 + q_0^2) z/p_0 \end{array}\right\} \tag{2.22}$$

Using the notation of Fig. 7, (which should not be confused with that used elsewhere in this paper) Hereward (1959) defined the quantities

$$\left.\begin{array}{l} G = b/a = 1/p \quad , \quad B = -c/a = -q/p \\ R = e/d = p/(1+q^2) \quad , \quad X = f/d = pq/(1+q^2) \end{array}\right\} , \tag{2.23}$$

so that

$$G + iB = 1/(R + iX) , \tag{2.24}$$

where i is the usual square root of -1. From Eqs. (2.21) - (2.23), we see that for a drift length z ,

$$R = R_0 , \quad X = X_0 + z , \tag{2.25}$$

while for a point lens, strength C

$$G = G_0 , \quad B = B_0 + C . \tag{2.26}$$

Thus we see that when the system contains only drift lengths and point lenses, the transformations using (R , X) and (G , B) are particularly simple. Montague (1960) has built an interesting analogue based on these ideas. When the transport system includes more complicated elements, however, the (p , q) notation is more useful.

When the matrix $\vec{A}$ for the transformation is known, Eq. (2.19) gives the general solution for the parameters of the transformed ellipse. When they are not known, and have to be determined, it is often as easy to solve the equations for (p , q) directly from the equations of motion as to solve first for $\vec{A}$ and then determine p , q .

The equations of motion are, from Eqs. (2.1) and (2.5),

$$dx/dz = x' \quad , \quad dx'/dz = -f(z)x . \tag{2.27}$$

Substitution of Eq. (2.17) into Eq. (2.27) yields the differential equations for p , q , θ

$$p' = 2q \quad , \quad p\theta' = 1 , \quad q' = pf + (1+q^2)/p . \tag{2.28}$$

It is easily verified that the p and q of Eq. (2.20) satisfy Eq. (2.28) with constant f .

Finally, we will require in part III the variation of a function given parameterically in terms of θ and z . If $F(\theta, z)$ is such a function, then from Eq. (2.28) we see that

$$\frac{dF}{dz} = \frac{1}{p}\frac{\partial F}{\partial \theta} + \frac{\partial F}{\partial z} . \tag{2.29}$$

3. The Identification of Pairs of Apertures with their Optimum Circumscribed Ellipse

A pair of apertures, summetrically placed about the centre, at a specified axial distance z, is represented in the phase plane by the pair of vertical lines AA' of Fig. 8a. From the discussion of the previous section, the lines AA' will transform, at a different axial plane z, into another pair of parallel lines characterised by (m,n) as shown in Fig. 8b. The transformations laws for (m,n) have been derived in Eq. (2.16). In the design of beam transport systems, it is convenient to define an ellipse as well as possible by two pairs of apertures. For our purposes, a parallelogram defines an ellipse as well as possible if no part of it lies outside the ellipse, and the ratio of its orea to that of the ellipse is as large as possible. Such a parallelogram will clearly be in - cribed in the ellipse, and we will show that the ratio of its area to that of the ellipse is $2/\pi$. For the purposes of this discussion, we will consider only parallelograms of the type shown in Fig. 9, i.e. we will consider the situation only at the plane of the apertures. Under these conditions there is a particularly simple relation between the numbers defining the apertures, and the parameters of the ellipse. Moreover, if we restrict ourselves to parallelograms with one pair of sides vertical, there is a one to one relation between the two pairs of apertures and the ellipse.

At the axial position of one of the pairs of apertures, these apertures will be represented in the phase plane by a pair of vertical lines with width $2a$, the gap between the apertures. The other pair of apertures are represented by a pair of parallel lines with parameters (m,n). This is illustrated in Fig. 9. The vertices of the parallelogram are the points P_i where P_i are the points

$$P_1,\ P_2 \equiv [\,a,\ (an \pm 1)/m\,]\ ,\ P_4\ ,P_3 \equiv [\,-a,\ (-an \pm 1)/m\,]. \tag{2.30}$$

Any ellipse may be expressed in the form

$$x = \sqrt{Ep}\ \sin\theta\ ,\quad y = \sqrt{E/p}\ (\cos\theta + q\sin\theta\,)\ . \tag{2.31}$$

If this ellipse passes through the vertices P_i of Eq. (2.30) then

$$\sqrt{Ep}\quad \sin\theta = a \qquad and \qquad 1/m = \sqrt{E/p}\quad \cos\theta\quad , \tag{2.32}$$

so that

$$p/m^2 + a^2/p = E. \tag{2.33}$$

This ellipse has minimum area E if

$$p = ma\ , \tag{2.34}$$

in which case, from Eqs. (2.30),(2.31) and (2.33),

$$E = 2a/m \qquad and \qquad q = na\ . \tag{2.35}$$

Conversely, given p , q , E , there exist a unique a , m , n which may be obtained from Eqs. (2.34) and (2.35), namely

$$a = \sqrt{Ep/2} \;, \quad m = \sqrt{2p/E} \;, \quad n = q\sqrt{2/(pE)} \tag{2.36}$$

The m , n , a may be related to the E , G , B of Hereward's notation by means of Eq. (2.23).

Having determined the values of (m , n) required to best define a given ellipse, it is a relatively straightforward matter to determine where the other apertures must be placed. Since (m,n) transform by Eq. (2.16), and since at the plane of the aperture $m = 0$, the aperture must be placed at a plane where

$$a_{11} m_0 + a_{12} n_0 = 0 \, . \tag{2.37}$$

In Eq. (2.37), m_0, n_0 are the solutions of Eq. (2.36) for a particular ellipse, and $\vec{\vec{A}}$, with components a_{ij} ,is the transfer matrix to some new axial plane where we wish to place the second aperture pair. The required half-width of the aperture will then, from Eq.(2.16), be $|1/n|$, where

$$n = a_{21} m_0 + a_{22} n_0 \tag{2.38}$$

We will give a practical example of the procedure. Let us suppose that we have determined the m_0 , n_0 , a required to define a given ellipse, and wish to position the second aperture after a drift length L_1 , lens C , and drift length L_2 as in Fig. 10. The transfer matrix of this system is, from repeated use of Eqs. (2.11), (2.13), and (2.14),

$$\begin{aligned} \vec{\vec{A}} &= \begin{pmatrix} 1 & L_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -C & 1 \end{pmatrix} \begin{pmatrix} 1 & L_1 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 - L_2 C & L_1 + L_2 - L_1 L_2 C \\ -C & 1 - L_1 C \end{pmatrix} . \end{aligned} \tag{2.39}$$

From Eqs (2.37) and (2.39), the lens strength C must be given by

$$C = 1/L_2 + n_0/(m_0 + n_0 L_1) \;, \tag{2.40}$$

and the gap $|2n|$ by

$$|1/n| = L_2 / |m_0 + n_0 L_1| . \tag{2.41}$$

By a similar technique the author has shown elsenwhere/Kirstein (1962)/, how apertures may be located in the x and y planes simultaneously.

The use of the methods sketched in this part both considerably reduce the work of computing beam transport systems, and also lay a groundwork for the non-linear theory of the next part.

III. The Effect of Slightly Non-Linear Fields such as Space-Charge

Introduction

In the second part of this paper, we have investigated the effect of linear electromagnetic fields on a distribution of particles which originally lie inside an ellipse in phase space. It has been established that the particles will be always bounded by an ellipse whose area will remain constant, though its other parameters will vary. The equations specifying the parameters were set up, and solved for some special cases. In this part we will discuss the initial departure from the elliptical boundary caused by the introduction of non-linear fields. The general formalism is presented in section 2. This formalism can be applied to the effect of nonlinear fields on initial distributions which are not quite elliptical, but the solution of this problem will not be discussed. In the last part, we showed that in general electromagnetic field the density in phase space is preserved. Hence whatever the fields, the area in phase space occupied by the beam will be constant. The useable part of this area, which we may call the effective emittance, may be considerably larger (as illustrated earlier in Fig. 5). For this reason we define the effective emittance as the emittance of the smallest ellipse which completely surrounds the region occupied by the beam. Expressions are also derived in section 2 for the effective emittance of a beam.

In section 3, we recall the familiar results of beam-spreading due to space-charge, in a strip and round beam. The equations are always valid for the edge particles, but those for particles inside the beam assume a uniform space-charge distribution and a zero transverse temperature (in which the beam occupies a straight line in phase space). The results of this section, although in no way original, are required for the finite emittance treatment of section 4. There the effects of space-charge on strip and round beams are considered. In this treatment, it is assumed that the beam is originally uniformly distributed in phase-space inside an ellipse.

All the treatment of the last part and most of this part is restricted to beams with constant energy. However most of the results apply equally well in accelerating systems, providing (x , p_x) are used as phase-space coordinates instead of (x , x') - where p_x is the transverse momentum of the particle. In section 5, we discuss the increase of effective emittance of particles in a round beam in an accelerating field. Formulae are developed for this increase in an accelerator, and some numerical examples are given.

2. The Formalism with Slightly Non-Linear Fields

In the last part of this paper, we discussed the effect of linear applied fields. A parametric description of an elliptic boundary was given, and the equations for the variation of the parameters in phase-space were derived. In this section, we will use the same parametric description of the boundary, and will show the perturbations which arise if non-linear fields are superimposed. The assumption is made that the perturbations are small.

