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EXACT RESULTS FOR THE MANY-BODY MODEL SYSTEMS INTERACTING WITH THE BOSON FIELD

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1. Introduction

In statistical mechanics most problems of physical interest are rather complicated to be solved directly. Model systems permitting a mathematical treatment of these problems are therefore acquiring considerable interest. Suffice it to say, that it has been the felicitously chosen models that enabled one to make a fundamental contribution to our understanding of such extremely phenomena as superfluidity, superconductivity, ferromagnetism and some others (see, e.g., refs. [1-7]).

A considerable interest represent mathematically rigorous methods for studying model Hamiltonians, which do not make use of any version of the perturbation theory or other similar approaches. The fact is, that usually only such methods enable one to get complete assurance in the adequate correspondence of the obtained solution with the model by itself. On the other hand, rigorous results can be used as a reliable basis for further (and may be less rigorous) investigations.

Just one of such approaches (the so-called "Approximating (trial) Hamiltonians" method) has been developed by one of the authors (N.N.B.,Jr.) in a number of works (see, e.g., refs. [6,7]), where a special mathematical technique has been worked out in order to obtain solution exact in the thermodynamical limit *) for certain types of model Hamiltonians. The approach is based on the replacement of the model Hamiltonian H, which is insoluble for finite system, with a special trial Hamiltonian $H_0(C)$ permitting an exact solution. A range of assertions on the asymptotical (in the thermodynamical limit) closeness of the free energies and statistical averages corresponding to the model Hamiltonian H on the one hand, and the trial Hamiltonian $H_0(C)$ taken by special choice of the trial parameters C, on the other, has been proved for different types of model problems. The trial Hamiltonian method enjoys many applications on modern many-body theory. Among concrete model problems which have been investigated on the basis of this technique one can find, in particular, the BCS-type model systems in the theory of superconductivity [6,7], some ferro- and anty-ferromagnetic models with the long-range interaction of J/N type [8,9], the Dicke maser model [10], some model problems for ferroelectrics of the KDP-type [11], for metal-insulator phase transition [12] and for superconductor with

*) I.E. in the limit $N \to \infty$, $V \to \infty$, $N/V = \text{const}$, where N is the number of particles, V is the volume of a system.

electron-hole coupling [13]*). What is more, the trial Hamiltonian method enabled one to investigate some classes of model Hamiltonians as a whole, from a unique standpoint (see [7,14] and the next sections of the present work).

It should be noted that a variety of techniques has been proposed recently in order to construct exact solutions in the thermodynamical limit for different model problems in the many-body theory. For example, a method for the asymptotically exact calculation of the free energy for a set of model Hamiltonians that contain a one-particle, part and separable two-particle operators has been developed by Tindemans and Capel in refs. [15,16]. A very interesting rigorous investigation of the thermodynamics of the Dicke-Haken-Lax maser model has been performed by Hepp and Lieb by means of modern algebra and analysis in ref. [17] (see also Sec. 2, (4)). A somewhat different type of approach is based on the C^*-algebra technique and related methods, when both model and trial Hamiltonians are treated for infinite system ab_ovo_ (see, e.g., [18-20]). Some more references can be found, e.g., in ref. [15].

It should be noted, however, that all these techniques make essential use of the structure of the Hilbert space on which the corresponding model Hamiltonian is definite, so being well adapted only to one or another concrete model or, at best,

*) For models considered in refs. [10-13] see also Sec. 2 of the present paper.

to definite groups of models with the identical structure of the corresponding Hilbert spaces. At the same time the trial Hamiltonian method does not make any use of the concrete Hilbert space structure, that enabled one to investigate extremely wide classes of model Hamiltonians from unique standpoint [7,14]

Just one of such classes - a general class of many-body model Hamiltonians with the interaction of substance and boson field - has been investigated in previous paper [14], where the existence of exact solution in the thermodynamical limit has been established for the whole set of models simultaneously. In the present paper we develop a somewhat different approach to the problem, which possesses good potentialities and is more convenient, in a sense, then previous one. We give the detailed formulation of the problem in the next section.

