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ON THE PROBLEM OF THE CALCULATION OF TWO-DIMENSIONAL ELECTROSTATIC FIELDS

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The methods for solving the problems of electrical field shaping may be divided into two large groups. The first group is based on the modelling (mathematical modelling by using the computer, electrolytic modelling using an electrolytic tank, etc.) of fields produced by the electrodes of the known shape. This method, as a rule, is more or less time-consuming. The second approach to the problem is the analytical solution. The basic drawback of this method is its limited applicability. As is known, the form of boundary conditions does not allow one quite often to obtain an exact analytic solution, which makes one use the methods of the first group.

Below is given a version for solving two-dimensional problems. Two approximated methods giving sufficient accuracy for a number of practical problems are described.

The conventional two-dimensional problems require either the certain dependence of the potential or its derivative along one or some curves, or the distribution of all the charges generating the field in the given region should be given.

In the first case one must calculate either integrals defining the series coefficients in the solution by the method of variable separation or the integral in the Green formula. This causes difficulties in solving the problems practically.

Nevertheless, very often it proves possible to determine sufficiently accurately the pattern of the field in the given region, conserving only the final number of the series terms and solving in order to determine coefficients, the same number of equations obtained by selecting the proper number of points of the boundary curve with the known values of the potential or its derivatives /1/. Then the general solution of the Laplace equation can be written in the approximated form as the sum of N particular solutions $u_k(x,y)$ and the expression for the potential $U(x,y)$ can be written as follows:

$$U(x,y) = \sum_{k=1}^{N} a_k u_k(x,y) . \qquad (1)$$

The number of the functions $u_k(x,y)$ must be equal to the number of points at which it is necessary to satisfy the boundary conditions. Each condition is a linear equation, therefore, the problem is confined to the solution of the system of N linear equations having the unknown values of a_k. As a result one finds expression (1) having the determined coefficients a_k. It is a solution of the Laplace equation which satisfies the conditions at N points of the boundary.

When applying this method practically one must pay attention to two facts. This method gives a rich choice of the number and the positions of boundary points, which, of course, affects the form of the solution. As experience shows, with increasing the number of conditions N the system of equations becomes weakly conditioned. In this case the values of the coefficients greatly increased and the values of $U(x,y)$ within the mastering points can be inadmissably large. Apparently, $N \lessapprox 12$ is reasonable.

It is well-known that both real and imaginary parts of each regular function $F(z)$ of the complex variable $z = x + iy$ are harmonic and, hence, satisfy the Laplace equation. Thus, any of them may be used as $u_k(x,y)$. Here we confine ourselves to considering only one of important cases which proves sufficient for a majority of actual problems.

The regular function $F(z)$ in the region without special points can always be presented as a power series. For the symmetric field $U(x,y) = U(x,-y)$ it leads to the following expression for the potential

$$U_A(x,y) = \sum_{k=0}^{N-1} a_k (x^2+y^2)^{k/2} \cos k\theta , \qquad (2)$$

where $$\theta = \arg(x+iy) . \qquad (3)$$

The disadvantage of expression (2) is as follows: the field $U_A(x,y)$ satisfying the relation $\mathcal{E}A = 0$ has no closed equipotentials. Therefore, it is possible to use any of the equipotentials as a surface electrode only in some restricted region where the edge effects of the actual system can be neglected.

Expression (2) at the symmetry axis has the following form:

$$U_A(x,0) = \sum_{k=0}^{N-1} a_k x^k . \qquad (4)$$

Having as a boundary condition the field on the symmetry axis it can be expressed as a power series (4) and the obtained coefficients substituted in (2). All the calculations are easily made by means of a small computer.

Example 1. In the centre of coordinates ($x=0, y=0$) it is necessary to produce the electric field of intensity $\mathcal{E}_x = 80$ kV/cm having a constant slope $\partial\mathcal{E}_x/\partial x$ -80 kV/cm in a region $-0.5 < x < 0.5$. Let $U_A(-0.5;\ 0) = 0$. Hence, three conditions are applied and therefore N must equal 3. By solving the system of these equations, one obtains $a_0 = -50$ kV, $a_1 = -80$ kV/cm, $a_2 = 40$ kV/cm^2. Figure 1 shows the corresponding equipotentials.