In part II we discussed the general equation for the motion of a particle

$$d^2x/dz^2 + f(x,z) = 0 , \qquad (2.2)$$

and then concentrated on its special form

$$d^2x/dz^2 + f(z)x = 0 . \qquad (2.4)$$

In this section we will discuss the generalisation of Eq. (2.4) in the form

$$d^2x/dz^2 + f(z)x = \Lambda(x,z) , \qquad (3.1)$$

where Λ is small. The general motion of a point in the phase plane is then given not by Eq. (2.27), but by the pair of equations

$$dx/dz = x' , \quad dx'/dz = \Lambda(x,z) - f(z) - f(z)x . \qquad (3.2)$$

Now we assume that the non-linear term Λ is small. In this case for $\Lambda(x, z)$ we may write $\Lambda(x_0, z)$, where x_0 is the value that x would have at this point if Λ were zero. Using the parametric representation of Eq. (2.17), a point on the boundary will now have the form

$$x = \sqrt{Ep}(\sin\theta + \xi) , \quad x' = \sqrt{E/p}(\cos\theta + q\sin\theta + \zeta) , \qquad (3.3)$$

while the equations of motion of the boundary will become, from Eq. (3.2)

$$dz/dx = x' , \quad dx'/dz = \Lambda(\sqrt{Ep}\sin\theta, z) - f(z)\sqrt{Ep}\sin\theta . \qquad (3.4)$$

The p, q, θ in Eqs. (3.3) and (3.4) will still obey Eq. (2.28); in particular θ will still satisfy the equation

$$d\theta/dz = 1/p. \qquad (3.5)$$

By successive approximations, it may be verified that the solution of Eq. (3.4) is given by Eq. (3.3), where ξ, ζ have the form

$$\left.\begin{aligned} \xi &= \xi_0 + \zeta_0 z + \int(\int\Lambda\, dz)\, dz \\ \zeta &= \zeta_0 + \int[\Lambda - z(\partial\Lambda/\partial\theta)(\partial\theta/dz)]\, dz\} , \end{aligned}\right\} \qquad (3.6)$$

which may be written, using Eq. (3.5),

$$\left.\begin{aligned} \xi &= \xi_0 + \zeta_0 z + \int(\int\Lambda\, dz)\, dz \\ \zeta &= \zeta_0 + \int[\Lambda - (z/p)\,\partial\Lambda/\partial\theta]\, dz \end{aligned}\right\} . \qquad (3.7)$$

In the applications of this paper Λ has the simple form

$$\Lambda = Z(z)\,\Theta(\theta). \qquad (3.8)$$

If Λ has the form of Eq. (3.8), we may define three quantities σ_1, σ_2, σ_3 by the relations

$$\sigma_1 = \int_0^z Z(z_1)\, dz_1 , \quad \sigma_2 = \int_0^z \sigma_1\, dz_1 , \quad \sigma_3 = \int_0^z z_1 Z(z_1)\, dz_1 . \qquad (3.9)$$

It is to be noted that σ_1 is of first order in z, while σ_2 and σ_3 are of second order. Using the σ_1 of Eq. (3.9), Eq. (3.7) may be written in the form

$$\xi = \xi_0 + \zeta_0 z + \sigma_2 \Theta , \; \zeta = \zeta_0 + \sigma_1 \Theta - \sigma_3 \, d\Theta/d\theta . \tag{3.10}$$

The solution for ξ , ζ is correct to the second order in the perturbation. In Eq.(3.10), the ξ_0 , ζ_0 terms represent the original deviation of the boundary curve from the elliptical; the z , σ_1 terms represent the first order corrections; and finally the σ_2 , σ_3 terms give the second order corrections.

In the special case that the original boundary curve is perfectly elliptical, the zero order terms vanish, so that

$$\xi_0 = \zeta_0 = 0 . \tag{3.11}$$

In this case the perturbed boundary has the form

$$\left.\begin{aligned} x &= \sqrt{Ep}\,(\sin\theta + \sigma_2 \Theta) \\ x' &= \sqrt{E/p}\,(\cos\theta + q\sin\theta) + \sigma_1 \Theta - \sigma_3 d\Theta/d\theta \end{aligned}\right\} . \tag{3.12}$$

Equation (3.12) may be put in a compact form by writing

$$\left.\begin{aligned} K_1(z) &= \sigma_1/(q\sqrt{E/p}) , \quad K_2(z) = \sigma_2/\sqrt{Ep} \\ K_3(z) &= \sigma_3/\sqrt{E/p} \end{aligned}\right\} \tag{3.13}$$

and

$$\Delta = q\sqrt{E/p}\,(\sin\theta + K_1 \Theta) , \quad \psi = \sqrt{E/p}\,(\cos\theta - K_3 \, d\Theta/d\theta) , \tag{3.14}$$

to give

$$x = \sqrt{Ep}\,(\sin\theta + K_2 \Theta) , \quad x' = (\Delta \pm \psi) , \; -\pi/2 \le \theta \le \pi/2 . \tag{3.15}$$

Equation (3.15) arises from Eq. (3.12) by using the fact that Θ depends only on x and z and therefore $\sin\theta$, z . We know, therefore

$$\Theta(\theta) = \Theta(\pi - \theta) , \quad d\Theta(\theta)/d\theta = - d\Theta(\pi - \theta)/d\theta . \tag{3.16}$$

Equation (3.14) allows a simple interpretation of the more important features of the boundary. The current density at any point, assuming constant density in phase space, is given by 2ψ , while the mean value of the slope x' is given by Δ . To first order ψ is constant and Δ varies.

We may easily verify that the total area A in phase space is conserved by the representation of Eq. (3.15) up to the second order. The area is given by

$$A = \int_{-\pi/2}^{\pi/2} 2\psi \, dx ,$$

and using Eqs (3.13) and (3.14), this becomes

$$A = \int_{-\pi/2}^{\pi/2} 2E\,(\cos\theta + K_2\, d\Theta/d\theta)(\cos\theta - K_3\, d\Theta/d\theta)\, d\theta\ . \qquad (317)$$

Now to second order in z, $K_2 \approx K_3$, so that to this order the area is conserved as expected. Actually we know that this area is conserved to any degree of approximation.

Although the total area occupied by the beam in phase space is conserved, its useful area may well increase. Even if some of the elements of the transport system, such as space-charge, are non-linear, most will be linear. Linear forces preserve ellipses in phase space, and will probably unless extraordinary care is taken irretrievably intertwine the non-linear boundary into a larger linear one. For this reason we define the Effective Area of the beam in phase space, as the smallest ellipse completely surrounding the curve of Eq. (3.15). We will now derive expression for the growth of effective phase space area for small K_i.

The ellipse given parametrically by the curve

$$X = \sqrt{E_1 p_1}\ \sin\theta_1\ , \quad X' = \sqrt{E_1/p_1}\ (\cos\theta_1 + q\sin\theta_1)\ , \qquad (3.18)$$

will completely surround that of Eq. (3.15) if at the same value of x, given by

$$\sqrt{E_1 p_1}\ \sin\theta_1 = \sqrt{Ep}\ (\sin\theta + K_2\Theta)\ , \qquad (3.19)$$

the ordinates satify the inequalities

$$X'(\theta) \geq x'(\theta)\ , \quad X'(\pi-\theta) \leq x'(\pi-\theta)\ , \quad -\pi/2 \leq \theta \leq \pi/2\ . \qquad (3.20)$$

Comparing Eqs. (3.15) and (3.18), we see that Eq. (3.20) is equivalent to

$$\left.\begin{aligned} \cos\theta_1 + q_1\sin\theta_1 &> \sqrt{E p_1/(E_1 p)}\ (\Delta + \psi) \\ \cos\theta_1 - q_1\sin\theta_1 &> \sqrt{E p_1/(E_1 p)}\ (\Delta - \psi) \end{aligned}\right\} \qquad (3.21)$$

Writing now

$$F_1 = E(1+\delta E)\ , \quad p_1 = p(1+\delta p)\ , \quad q_1 = q(1+\delta q)\ , \qquad (3.22)$$

and remembering that δE, δp, δq, K_1, K_2, K_3 are small so that quadratic terms in these may be neglected, Eqs. (3.14) (3.19), (3.20) and (3.21) yield

$$\sin\theta_1 = (1 - \tfrac{1}{2}\delta E - \tfrac{1}{2}\delta p)\ \sin\theta + K_2\Theta\ , \qquad (3.23)$$

and

$$[\cos^2\theta + (\delta p + \delta E)\ \sin^2\theta - 2K_2\Theta\ ]^{\frac{1}{2}} - [1 - \tfrac{1}{2}\delta E + \tfrac{1}{2}\delta p]\cos\theta - \qquad (3.24)$$

$$- K_3\, d\Theta/d\theta \geq |q| \cdot |K_1\Theta - \delta q \sin\vartheta|\ .$$

It may be shown that Eqs. (3.23) and (3.24) can only be satisfied with $\delta E = 0$ if

$$K_2 = K_3 = 0 \;, \qquad \Theta = \sin\theta \;, \tag{3.25}$$

which is equivalent to having no non-linearities in the system. For any other K_1 , Θ it can be shown that ΔE is always greater than zero, showing that the effective emittance, which is E_1 is increased. The minimum value of $(E\,\delta E)$ satisfying Eqs (3.23) and (3.24) is the increase of the emittance. To first order we may neglect K_2 and K_3 this approximation will greatly simplify Eq. (3.24) and will always be used in this paper.

Specific examples of this formalism will be given in sections 4 and 5.