2. Preliminaries.

First of all, let us describe the models we are going to deal with below. We shall consider a set of model Hamiltonians of the form [14]:

$$H = \sum_{\alpha=1}^{S} \omega_\alpha \overset{+}{a}_\alpha a_\alpha + \sqrt{N} \sum_{\alpha=1}^{S} (\overset{*}{\lambda}_\alpha a_\alpha \overset{+}{L}_\alpha + \lambda_\alpha \overset{+}{a}_\alpha L_\alpha) + T - N \sum_{\alpha=1}^{S} \varkappa_\alpha \overset{+}{L}_\alpha L_\alpha , \tag{1}$$

where:

a) $\overset{+}{a}_\alpha$ and a_α are creation and annihilation operators for the α-th mode of quantized boson field; $\overset{+}{a}_\alpha$, a_β satisfy the well-known Bose commutation relations:

$$a_\alpha \overset{+}{a}_\beta - \overset{+}{a}_\beta a_\alpha = \begin{cases} 1, \alpha=\beta \\ 0, \alpha \neq \beta \end{cases}, \quad \begin{matrix} \overset{+}{a}_\alpha \overset{+}{a}_\beta - \overset{+}{a}_\beta \overset{+}{a}_\alpha = 0 , \\ a_\alpha a_\beta - a_\beta a_\alpha = 0 . \end{matrix} \tag{2}$$

b) $T = \overset{+}{T}$, L_α, $\overset{+}{L}_\alpha$ $(1 \leq \alpha \leq S)$ are operators representing "substance" or " L -subsystem" *). These operators are of different concrete substructure for different concrete models. The only extremely general additional conditions are following:

$$\| L_\alpha \| \leq M_1 , \tag{3a}$$

$$\| L_\alpha T - T L_\alpha \| \leq M_2 , \tag{3b}$$

$$\| L_\alpha L_\beta - L_\beta L_\alpha \| \leq \frac{M_3}{N} , \quad \| L_\alpha \overset{+}{L}_\beta - \overset{+}{L}_\beta L_\alpha \| \leq \frac{M_3}{N} , \tag{3c}$$

where $\| \dots \|$ denote the norm of the operators and M_1, M_2, M_3 are constants independent of N (the number of particles in the "substance") as well as of α, β $(1 \leq \alpha, \beta \leq S)$. Besides the free energy (per particle) $f[T]$ **) should exist by N finite as well as in the limit $N \to \infty$.

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*) This name comes from the notation of the operators L_α , $\overset{+}{L}_\alpha$.

**) For the definition $f[\dots]$ see below (16a).

c) ω_α $(1\le\alpha\le s)$ are real positive parameters, $\omega_\alpha > 0$;
$\varkappa_\alpha$ $(1\le\alpha\le s)$ are real nonnegative parameters, $\varkappa_\alpha \geqslant 0$;
$\lambda_\alpha, \lambda_\alpha^*$ $(1\le\alpha\le s)$ are, generally speaking, complex parameters.

d) N is the number of particles in the "substance". We shall be interested only in those properties of the model systems under consideration which are invariant with respect to the limit $N\to\infty$ (thermodynamical limit). But the whole treatment will be performed by finite fixed N, and the limit $N\to\infty$ will be carried out at the end of the calculations.