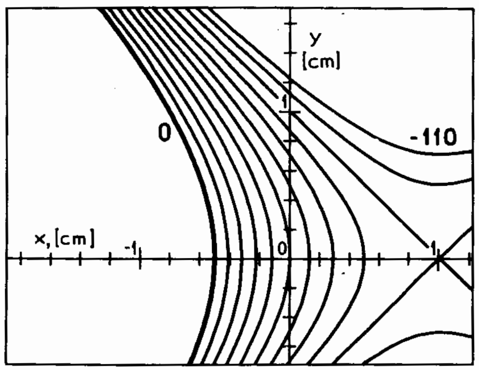


Fig. 1. The shape of equipotentials for a power series solution (solution of example 1).
Numbers are potentials (kV) at proper lines.

The second approach used in this investigation to solve practical electrostatic problems is based on the image method /2/. With certain conditions one can select such a system of charges having a proper value and properly located outside the given region that the effect of these charges provides correctly or with sufficeent approximation the required boundary conditions.

Thus, the actual problem having boundary conditions is replaced by an identical problem for the field determination in the expanded region without boundary conditions, but with the account of charges-images. Charges-images should be outside the given region since the potential of the field produced by them must satisfy the Laplace equation in this region. If, similar to the first method, one puts the boundary conditions only in the final number of points of the given curve, one obtains the final number of charges-images equal to that of boundary points.

Further, an identical problem is exactly solved and the comparison of the fields along the boundary curve both obtained from this solution and the given one determines how accurately the identical problem corresponds to the real one.

The plane system of N point charges has a potential which can be written as the following sum:

$$U_B(x,y) = \sum_{k=1}^{N} q_k \times \ell n [(x-x_k)^2 + (y-y_k)^2]. \quad (5)$$

The charges are located at the points (x_k, y_k). The main advantage of expression (5) is that the equipotentials of such a system are closed. With the addtional condition

$$\sum_{k=1}^{N} q_k = 0 \quad (6)$$

the potential at the infinity vanishes to zero.

The last expressions have one more free choice in addition to the above-said ones, namely, before composing the equations corresponding to the boundary conditions it is necassary to distribute N charges, i.e., it is

necessary to set their coordinates x_k, y_k. We performed it in an experimental way. One of the criteria for comparing the various results of the same problem can serve the sum $\Sigma |q_k|$, which should be near the minimum. Despite the described arbitrary approach, the results of applying formula (5) are rather promising which is proved by below calculations of the focusing section of the electrostatic deflector of the U-120M isochronous cyclotron /3/ and the electrostatic peeler of the extraction system of the ,,F'' machine /4/.

Example 2. It is necessary to find the shape of electrodes producing the electric field described in example 1. In addition to the mentioned conditions the following is required: in order to have the proper linearity of the electric field at the symmetry line we take $\partial^3 U/\partial x^3 = 0$, $\partial^4 U/\partial x^4 = 0$. In order to provide the grounded screen at the height $y = 3.8$ cm let $U(-.05; 3.8) = 0$, $U(0.5; 3.8) = 0$. Require also that condition (6) is fulfilled. On the whole 8 conditions have been applied. In order to satisfy these boundary conditions we distribute 8 charges in the region occupied by the supposed electrodes: (-0.8; 0.2), (-0.9; 0.45) (0.6;1), (0.8; 0.8), (1; 0.6), (0; 5.5), (0; 7.5), (1.5; 7) and their conjugated pair charges in the hemisphere $y < 0$ with equal q_k, respectively. The results are shown in Fig. 2, where the shape of the electrodes producing the given electric field at the symmetry line is seen.

Example 3. It is required to produce an electric field of intensity $\mathcal{E}_x(x,0) = 0$ with $x \leq 0$, the linear increase of intensity being $\partial \mathcal{E}_x/\partial x|_{y=0} = 6.6$ kV/cm^2 at a cut of $0 < x < 6$ cm. It is required to provide a grounded screen at the height $y = 5$ cm and the hole in the left electrode at least 2 cm for the proton beam to enter. After several solutions the best result has been taken: $N = 12$; the charges are located at the points (-0.095; 2.063), (7.5; 1.623), (-15; 9.07), (-12.91888; 9.142), (-10.85777; 9.08), (-8.66666; 9.07), (-6.55555; 9.07), (-4.44444; 9.07), (-2.33333; 9.07), (-0.2222; 9.04), (2.09888; 9.085), (4; 9.22) and the corresponding symmet-