3. The Zero Temperature Space-Charge-Spreading Theory

In this section we will recall the familiar equations for the spreading of infinite strip beams and axially symmetric ones. These equations have been derived by many authors, e.g. Harrison (1958). The equations are correct for the particles at the edge of the beam, irrespective of the distribution of the particles in phase space. However for the particles inside the beam, their distribution in real space becomes important, and this will be assumed always uniform; this assumption implies that the beam lies on a straight line in phase space. In as much as the original distribution occupies a finite area in phase space, the predictions of this theory will be in error and those of the next must be used.

a) Sheet Beam. We will first assume that the space-charge distribution in the y and z -directions is infinite and uniform, though it may vary in the x -direction (all in physical space). In this case the electric field due to a planar change Q per m^2 at x_0 is given by ($\mathcal{E}_s$, 0 , 0) where

$$\begin{aligned} \mathcal{E}_s &= -\,Q/(2\epsilon_0) \;, \qquad x > x_0 \\ &= Q/(2\epsilon_0) \;, \qquad x < x_0 \end{aligned} \tag{3.26}$$

where ϵ_0 is the dielectric constant of free space. For a sheet beam of thickness $2a$, width in the y -direction b , axial-velocity v (assumed constant through the beam), and current density $i(x)$, $\mathcal{E}_s$ is given by

$$\mathcal{E}_s(x) = -\left[\int_{-a}^{x} i\,dx_1 - \int_{x}^{a} i\,dx_1\right]/(2\epsilon_0 v) \,. \tag{3.27}$$

Moreover the current density i is related to the total current I by the relation

$$I/b = \int_{-a}^{a} i\,dx \;. \tag{3.28}$$

For the edge particle, at $x = a$, Eq. (3.27) becomes

$$\mathcal{E}_s(a) = -I/(2\epsilon_0 v b) \quad . \tag{3.29}$$

Writing in the usual way

$$v = \beta c \ , \quad \gamma = (1-\beta^2)^{-1/2} \quad , \quad m = m_0 \gamma \quad , \tag{3.30}$$

where m_0 is the rest mass of the particle and e its charge, the relativistic Lorentz Force Law may be written

$$(m \beta^2 c^2)\, d^2x/dz^2 = (1-\beta^2)\, e\, \mathcal{E}_s + e\, \mathcal{E}_F \ , \tag{3.31}$$

where $\mathcal{E}_F$ is the applied electric field; this equation may be written in the form

$$d^2x/dz^2 = \mathcal{E}_s/[(m_0 c^2/e)\,\beta^2 \gamma^3] + \mathcal{E}_F/[(m_0 c^2/e)\,\beta^2 \gamma] \ . \tag{3.32}$$

For the edge particle, at $x = a$, Eq. (3.22) may be written

$$d^2a/dz^2 = \Lambda_0 + F(a,z) \ , \tag{3.33}$$

where

$$F(x,z) = \mathcal{E}_f(x,z)/[(m_0 c^2/e)\,\beta^2\gamma] \ , \tag{3.34}$$

and Λ_0 is given, using Eqs. (3.29) and (3.32), by

$$\Lambda_0 = -(I/b)/[2\epsilon_0 c (m_0 c^2/e)\,\gamma^3 \beta^3] \ . \tag{3.35}$$

Λ_0 is positive for all particles if we take the magnitude of the current for I; using Eq. (3.30) Λ_0 may be written in the form,

$$\Lambda_0 = \lambda (I/b)/(\gamma^2 - 1)^{-3/2} \quad , \tag{3.36}$$

where λ has the value, in MKS units,

$$\left.\begin{aligned} \lambda &= 3.82 \times 10^{-4} && \text{for electrons} \\ &= 2.08 \times 10^{-7} && \text{for protons} \end{aligned}\right\} \ . \tag{3.37}$$

For non-relativistic beams,

$$\gamma^2 - 1 = 2V/\mathcal{E}_0 \quad , \tag{3.38}$$

where V is the voltage and $\mathcal{E}_0$—the rest energy of the particle, so that Eq.(3.36) takes the form

$$\Lambda_0 = (\lambda'/b)\,(I/V^{3/2}) \quad , \tag{3.39}$$

where λ' has the numerical values

$$\left.\begin{aligned} \lambda' &= 4.78 \times 10^{4} && \text{for electrons} \\ &= 2.05 \times 10^{6} && \text{for protons} \end{aligned}\right\} \ . \tag{3.40}$$

Equations (3.33)-(3.40) have been derived by Harrison (1958) and others, though it has usually been solved with zero applied field $\mathcal{E}_F$. The detailed solution of the equations, or the range of applicability do not concern us at this point. We will return to these equations in the next section.

While the equation for the motion of the edge particle does not depend on the distribution of i across the beam, this distribution does affect the motion of particles inside the beam. If, and only if, we assume that the current distribution is unform across the beam, so that $\mathcal{E}_s$ varies linearly, the equation for any particle becomes

$$d^2x/dz^2 = \Lambda_0'(x/a) + \mathcal{E}_F(x,z) \, , \qquad (3.41)$$

where again a is the width of the beam. The effect of space charge is therefore the same as that of a uniform defocusing lens of strength Λ_0/a per metre. If $i(x)$ is not uniform, the effect of space charge is non-linear in x, and can only be estimated by the methods of section 2.

b) The Circular Beam. In this case we will assume an axially symmetric beam, uniform in the z -direction. The electric field at r due to a ring of charge Q at r_0 is then ($\mathcal{E}_s$, 0 , 0) in cylindrical polar coordinates, where

$$\left.\begin{aligned} \mathcal{E}_s(r) &= -Q/(2\pi r\epsilon_0) \qquad & r > r_0 \\ &= 0 & r < r_0 \end{aligned}\right\} . \qquad (3.42)$$

Using the same procedure as in paragraph (a), the field at r due to a current distribution $i(r)$ in the region $0 \le r \le a$ is given by

$$\mathcal{E}_s(r) = \left(\int_0^r i\,dr_1\right) / (2\pi r\epsilon_0 v_z) \, . \qquad (3.43)$$

For the edge particle, Eq.(2.38) gives

$$\mathcal{E}_s(a) = I/(2\pi\epsilon_0 v_z a) \, . \qquad (3.44)$$

The exact procedure of paragraph (a) may therefore be followed for the edge particle, with (πa) replacing b . This means that Eq. (3.33) is satisfied again but that now instead of Eq. (3.36), Λ_0 takes the form,

$$\Lambda_0 = \lambda\,[I/\pi a)]\,(\gamma^2 - 1)^{-3/2} \, . \qquad (3.45)$$

Equation (3.33) with this form for Λ_0 has been derived by Harrison (1958) and solved for $F = 0$. We will not repeat this solution.

Again Eq. (3.33) is independent of the distribution of i across the beam, while the motion of particles inside the beam does depend on i . If we assume that the current density is constant in the beam, so that

$$i(r) \propto r \, ,$$

then the motion of particles inside the beam may be again expressed by Eq. (3.41), with r

replacing z and Λ_0 given by Eq. (3.45). The effect of space-charge is therefore that of a uniform lens of strength $\lambda (I/\pi)(\gamma^2 - 1)^{-3/2}/a^2$ per metre, where λ is a universal constant. The variation of lens strength with a^{-2} will prove to be of great importance in estimating the increase of effective emittance in the (r , r') space.

4. The Finite Temperature Space-Charge Spreading Theory

In the previous section, we developed the equations of space-charge spreading of a beam under the assumptions that the beam originally occupied a straight line in phase space. The equations for the edge particles, however, were independent of the distribution of particles in phase space. In this section, we will extend the theoryto thecase where the beam originally occupies an elliptic region in phase space. This assumption is approximately satisfied in practical beams. In cylindrical beams, this assumption requires the convention that the point ($-r$, $-r'$) in phase space is identical to the point (r , r'), with the appropriate simplification of certain formulae. We will use the non-linear theory developed in section 2, and discuss the differences which occur between the predictions of this theory and those of section 3.

a) Sheet Beam Case. The general expression for the motion of a particle, with coordinates (x , x') in phase space, under the influence of slightly non-linear fields has been given in Eq. (3.2). If the non-linear fields arise only from space charge, the value of Λ at the point $x = a$ is given by Eq. (3.36). The Λ at other points may be obtained, from Eqs (3.22), (3.32) and (3.33) in the form

$$\Lambda = b\,\Lambda_0 \left[\int_{-a}^{x} i\,dx_1 - \int_{x}^{a} i\,dx_1 \right] / I \ . \tag{3.47}$$

We will assume that the applied field is linear, so that F has the form

$$F = -f(z)\,x \ . \tag{3.48}$$

With the F and Λ of Eqs (3.47) and (3.48), the equations of motion are given by Eq. (3.2).