e) Let us describe exactly the structure of the Hilbert space on which the Hamiltonian (1) is defined. By given fixed α the Bose operators $\overset{+}{a}_\alpha, a_\beta$ are defined on the Fock space, which we denote as $\mathcal{H}_B^{(\alpha)}$; we also denote $\mathcal{H}_B = \bigoplus_{\alpha=1}^{s} \mathcal{H}_B^{(\alpha)}$. The operators T, L_α, $\overset{+}{L}_\alpha$ $(1\le\alpha\le s)$ assumed to be defined in the separable (by finite N) Hilbert space $\mathcal{H}_L$ independent of the space $\mathcal{H}_B$, i.e., $\mathcal{H}_L$ and $\mathcal{H}_B$ should not contain any common vectors $(\mathcal{H}_L \cap \mathcal{H}_B = \emptyset)$. This condition conforms to the physical assumption that the "substance" and the "boson field" are of different physical nature. In particular, owing to the independence of the spaces $\mathcal{H}_L$ and $\mathcal{H}_B$ the operators T, L_α, $\overset{+}{L}_\alpha$ commute with the operators $\overset{+}{a}_\beta$, a_β by every α and β $(1\le\alpha,\beta\le\le s)$. So the Hamiltonian H (1) is defined in the space $\mathcal{H} = \mathcal{H}_L \otimes \mathcal{H}_B$ and describes L-subsystem coupled to finitely many boson field modes *).

*) Under certain additional conditions the situation of the infinite (but numerical) number of modes "$s=\infty$" is also allowable; for details see ref. [14].

We finish the description of the Hamiltonian (1) by the notation, that the first term in (1) represents the free boson field, the second one describes the interaction, and the last two terms represent the "substance".

Let us now consider some concrete examples of the model problems covered by our class (1) as special cases.

i) <u>the Dicke-Haken-Lax maser model</u> (see, e.g., Ref. [10,17] and references given therein). The model Hamiltonian is

$$H_D = \omega \overset{+}{a} a + \lambda\sqrt{N}\,(J^+ a + J^- a^+) + \epsilon N S^z, \quad (4)$$

where:

$$J^\pm = S^\pm + \mu S^\mp; \quad S^\pm = S^x \pm i S^y, \quad S^{x,y,z} = \frac{1}{2N}\sum_{i=1}^{N} \sigma_i^{x,y,z};$$

$\sigma^x, \sigma^y, \sigma^z$ are 2×2 Pauly matrices; $\overset{+}{a}$ and a are photon creation and annihilation operators; ω, ϵ, λ and μ are real parameters, $\omega>0$, $\epsilon>0$, $0\le\mu\le1$. The model (4) represents N two-level molecules coupled to one mode of a quantized electromagnetic field via a truncated dipolar interaction, it finds a set of applications in the laser and maser theory.

ii) <u>A model for ferroelectrics of the KDP-type</u> [11,14]: The corresponding model system consists of the subsystem of heavy ionic complexes connected via short hydrogen bonds and the subsystem of protons on these bonds. In order to describe the protonic motion in the effective potential, which resembles a double-minimum well, the quasi-spin formalism is used. The ferroelectric phenomena in such substances to a large measure are due to the interaction of the ionic and protonic subsystems.

In Ref. [11] a simplified model for the ferroelectrics of the KDP-type, which takes into account the interaction of protons only with the long-wave optical vibrations of the ionic subsystem, is under consideration. The essential part of the corresponding model Hamiltonian can be represented ultimately in the form [14]:

$$H_F = \omega_0 \overset{+}{a}_0 a_0 - \frac{K}{\sqrt{2M\omega_0}} \frac{\overset{+}{a}_0 + a_0}{\sqrt{N}} \sum_{i=1}^{N} \sigma_i^z - \Omega \sum_{i=1}^{N} \sigma_i^x - \frac{I}{2N} \sum_{i=1}^{N} \sum_{j=1}^{N} \sigma_i^z \sigma_j^z , \qquad (5)$$

where $\overset{+}{a}_0$ and a_0 are the long-wave $(K=0)$ phonon creation and annihilation operators; σ^x, σ^z are 2×2 Pauly matrices; M is the reduced mass of the ionic complex, ω_0 is the optical frequency of lattice vibrations in the long-wave limit $K \to 0$; Ω is the de Gennes tunneling frequency, $\Omega > 0$; K and I are real interaction parameters, $I > 0$.