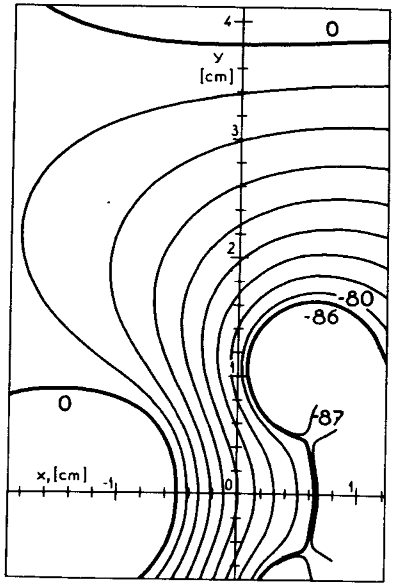


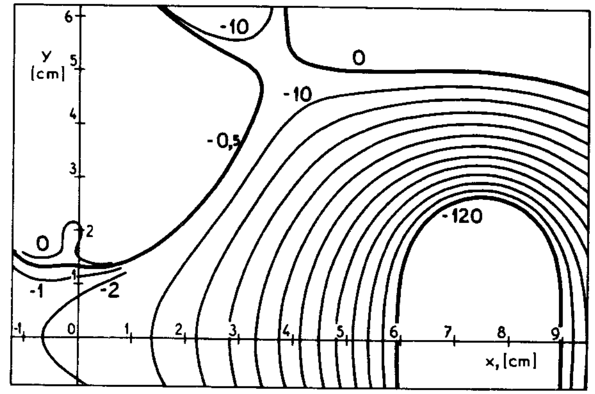
Fig. 2. Focusing section of the electrostatic deflector (solution of example 2). Numbers are potentials (kV) at proper lines.

rical points of the semi-plane $y < 0$; the following conditions have been assigned: $\partial U/\partial x = 0$, $\partial^2 U/\partial x^2 = 0$, $\partial^3 U/\partial x^3 = 0$ at the point (-3.2; 0); $\partial U/\partial x = -19.9$, $\partial^2 U/\partial x^2 = -3.3$ at the point (3; 0); $\partial^3 U/\partial x^3 = 0$ at the point (2.81; 0); $U = 0$ at the points (-0.04; 1.58), (-0.501; 1.5), (5; 5), (5.96; 5), (7.65; 5) and condition (6) is set. The result of calculation is shown in Figs. 3 and 4. The effect of the fields in the vicinity of the point (3.7; 5) and $x < -1$ is neglected. The equipotential $U = -0.5$ is taken as a grounded electrode.

All the above calculations have been performed by means of the NAIRI-2 computer modified for the operation with a plotter /5/.

The described method allows one to find the accurate solution of the 2-dimensional Laplace equation with boundary conditions assigned in the finite number of points. If one knows the general properties of particular solutions one can select a proper form of mathematical

Fig. 3. Electrostatic peeler of the extraction system of the "F" machine (solution of example 3). Numbers are potentials (kV) at proper lines.



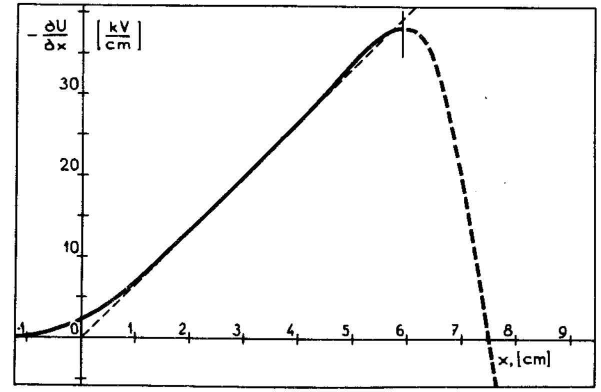


Fig. 4. Electric field intensity at the symmetry axis produced by the electrodes shown in Fig. 3.

solution. The first method is especially convenient for reconstructing the electrical field by means of the field at the symmetry axis. Practically all the processing of data may be performed with a small computer. The second method is more convenient for practical determination of the shape of electrodes though its drawback is a great freedom for selecting some parameters. All the equipotentials make closed figures of finite sizes and each of them can be replaced by the electrode surface which is the main advantage of that method. The important fact is that the obtained result takes into consideration the edge effects on the field of the working region. The use of both the plotter and graph display makes it possible to correct boundary conditions when solving the problem.

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