We will assume a constant distribution of current inside an elliptical area defined in the usual way in the parametric form

$$x = \sqrt{Ep}\ \sin\theta \ , \quad x' = \sqrt{E/p}\,(\cos\theta + q\sin\theta) \ . \tag{3.49}$$

This is the form of boundary assumed in Part II, and the equations for p , q were derived in Eq. (2.28), while E is constant. The current density, assuming a uniform distribution of current inside the curve of Eq. (3.49) is given by

$$i\,dx = 2E\cos^2\theta\ d\theta\,, \tag{3.50}$$

so that, using Eqs. (3.28) and (3.46),

$$\Lambda(\theta, z) = \Lambda_0(z)[(2\theta + \sin 2\theta)/\pi]\,,\ -\pi/2 \le \theta \le \pi/2\,, \tag{3.51}$$

while again, since Λ can only depend on x, z

$$\Lambda[\theta, z] = \Lambda[\pi - \theta, z]\,,\ \pi/2 \le \theta \le 3\pi/2\,. \tag{3.52}$$

These expressions for Λ are in the form of Eq. (3.8) so that we may directly use the formalism developed in section 2. Without space charge, the motion of the boundary is given directly by Eq. (2.28). Since Λ is in the form of Eq. (3.8), we may write, from Eq. (3.36),

$$Z(z) = \Lambda_0(z) = \lambda(l/b)(\gamma^2 - 1)^{-3/2}\,, \tag{3.53}$$

and

$$\Theta(\theta) = (2\theta + \sin 2\theta)/\pi\,.$$

With this form of Θ, $d\Theta/d\theta$ is given by

$$d\Theta/d\theta = (2/\pi)(1 + \cos 2\theta)\,, \tag{3.54}$$

and the σ_i, K_i of section 2 must be used with these values of Z, Θ. It is to be noted that in this case Z does not depend on z, but is constant.

With these values of K_i, the phase-space boundary is given by Eqs (3.14) and (3.15), namely

$$x = \sqrt{Ep}\,(\sin\theta + K_2)\,,\ x' = \Delta \pm \psi\,, \tag{3.55}$$

where

$$\Delta = q\sqrt{Ep}\,(\sin\theta + K_2\Theta)\,,\ \psi = \sqrt{E/p}\,[\cos\theta + K_3 d\Theta/d\theta]\,. \tag{3.14}$$

In comparing these results with those of section 3, we see that, since

$$\Theta = 1\,,\ d\Theta/d\theta = 0\,,\ \text{at}\ \theta = \pi/2\,, \tag{3.56}$$

while Θ is odd in θ, the motion of the edge particle is identical in the two sections as expected, if we put

$$\sqrt{Ep} = a\,. \tag{3.57}$$

For small z we may take from the definitions of σ_i and K_i of Eq. (3.39) and (3.13),

$$K_2 \approx K_3\,. \tag{3.58}$$

Typical curves of how Δ varies across the cross-section of the beam are given for $K_1 = 0$, 1 and ∞ in Fig. 11. The $K_1 = 0$ curve is also that arising from the theory of section 3 (since the space-charge term would be included in the change of q). The variation of current density ψ across the cross-section is shown in Fig. 12, for $K_2 = 0, 0\cdot25, 0\cdot5$;

the heavy curve is the theoretical one based on the theory of section 3. In most applications, the model of this section is considerably better than that of the last.

Finally, it is of importance to estimate the area of the smallest ellipse which will completely surround the curve of Eq. (3.58). This ellipse was shown in section 2 to have area $\pi E(1+\delta E)$ where δE, δp, δq satisfy eq. (3.24). For a small growth of emittance, we may neglect the second order terms K_2, K_3, and obtain from Eq. (3.24)

$$[\cos^2\theta + (\delta p + \delta E)\sin^2\theta]^{1/2} - (1 + \tfrac{1}{2}\delta q)\cos\theta \geq |q| \cdot |K_1 \Theta - \delta q \sin\theta| \ . \tag{3.59}$$

It is difficult to find the minimum δE for all values of δp, δq, so that Eq.(3.59) is satisfied. However, by an iterative procedure, we find that an approximate minimum of δE is obtained, for small δp, δq, when

$$\delta q \approx 7/6\, K_1 ,\ \delta p \approx 0 ,\ \theta \approx \pi/4 , \tag{3.60}$$

and the equality is satisfied in Eq. (3.59). Under these conditions,

$$\delta E_{min} \approx 1/15\, (K_1 q) \ . \tag{3.61}$$

Returning to the definition of K_1 and δE, we see, from Eqs. (3.9),(3.13) and (3.22) that the charge in E, ΔE, is given by

$$\Delta E = E\,\delta E_{min} \approx 1/15\, \sigma_1 \sqrt{Ep} \ . \tag{3.62}$$

Now $\sqrt{Ep}$ is just the half width a of the beam; the formula of Eq. (3.62) is only correct to first order in z. A more accurate expression, using the definition of σ_1, a of Eqs. (3.9), (3.53) and (3.57) would be

$$\Delta E \approx 1/15 \int_0^z \Lambda_0\, a\, dz_1 \ . \tag{3.63}$$

Equation (3.63) is probably the most important result of this paper, since it gives in a very compact form an approximation for the rate of growth of emittance. Several properties of the solution are unexpected. First of all the growth of effective phase area does not depend on the original emittance or phase configuration of the beam, but only on the space-charge parameter Λ_0, and the beam size a. Secondly, although the space-charge forces at the edge of the beam do not depend on a, the rate of increase of useful emittance does. Let us give a numerical example of the magnitude of this effect. In a $1\,A$ electron beam with the dimensions 1cm x 2mm at 2MeV,

$$a \approx 10^{-3} \ , \quad \gamma \approx 5 \ , \quad \Lambda_0 \approx 3.82 \times 10^{-4}/1.25 \approx 3 \times 10^{-4} ,$$

so that

$$\Delta E \approx 1/5 \times 10^{-7} z \ . \tag{3.64}$$

The effective emittance would therefore increase by 0.02 mm mrads/m. By applying a similar procedure in an accelerating beam, a procedure to be described in section 5, it is possible to estimate the minimum emittance of a beam.

b) Circular Beam. For an axially symmetrical beam, the argument is almost identical to that of paragraph (a). The only difference comes in the formulae to be used for Λ. All other formulae remain the same. From the space-charge potential of Eq. (3.43), the expression for Λ becomes

$$\Lambda = \Lambda_0 (a/r) [\int_0^r i \, dr_1] / I , \tag{3.65}$$

where now Λ_0 is given by Eq.(3.45), and a is the radius of the beam. If we again assume that a point in (r , r') space occupies an ellipse, with the point ($-r$, $-r'$) $\equiv$ (r , r') then the boundary of the phase ellipse, in the absence of space charge, may be written in the usual form

$$r = \sqrt{Ep} \; \sin\theta \;, \quad r' = \sqrt{E/p} \;\; (\cos\theta + q \sin\theta) \, , \tag{3.66}$$

where p , q again satisfy Eq. (2.28), and E is constant. In paragraph (a), we assumed that the density of current in (x , x') space was constant, and that it was uniform in the y - direction; by analogy, since the circumference of a tube of current increases linearly with r in real space, we will assume

$$i(r) \propto r(\Delta r') , \tag{3.67}$$

where $\Delta r'$ is the length of ordinate inside the curve of Eq.(3.61); this $i(r)$ can be expressed as

$$i(r) \, dr \propto \sqrt{E^3 p} \;\; \cos^2\theta \; \sin\theta \, d\theta \; , \tag{3.68}$$

so that the Λ Eq.(3.65) becomes

$$\Lambda = \Lambda_0 \Theta , \tag{3.69}$$

where now Λ_0 is given by Eq. (3.45), and Θ by the relation

$$\Theta = (1 - \cos^3\theta) / \sin\theta . \tag{3.70}$$

Equation (3.70) is derived by integrating Eq. (3.68), substituting for r from Eq.(3.65), and putting for I the value of the integral at $\theta = \pi/2$. All the equations of Eqs. (3.52)-(3.59) will follow identically, if the Θ of Eq. (3.70) replaces that of Eq.(3.54), and we write (r , r') for (x , x'). The variation across the beam of Δ and ψ of Eqs. (3.58) are given in Fig. 13 and 14, the results of section 3 are superimposed in these figures in which the same approximation is made as in paragraph (a) that $K_2 \approx K_3$. The curves have the same significance as those of Figs. 11 and 12. It is to be noted, from Fig. 13, that the divergence may no longer be maximum at the outside of the beam for sufficiently large values of K_2 . This result is a consequence of the fact that the integral of the current density grows more slowly than the (i/r) term in Eq. (3.65). Actually the $K_2 = 0.5$ curve is too large for the theory to be valid, so that in practice the divergence will probably still be maximum at the beam edge. However, Fig. 14 shows that even from moderate K_2, the current density is reduced in the center of the beam. This is consistent with the 'hole' which in practice often occurs in round beams.

The rate of growth of effective phase space area will again be derived from the minimum δE satisfying Eq. (3.59). For small δp, δq, K_1, the minimum is again near that of Eq. (3.60), and the minimum value of δE given by

$$\delta E_{min} = K_1 q/8 \quad , \tag{3.71}$$

where the factor 1/8 is again only approximate. Now K_1 is defined by Eq. (3.13), and σ_1 by Eqs (3.8) and (3.9). Hence we see that the change in emittance, ΔE, is given, by analogy with Eq. (3.63), by the expression

$$\Delta E = E\,\delta E_{min} = 1/8 \int_0^z \Lambda_0\, a\, dz_1 \quad ;$$

this becomes, using the definition of Λ_0 of Eq. (3.45),

$$\Delta E = 1/8\ \lambda\ (I/\pi)\ (\gamma^2 - 1)^{-3/2} z \ . \tag{3.72}$$

Equation (3.72) is very interesting, in that it indicates that the rate of growth of emittance in a cylindrical beam depends, to a first approximation, only on the total current and energy of the beam, not on its physical size, or current distribution in phase space. In non-relativistic beams, Eq. (3.72) shows, using Eq. (3.38), that the rate of growth of emittance depends only on the value ($I/V^{3/2}$), the perveance of the beam.