The model (5) permits the exact solution in the thermodynamical limit and exibits, by a certain critical temperature $\theta = \theta_c$, the second order phase transition from disordered into ordered state. The spontaneous polarization here is due to the protonic ordering accompanied by the macroscopic displacements of the heavy ions (for more details see refs. [11,14] and references therein).

iii) <u>A model for metal-insulator phase transition</u> (e.g., [21], see also [13] and references given there). The model is based on the Fröhlich Hamiltonian for a coupled electron-phonon system allowing for the interaction of electrons with the only lattice vibration mode Q singled out in a special way *):

$$H_{MI} = \sum_{K,\sigma} (\epsilon_K - \mu) \overset{+}{C}_{K,\sigma} C_{K,\sigma} + \omega_Q \overset{+}{b}_Q b_Q + \frac{g}{\sqrt{N}} \sum_{K,\sigma} \left(\overset{+}{C}_{K+Q,\sigma} C_{K,\sigma} b_Q + \overset{+}{C}_{K,\sigma} C_{K+Q,\sigma} \overset{+}{b}_Q \right), \qquad (6)$$

where $\overset{+}{C}$, C and $\overset{+}{b}$, b are creation and annihilation operators for electrons and phonons, respectively.

The model (6) is also covered by the general class of model Hamiltonians (1), and the corresponding approximating Hamiltonian (see below (7)) is exactly soluble, that enabled one to investigate closely thermodynamics of the model. It has been shown (see e.g. [21]), in particular, that under certain critical temperature θ_c the model (6) exhibits the second-order metal-insulator phase transition (due to lattice instability and the unit-cell doubling).

It should be also noted, that some model systems allowing for metal-insulator and metal-superconductor phase transitions simultaneously has been studied recently by a number of authors (see e.g., [15,21] and refrences given there) on the bases of the Hamiltonian composed of H_{MI} (6) and the BCS-type model

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*) $Q = (\pi/a)(\pm 1, \pm 1, \pm 1)$, where a is lattice constant; in (6) the electron energy spectrum has the property: $\epsilon_{K+Q} = -\epsilon_K$ (for s.c. and b.c.c. lattices in the tight-binding limit). For details see refrences mentioned above.

Hamiltonian (the last one is of the same structure as "$\mathcal{H}$-term" in (1)):

$$H_{MI-MS} = H_{MI} + H_{BCS} . \qquad (7)$$

Such models enabled one to investigate the influence of electron-hole coupling on the Curie point temperature of superconductor, that is of considerable interest in view of some problems of high-temperature superconductivity.

The model Hamiltonians (4)-(7), being special cases of the general Hamiltonian (1), permit of exact solution in the thermodynamical limit, as $N \to \infty$ (see below). The detailed discussion of their physical properties can be find in the literature given above. Some of these models, as well as the corresponding techniques proposed in the references indicated, are also under discussion in ref. [14].

Let us now return to our general Hamiltonian (1). The trial Hamiltonian $H_0(C)$, depending on complex parameters $C_\alpha \ (1 \leq \alpha \leq S)$, should be taken here as follows [14]:

$$H_0(C) = T + N \sum_{\alpha=1}^{S} [(\nu_\alpha + g_\alpha C_\alpha) \overset{+}{L}_\alpha + (\overset{*}{\nu}_\alpha + g \overset{*}{C}_\alpha) L_\alpha] + N \sum_{\alpha=1}^{S} g_\alpha C_\alpha \overset{*}{C}_\alpha , \qquad (8a)$$

where

$$g_\alpha = \mathcal{H}_\alpha + |\lambda_\alpha|^2 / \omega_\alpha \geq 0 . \qquad (8b)$$