As a numerical example, let us consider the growth of effective emittance in a 2MeV beam carrying 1A. Here

$$\gamma = 5 \ , \quad I = 1 \qquad \text{so that}$$

$$\Delta E = 1/8 \times 10^{-6}\ z \ .$$

Thus the effective emittance grows at approximately 1/8 mm millirads/metre.

5. The Expected Emittance of a Cylindrical Beam in an Accelerator with a Uniform Axial Field

In all the previous analyses of this paper, we have assumed that the energy is constant. This has allowed the use of (x , x') or (r , r') as independent variables in phase space obeying Eqs. (2.1) and (2.2). If the energy of the particles is allowed to vary with z, then we must use conjugate variables (x , p_x) or (r , p_r) instead of

(x, x'), (r, r'), where p_x, p_r are the momenta in the x- or r- directions. In this case areas in phase space will still be conserved, but the axial variation of the parameters which depend on energy will also have to be considered.

In this section, we will consider the axially symmetric problem with a constant axial field. We will analyse the physical problem shown schematically in Fig. 15. The first part, from the cathode K to the grid G, consists of a gun which accelerates particles from $\gamma = 1$ to $\gamma = \gamma_0$. The second part from the grid G to the anode A consists of an accelerating column with a constant axial field, which accelerates the beam from γ_0 to γ_F. Superimposed on the axial field may be some focusing, elements which slightly alter the axial variation of the field, but the axial effect of such variations will be ignored. The transverse effects would be linear in r near the axis of the acceleration column. This is the situation which occurs in practice in accelerators such a Van de Graaff generators.

The shape of the phase-plane diagram which will arise at the anode A depends on the conditions at the entrance of the accelerator column G, and the linear focusing fields experienced by the beam in the column. We will not attempt to predict this shape in this paper. The expected emittance, however, can be divided into two parts; the first part is the emittance of the beam at G, the second part is the increase in the effective emittance due to non-linear effects in the region $G-A$. In the gun region, the fields will in any case be highly non-linear; they will be designed to reduce the emittance of the beam at G as much as possible. In the accelerating region, however, the applied fields have linear local variation some distance from the axis. The linear fields will increase the effective emittance if the initial phase plot is not elliptic, or if any other non-linear fields perturb the elliptic shape. In this section we will assume that the beam, at entrance to the column, uniformly occupies an ellipse in phase space. We will investigate the increase of effective emittance due to space charge using expressions analogous to those of section 4. Of course the equation must be modified to allow for the use of new variables, but the results will be substantially the same. The initial ellipse at G will be assumed to be that due to thermal velocities at the cathode.

First let us introduce the relativistic transverse Lorentz Force Law which has the form /see Sturrock (1955) for example/,

$$dr/dt = p_r/(mc) \quad , \quad dp_r/dt = e\,\mathcal{E}_r\,. \tag{3.73}$$

Remembering that m is related to the rest mass m_0 by

$$m = m_0\,\gamma \tag{3.74}$$

and that γ now varies with distance, we may introduce the <u>proper time</u> τ given by

$$\gamma \, d\tau = dt \, ,$$

to obtain (3.75)

$$dr / d\tau = p_r /(m_0 c) \, , \quad dp_r / d\tau = e\mathcal{E}_r \gamma \, . \tag{3.76}$$

Defining arbitrarily a variable s by the relation

$$s = p_r /(m_0 c) \, , \tag{3.77}$$

we see that the equations of motion in phase space become

$$dr / d\tau = s \, , \quad ds / d\tau = - f(r, \tau) \, , \tag{3.78}$$

where now f is derived from $\mathcal{E}_r$ by the expression

$$f = - [e\gamma /(m_0 c)] \, \mathcal{E}_r \, . \tag{3.79}$$

Equation (3.78) is formally identical with Eq. (2.2), allowing all the earlier formulation to be used on the coordinates (r , s) in phase space. It is necessary, however, to know how f varies with r .

As in the previous work, f may be divided into two parts, one linear in r , the other non-linear. Since we are only interested in the increase in emittance, the only part of f which is interesting is the non-linear space-charge contribution. The introduction of constant or linearly varying axial fields will have no non-linear effect on the transverse field. The non-linear part of f , $\Lambda(r, \tau)$ will be similar to that of section 4, with a different factor of proportionality, due to the different units of the independent variable. By direct comparison of Eqs (3.79) and (3.31), and remembering that $\mathcal{E}_s$ is still given by Eq. (3.34), we see that

$$f = \Lambda_0 [\int_0^r i \, dr_1] / I \, , \tag{3.80}$$

where Λ_0 is not given by Eq. (3.45), but instead by

$$\Lambda_0 = \lambda c (I / \pi a)(\gamma^2 - 1)^{-1/2} \tag{3.81}$$

With this exception, all the analyses of section 4 apply. Hence we may again deduce the change of effective emittance in the (r , s) plane directly from Eq. (3.71), where K_1 is still defined by Eq. (3.13). We may therefore deduce that the change of emittance in the (r , s) plane, ΔE_{rs}, is given by

$$\Delta E_{rs} = (1/8) \int_0^z \Lambda_0 \, a \, d\tau \, , \tag{3.82}$$

where Λ_0 is given by Eq. (3.81), and λ again by Eq.(3.37).

To integrate Eq. (3.82), we must see how γ, and hence Λ_0, vary with τ. The axial equations of motion yield the relations

$$dz/d\tau = p_z/m\,,\quad dp_z/dt = e\mathcal{E}_z\,. \tag{3.83}$$

Now as in Eq. (3.30),

$$m = m_0\gamma\,,\quad dz/dt = \beta c\,,\quad \beta\gamma = \sqrt{\gamma^2-1}\,, \tag{3.30}$$

hence Eq. (3.83) yields the expression

$$d(\sqrt{\gamma^2-1})/d\tau = [e/(m_0 c)]\gamma\,\mathcal{E}_z\,. \tag{3.84}$$

Let us assume that the accelerating column in Fig. 15 accelerates particles from γ_0 at G to γ_F at A in a length L, Let us also assume that γ_0 is near unity, implying a non-relativistic beam at injection; this assumption is in no way necessary, but usually holds in practice. The axial field $\mathcal{E}_z$ of Eq. (3.84) is related to the rest energy of the particle $\mathcal{E}_0$, and the other parameters of the accelerator by

$$\mathcal{E}_z = (\gamma_F - \gamma_0)\,\mathcal{E}_0/L\,. \tag{3.85}$$

Substituting the $\mathcal{E}_z$ of Eq. (3.85) into Eq. (3.84), and integrating, we obtain

$$ch^{-1}\gamma = ch^{-1}\gamma_0 + \omega\tau\,, \tag{3.86}$$

where ω is given by

$$\omega = (\gamma_F - \gamma_0)\,c/L\,. \tag{3.87}$$

Using the fact that γ_0 is near unity, so that

$$ch^{-1}\gamma_0 \approx \sqrt{2(\gamma_0-1)}\,, \tag{3.88}$$

Eq. (3.86) can be written

$$\gamma \approx ch[\sqrt{2(\gamma_0-1)} + \omega\tau]\,, \tag{3.89}$$

and

$$\sqrt{\gamma^2-1} \approx sh[\sqrt{2(\gamma_0-1)} + \omega\tau]\,. \tag{3.90}$$

Combining Eqs (3.81), (3.82), and (3.90), we obtain the equation for the effective emittance, in (τ , s) coordinates namely

$$\Delta E_{\tau s} = \int_{\gamma=\gamma_0}^{\gamma=\gamma_F} [\lambda c I/(8\pi)]\,\mathrm{cosech}\,[\omega\tau + \sqrt{2(\gamma_0-1)}]\,. \tag{3.91}$$

Now for an arbitrary constant a, ω

$$\int \mathrm{cosech}\,(\omega\tau + a)\,d\tau = -(1/\omega)\log[\mathrm{cosech}\,(\omega\tau+a) + \cot(\omega\tau+a)]\,, \tag{3.92}$$

hence Eq. (3.91) may be integrated to yield

$$E_{rs}(r) - E_{rs}(0) = [\lambda c I/(8\pi\omega)] \log \chi \ , \tag{3.93}$$

where χ is given by

$$\chi(r) = \frac{1 + ch\sqrt{2(\gamma_0 - 1)}}{sh\sqrt{2(\gamma_0 - 1)}} \quad \frac{sh[\sqrt{2(\gamma_0 - 1)} + \omega r]}{1 + ch[\sqrt{2(\gamma_0 - 1)} + \omega r]}$$

$$\approx \frac{2}{\sqrt{2(\gamma_0 - 1)}} \cdot \frac{sh[\sqrt{2(\gamma_0 - 1)} + \omega r]}{1 + ch\sqrt{2(\gamma_0 - 1)}} \ . \tag{3.94}$$

Thus the final emittance at A in Fig. 15 is obtained by putting $r = r_F$ in Eq. (3.94), which gives, from Eq. (3.89) and (3.90),

$$\chi(r_F) = \frac{2}{\sqrt{2(\gamma_0 - 1)}} \cdot \frac{\sqrt{\gamma_F^2 - 1}}{1 + \gamma_F} = \sqrt{\frac{2}{\gamma_0 - 1}} \sqrt{\frac{\gamma_F - 1}{\gamma_F + 1}} \ . \tag{3.95}$$

The emittance E_{rs} of Eq. (3.93) is in rather arbitrary units. The more usual units are in the (r , r') plane. The transition from one to the other is made using the relation, derived from Eqs (3.75)-(3.77),

$$s = p_r/(m_0 c) = m_0[\beta\gamma c/(m_0 c)] r' = \sqrt{\gamma^2 - 1} \ r' \ . \tag{3.96}$$

Hence the emittance in the (r , r') plane is obtained, from Eqs (3.87),(3.93), and (3.96) in the form

$$E_{rr'}(r_F) = [\lambda/(8\pi)][IL/\{(\gamma_F - \gamma_0)\sqrt{\gamma^2 - 1}\}] \log \chi + E_{rr'}(0)\sqrt{(\gamma_0^2 - 1)/(\gamma_F^2 - 1)} \ , \tag{3.97}$$

where χ is given by Eq. (3.95). It is a little difficult to decide whether the emittance of Eq.(3.97) is a lower or an upper limit. It is certainly not a lower limit, since by suitably chosen non-linear lenses, the first terms in Eq. (3.97) could be eliminated. However in practice, the fields inside the column have to be linear, and the periodic focusing of the column ensures that much of the phase plane shape has become irretrievably entwined. The emittance of Eq. (3.97) is not an upper limit, since any accidential non-linearities in the transverse fields could increase the figure. Since such non-linearities do not usually occur, we may say that $E_{rr'}$ of Eq. (3.97) is the expected emittance. The theory is not strictly accurate for large r , but since this would mainly affect the χ term, which in any case only varies logarithmically , we would expect the result to be fairly accurate.

The second term can be reduced to the thermal one by using non-linear fields in the gun region. It will usually be possible to optimise the design for high current, and rely on the fact that at low current the space-charge increase of emittance is less important. For this reason the value of the second term can be estimated fairly accurately by using the value of emittance at the cathode which results from thermal effects.

The $E_{rr'}$ of Eq. (3.97) is then a reasonable estimate of the best beam that can be obtained. To find the value of the second term in Eq. (3.97) we use the relevent emittance at the cathode and assume that this is unchanged from the cathode to the anode in the units (r, s). Now in (r , s) space, the transverse distribution of the current j at the cathode is given from Pierce (1954) by the expression.

$$j(r,s) = (2rI/a^2)\sqrt{m_0/(2kT\pi)}\ \exp - [\,m_0 s^2/(2kT)\,] . \qquad (3.98)$$

In Eq. (3.98) we have used the fact that in non-relativistic beams the transverse velocity is given by s ; in Eq. (3.98) a is the radius of the cathode, I the total current emitted k is Boltzman's constant, T is the temperature of the cathode, and m_0 the rest mass of the particle. Integrating Eq. (3.98) with respect to s to the limits $\pm s$, we obtain

$$i(r) = (2r/a^2)\,I\ \mathrm{Erf}\,[s_0\sqrt{m_0/(2kT)}] , \qquad (3.99)$$

where Erf (x) is the error function of x .

If we arbitrarilly define the emittance as the area containing 90% of the beam, then for s_0 we should take the value which makes the error function equal to 0.9 . Since

$$\mathrm{Erf}\,(x) = 0.9 \quad \text{if} \quad x = 1.17 , \qquad (3.100)$$

the resulting s_0 is given by

$$s_0 = 1.65\sqrt{kT/m_0} . \qquad (3.101)$$

It is to be noted that the numerical factor is rather arbitrary for two reasons. First factors other than 90% be chosen for emittance; secondly, in the earlier definitions of emittance, constant current distribution inside an ellipse in phase space was assumed. However the figures resulting from Eq. (3.101), allow us to estimate fairly well the emittance we should use in Eq. (3.97). Using the s_0 of Eq. (3.101), and remembering that our usual definition of emittance is the area of phase space divided by π , we find

$$E_{rs} = 1.65 \times (4/\pi)\,a\sqrt{kT/m_0} . \qquad (3.102)$$

In the units of r , s , E_{rs} is invariant; its value at the point A is related to the emittance $E_{rr'}$, by

$$E_{rr'} = E_{rs}/(\beta_F\,\gamma_F) . \qquad (3.103)$$

Equation (3.103) can be written

$$E_{rr'} = K\sqrt{T}\ a/\sqrt{\gamma_F^2 - 1} \quad , \tag{3.104}$$

when K takes the numerical values

$$\left.\begin{array}{ll} K = 26 & \text{for electrons} \\ K = 0.14 & \text{for protons} \end{array}\right\} , \tag{3.105}$$

and $E_{rr'}$ is measured in mm millirads. As a practical case, let us consider a 1.5 MeV electron beam coming from a circular cathode of radius 4 mm and a temperature 900°C. In this case the $E_{rr'}$ due to thermal effects, the $E_{rr'}$ of Eq. (3.104), is given by

$$E_{rr'} = 26 \times \sqrt{1173} \times 4 \times 10^{-3} / \sqrt{63} = 0.45 \text{ mm millirads} . \tag{3.106}$$

The $E_{rr'}$ of Eq. (3.106) is the minimum possible obtainable emittance for 90% of the beam. It assumes that there is no increase in space charge due to space-charge spreading.

Let us now assume that the space charge spreading causes the emittance to increase by the amount given by the first term of Eq. (3.97). As a practical example let us consider an electron beam accelerated by a space-charge limited electron gun to 20 KV, and then accelerated up to 1.5 MeV with a constant field. Let us assume that the current in the beam is 1.5 A, the length of the column is 2m, and that there is no increase of emittance in the gun region. We may then use Eq. (3.97) with

$$\gamma_0 = 1.04 \ , \quad I = 1.5 \ , \quad \gamma_F = 4 \ , \quad L = 2 \ . \tag{3.107}$$

Under these conditions, using Eq. (3.95), $\chi = 5.5$, the first term of Eq. (3.97) becomes

$$\Delta E_{rr'} = \frac{3.82 \times 10^{-4} \times 3}{8\pi \times 3 \times \sqrt{15}} \quad \log 5.5 \ \text{m rads.} = 6.75 \text{ mm mrads.} \tag{3.108}$$

We thus find the total emittance to be the sum of the contributions from Eq. (3.106) and (3.108), namely 7.2 mm mrads. In an actual Van der Graaf generator with these characteristics the emittances measured for 90% of the beam were 6.2 and 8.3 mm mrads (J. Gale, Private Communication). In this case there were inhomogeneties in the fields which effected the beam differently in the two directions, so that it was not axially symmetric. The agreement between the theoretical values and the average of the two experimental values is much closer than the probable error in the theory. However the theory is certainly confirmed in an encouraging manner by this example.

Acknowledgement

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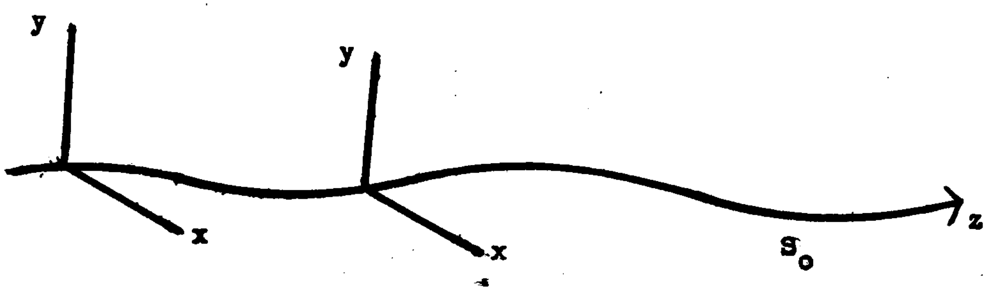
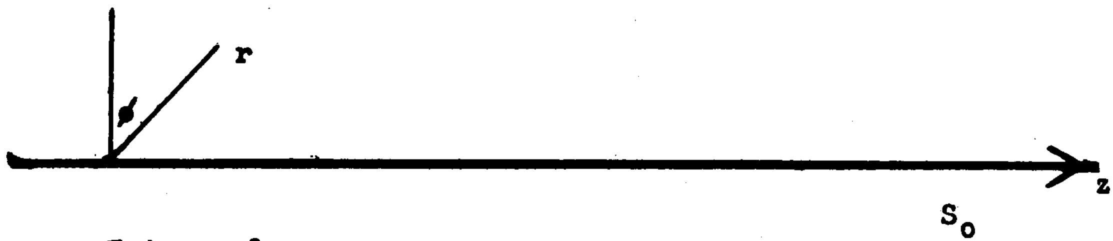


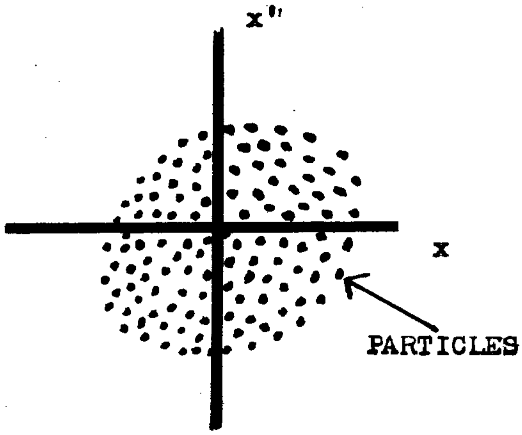
F i g. 1.

Notation in (x , y , z) coordinates.