In ref. 14 we have obtained the majorating bounds which prove the asymptotical closeness (as $N \to \infty$) of the free energies corresponding to the Hamiltonians H (1) and $H_0(C)$ (8) (the last one taken by special choice of the trial parameters C) for the whole class of models simultaneously:

$$\left| f[H] - \operatorname{abs} \min_C f[H_0(C)] \right| \xrightarrow[N \to \infty]{} 0 . \qquad (9)$$

As one easily see, the trial Hamiltonian $H_0(C)$ (8) is of considerably simpler structure than the basic Hamiltonian (1). Therefore for a set of concrete model problems (in particular for those of (4)-(7)) the free energy $f[H_0(C)]$ permits of exact calculation by N finite, and the

$$\lim_{N \to \infty} \{ \operatorname{abs} \min_C f[H_0(C)] \}$$

exists. Then, if (9) is valid, the free energy $f[H]$ exists, the problem of its calculation being solved with asymptotical accuracy *).

Another important result obtained in ref. [14] is a set of asymptotically exact relations for averages constructed from the Bose operators $\overset{+}{a}_\alpha$, a_β, on the one hand, and from the operators $\overset{+}{L}_\alpha$, L_β, on the other $(1 \leq \alpha, \beta \leq S)$. The key inequality hereby is the following one:

*) There are possibly also other reasons why $H_0(C)$ would be more convenient to study than H.

$$\sum_{\alpha=1}^{S} \omega_\alpha \langle \overset{+}{B}_\alpha B_\alpha \rangle_H \leq \varepsilon_N \xrightarrow[N\to\infty]{} 0, \quad (10a)$$

where

$$\overset{+}{B}_\alpha = \frac{\overset{+}{a}_\alpha}{\sqrt{N}} + \frac{\overset{*}{\lambda}_\alpha}{\omega_\alpha} \overset{+}{L}_\alpha, \quad B_\alpha = \frac{a_\alpha}{\sqrt{N}} + \frac{\lambda_\alpha}{\omega_\alpha} L_\alpha . \quad (10b)$$

The bound (10b) has been derived in ref. [14] on the basis of some intermediate relations obtained when proving (9).

In the present paper we develop a somewhat different approach to the problem. In Sec. 3 we start from the direct proof of the inequality (10a) without dealing with any relations for the free energies, but making use of special ineqiality for averages ferst obtained by one of the authors in ref. [6] (see below (17c)). In Sec. 4, making use of the auxiliary Hamiltonian $\tilde{H}$ of the form *):

$$\tilde{H} = T - N \sum_{\alpha=1}^{S} g_\alpha \overset{+}{L}_\alpha L_\alpha , \quad (11a)$$

where (see (8b))

$$g_\alpha = \varkappa_\alpha + |\lambda_\alpha|^2/\omega_\alpha \geq 0 \quad (1 \leq \alpha \leq S), \quad (11b)$$

we derive the inequalities:

$$-\xi_N \leq f[H] - f[\tilde{H}] \leq \eta_N , \quad (12)$$

where $\xi_N, \eta_N \to 0$ as $N \to \infty$.

*) The auxiliary Hamiltonian (11a) has been essentially used also in ref [14].

Such succession of operations is more convenient in a sense. In particular, the majoration bound in (10a) appears to be stronger in powers of $1/N$ than the previous one [14] (see Sec. 3). Such an improvement is likely to be usefull when estimating fluctuations in the systems under consideration (say, in the Dicke-Haken-Lax maser model, where the fluctuations are of direct physical interest).

The bounds (12) prove the thermodynamical equivalence (on the level of free energies) of the Hamiltonians H (1) and $\tilde{H}$ (11). Now it is just the right moment to note, that the thermodynamical equivalence of the Hamiltonians $\tilde{H}$ (11) and $H_0(\bar{C})$ (8) (where $f[H_0(\bar{C})] = \text{abs} \min_{C} f[H_0(C)]$) is a well-known fact due to original investigations [6,7]. Namely, the following statement is valid:

<u>THEOREM 1</u> (N.N.BOGOLUBOV, JR., [6,7]). Let the Hamiltonians $H_0(C)$ (8a) and $\tilde{H}$ (11a) be given, wherein $g_\alpha \geq 0$ $(1 \leq \alpha \leq S)$. Let the operators T , L_α, $\overset{+}{L}_\alpha$ satisfy the conditions (3a-c) and other requirements formulated in Sec. 1 (see description of the Hamiltonian (1), b and e). Then the free energy $f[H_0(C)]$ attains an absolute minimum with respect to the trial parameters C_α $(1 \leq \alpha \leq S)$ in certain finite points $\bar{C}_\alpha$, which therefore obey a set of equations

$$C_\alpha = \langle L_\alpha \rangle_{H_0(C)} \quad (1 \leq \alpha \leq S), \quad (13)$$

and the inequalities hold:

$$0 \le f[H_0(\bar{C})] - f[\tilde{H}] \le \frac{S P_1}{N^{2/5}} + \frac{S \theta P_2}{N^{3/5}} , \qquad (14)$$

where P_1 and P_2 are simple combinations of the quantities M_1, M_2, M_3 (see 3a-c) and $G = \max_\alpha g_\alpha$.

In particular, the inequalities (12) and (14) proves (9).

In Sec. 4 we shall also make use of the following general statement:

<u>THEOREM 2</u> (N.N.BOGOLUBOV, for proof see, e.g.,refs. 6 and 7). The inequalities hold:

$$\frac{1}{N}\langle \mathcal{U}_1 - \mathcal{U}_2 \rangle_{\mathcal{U}_1} \le f[\mathcal{U}_1] - f[\mathcal{U}_2] \le \frac{1}{N}\langle \mathcal{U}_1 - \mathcal{U}_2 \rangle_{\mathcal{U}_2}, \qquad (15)$$

where $\mathcal{U}_1 = \overset{+}{\mathcal{U}}_1$ and $\mathcal{U}_2 = \overset{+}{\mathcal{U}}_2$. Here the free energy (per particle) $f[\mathcal{U}]$ is a well-known construction

$$f[\mathcal{U}] = -\frac{\theta}{N} \ln \mathrm{Tr}\, e^{-\frac{\mathcal{U}}{\theta}}, \quad \mathcal{U} = \overset{+}{\mathcal{U}}, \qquad (16)$$

and the averages taken over the "Hamiltonian" $\mathcal{U} = \overset{+}{\mathcal{U}}$ are definite by a familiar expression

$$\langle \dots \rangle_{\mathcal{U}} = \mathrm{Tr}\left(\dots e^{-\frac{\mathcal{U}}{\theta}}\right) / \mathrm{Tr}\, e^{-\frac{\mathcal{U}}{\theta}}, \quad \mathcal{U} = \overset{+}{\mathcal{U}} . \qquad (16b)$$

One should keep in mind, that the Hamiltonians $\mathcal{U}_1$ and $\mathcal{U}_2$ in (15) are both assumed to be definite in the same Hilbert space and by the calculation of any term in (15) the procedure "Trace" should be taken over the whole space.

We finish this section by special inequality (upper bound) for averages of the form $\langle \overset{+}{B} B \rangle$, first obtained in ref. [6] when proving Theorem 1.

LEMMA (N.N.BOGOLUBOV, JR., [6,7]): Let us denote

$$\overset{+}{B}(\tau) = e^{\tau \frac{H}{\theta}} B e^{-\tau \frac{H}{\theta}} , \qquad (17a)$$

$$R = BH - HB , \quad \overset{+}{R} = -\overset{+}{B} H + H \overset{+}{B}, \qquad (17b)$$

then the following inequality holds *):

$$\langle \overset{+}{B} B \rangle_H \le \int_0^1 \langle \overset{+}{B}(\tau) B \rangle_H d\tau + \left(\int_0^1 \langle \overset{+}{B}(\tau) B \rangle_H d\tau \right)^{2/3} \left(\frac{\langle R \overset{+}{R} + \overset{+}{R} R \rangle_H}{2\theta^2} \right)^{1/3}, \qquad (17c)$$

where H is the Hamiltonian of the corresponding system, θ is the temperature; for $\langle \dots \rangle_H$ see (16b).