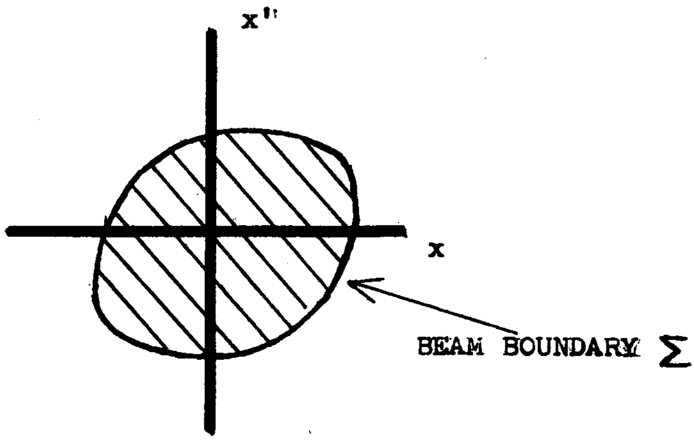
F i g. 2.

Notation in (r , ϕ , z) coordinates.



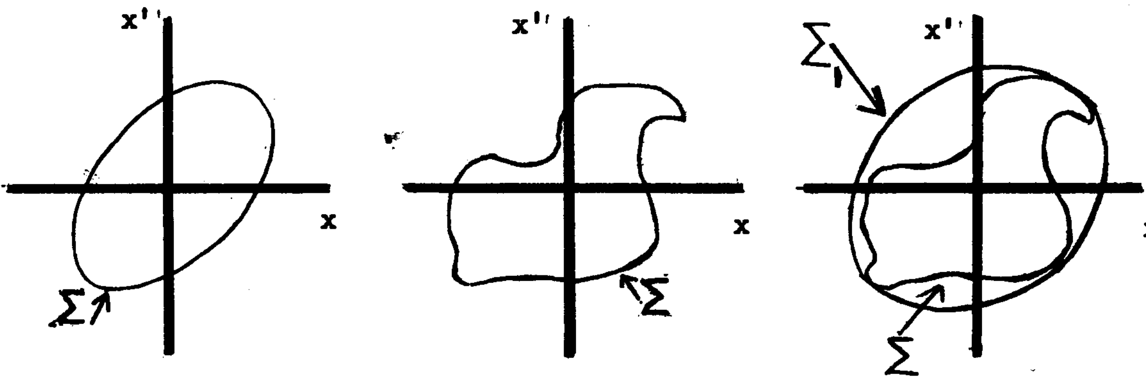
F i g. 3.

Representation of particles in phase space.



F i g. 4.

Representation of a continuous beam in phase space.



a)Before element b)After element c)Equivalent elliptical area

F i g. 5.

Sketch of striations in phase space

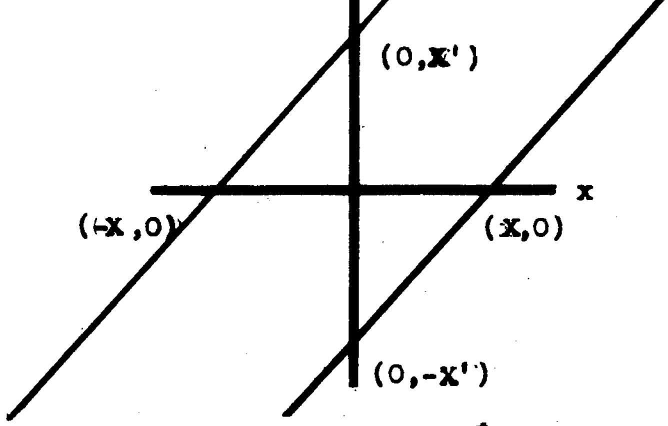


Fig. 6.

The representation of parallel lines in phase space.

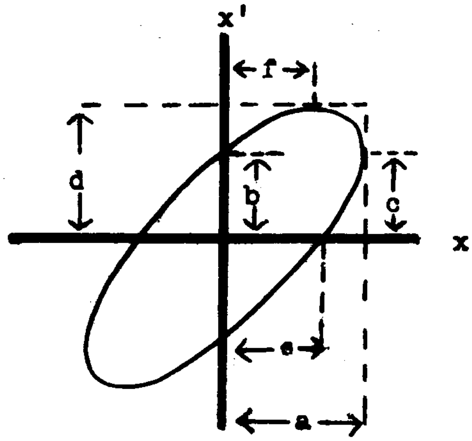
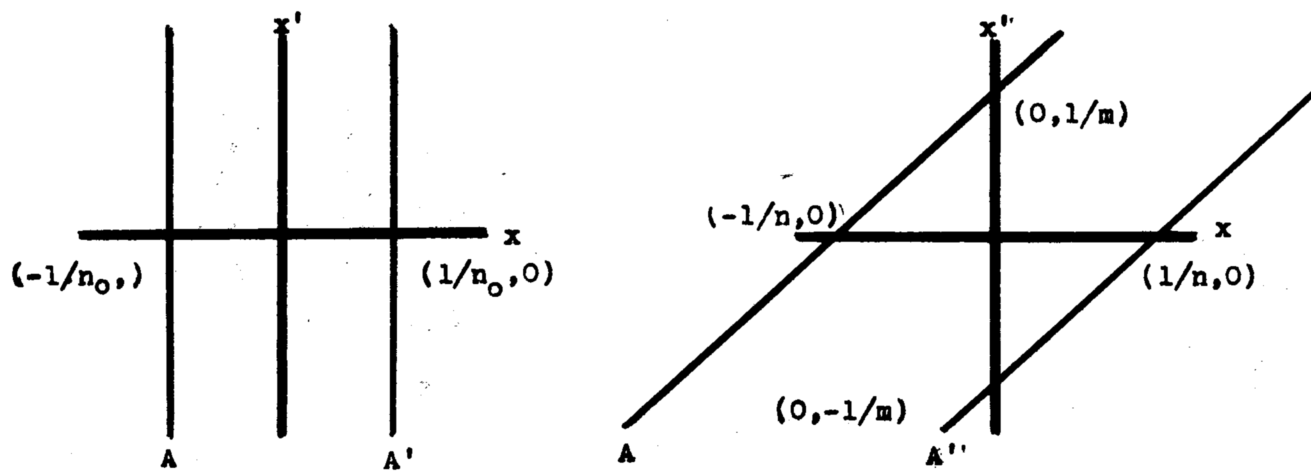


Fig. 7.

The representation of ellipses in phase space.

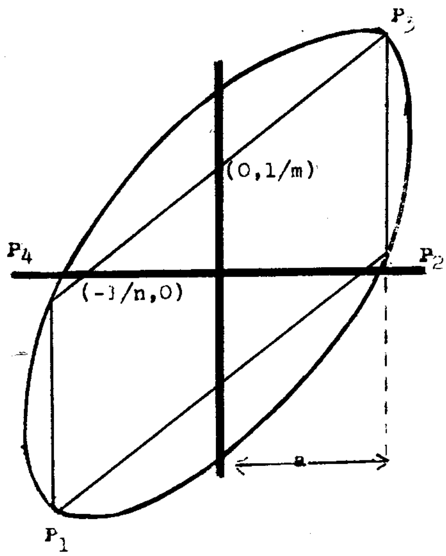


a) at the plane of the aperture

b) at another plane

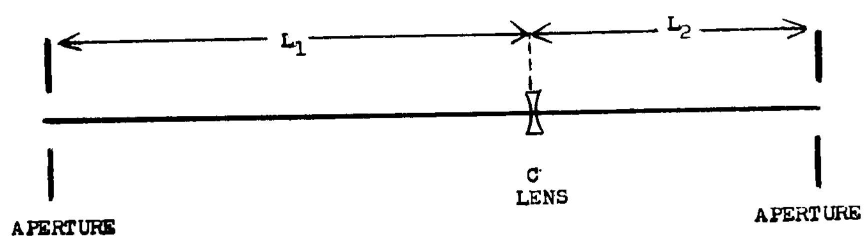
Fig. 8.

The representation of a pair of parallel apertures in phase space;



F ig. 9.

The representation of two pairs of apertures and the circumscribing ellipse, in phase space, at the plane of one aperture.



F i g. 10.

Schematic of a possible beam transport system to locate the apertures required to define a given ellipse.

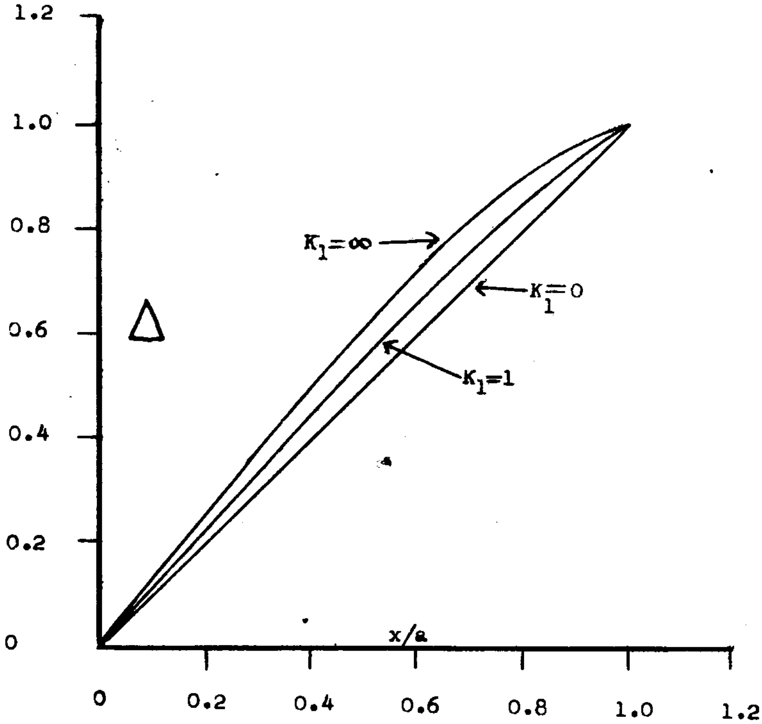


Fig. 11.

Variation of the mean x', Δ across the beam cross-section for a sheet beam.

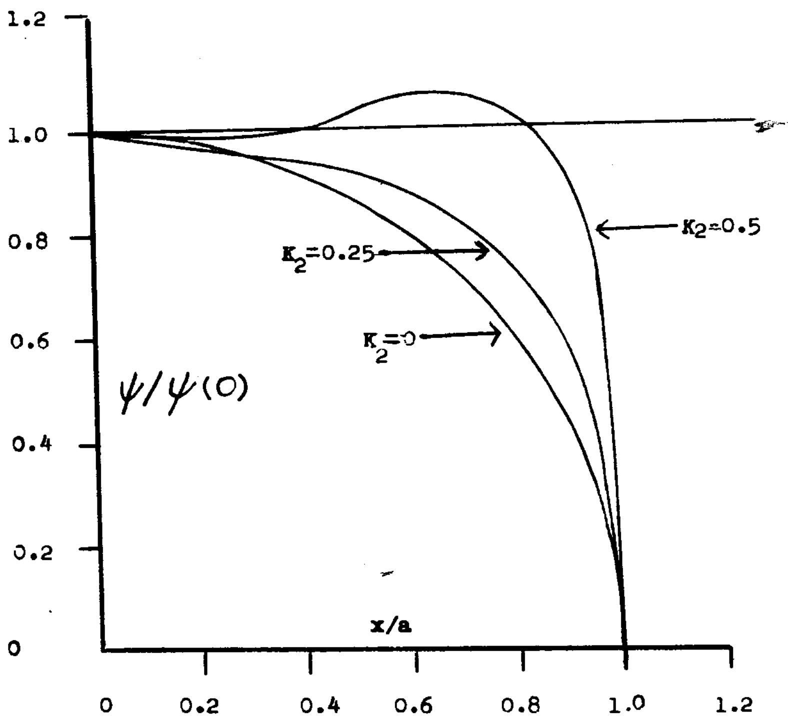
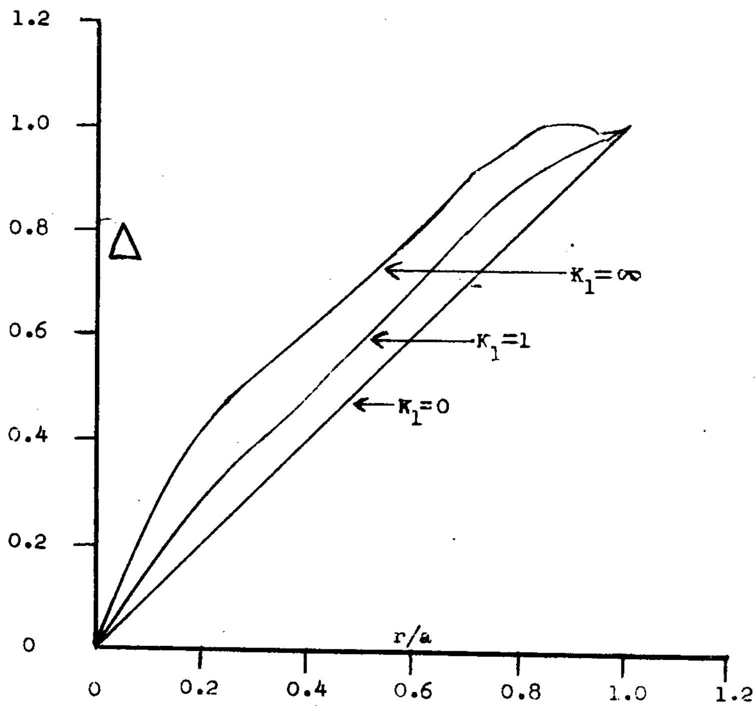


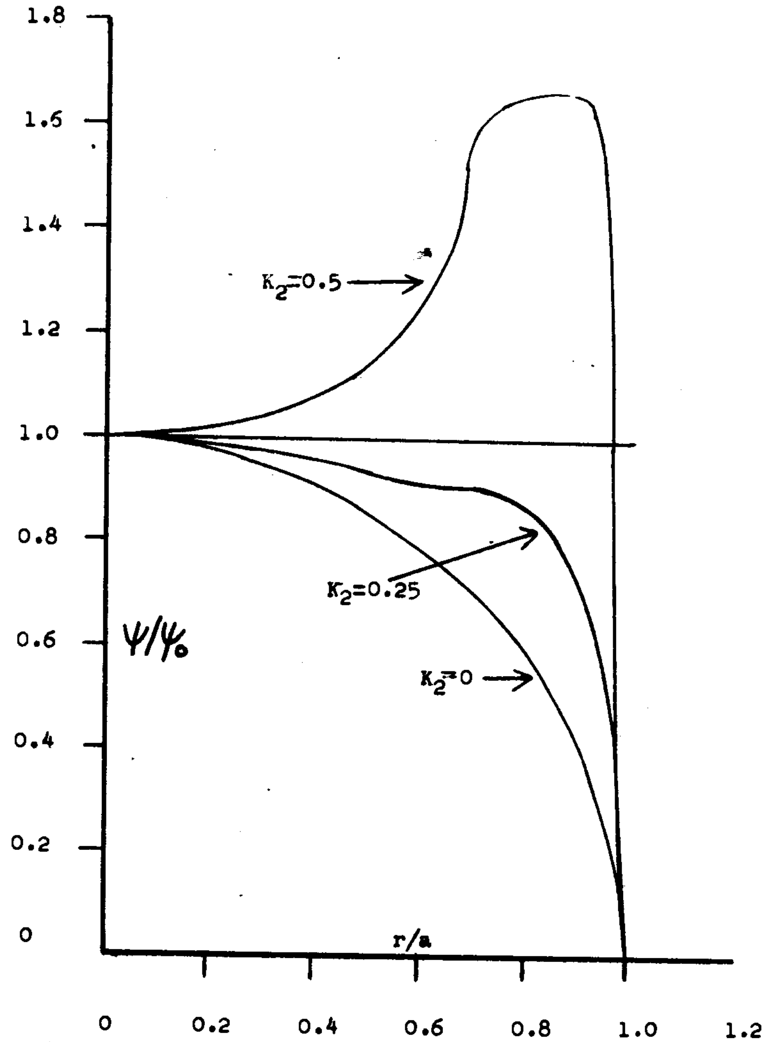
Fig. 12.

Variation of current density across the beam cross-section for a sheet beam. Here ψ is the current density, ψ_0 its value at the centre of the beam.



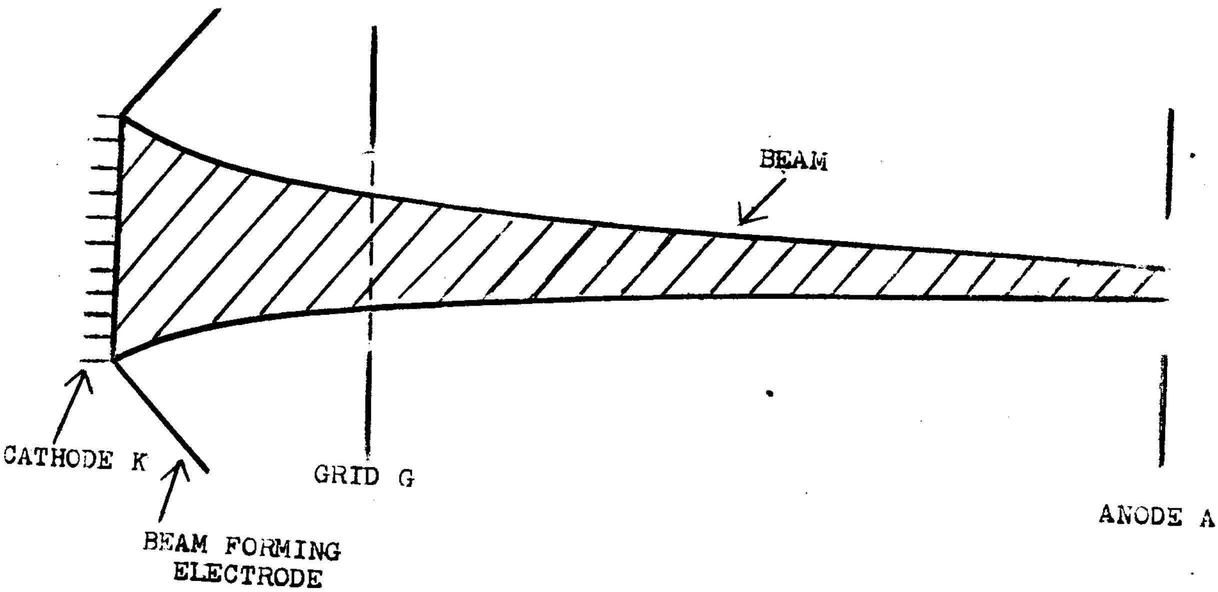
F i g. 13.

Variation of the mean r' , Δ across the beam cross-section for a round beam.



F i g. 14.

Variation of current density across the beam cross-section for a round beam. ψ is the current density, ψ_0 its value at the middle of the beam.



F i g. 15.

Schematic of a Van de Graaff generator.