3. Basic Inequality.

In this section we prove the bound (10a) by way of direct employment of the general inequality (17). Putting in (17c) $B^+ = B^+_\alpha$, $B = B_\alpha$ (see (10b)) and summarizing over α from 1 to S with the weight ω_α , we get **):

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*) One can find also some inequalities which generalize and improve (17c) in refs. [22] .

**) For convenience, in this section we omit the subscript in statistical averages over the Hamiltonian H (1), i.e.,we write $\langle \dots \rangle \equiv \langle \dots \rangle_H$.

$$\sum_{\alpha=1}^{s}\omega_\alpha\langle \overset{+}{B}_\alpha B_\alpha\rangle \le \sum_{\alpha=1}^{s}\omega_\alpha\int_0^1\langle \overset{+}{B}_\alpha(\tau)B_\alpha\rangle d\tau +$$

$$+\sum_{\alpha=1}^{s}\left(\frac{\omega_\alpha}{\theta}\int_0^1\langle\overset{+}{B}_\alpha(\tau)B_\alpha\rangle d\tau\right)^{2/3}\left[\frac{\omega_\alpha}{2}\langle R_\alpha \overset{+}{R}_\alpha + \overset{+}{R}_\alpha R_\alpha\rangle\right]^{1/3}, \tag{18a}$$

where

$$R_\alpha = B_\alpha H - H B_\alpha, \quad \overset{+}{R}_\alpha = H\overset{+}{B}_\alpha - \overset{+}{B}_\alpha H, \tag{18b}$$

H is the Hamiltonian (1). In order to estimate the right-hand side of the inequality (18a) we first calculate the quantity $\int_0^1\langle\overset{+}{B}_\alpha(\tau)B_\alpha\rangle d\tau$. It is not difficult to verify, that the operator $\overset{+}{B}_\alpha(\tau)$ (see (10a) and (17a)) is expressible as follows:

$$\overset{+}{B}_\alpha(\tau) = \frac{1}{\sqrt{N}}\,\frac{\theta}{\omega_\alpha}\frac{d}{d\tau}\left(e^{\tau\frac{H}{\theta}}\overset{+}{a}_\alpha e^{-\tau\frac{H}{\theta}}\right), \quad 0\le\tau\le 1. \tag{19}$$

Making use (19) and taking into account (2), (16b) as well as the possibility of cycle permutations under the sign of "Trace", we finally obtain:

$$\frac{\omega_\alpha}{\theta}\int_0^1\langle\overset{+}{B}_\alpha(\tau)B_\alpha\rangle d\tau = \frac{\langle B_\alpha\overset{+}{a}_\alpha - \overset{+}{a}_\alpha B_\alpha\rangle}{\sqrt{N}} = \frac{1}{N}. \tag{20}$$

Consider now the average $\langle R_\alpha\overset{+}{R}_\alpha + \overset{+}{R}_\alpha R_\alpha\rangle$. One can derive on the basis of (2) and (3a-c) with making use of the Bogolubov inequality [23]:

$$|\langle \mathcal{U}_1\mathcal{U}_2\rangle| \le \sqrt{\langle\mathcal{U}_1\overset{+}{\mathcal{U}}_1\rangle\langle\overset{+}{\mathcal{U}}_2\mathcal{U}_2\rangle}, \tag{21}$$

the following bound (for details see Appendix, (A10)):

$$\sum_{\alpha=1}^{s}\left[\frac{\omega_\alpha}{2}\langle R_\alpha\overset{+}{R}_\alpha + \overset{+}{R}_\alpha R_\alpha\rangle\right]^{1/3} \le$$

$$\le (sd_1)^{2/3}\left(\sum_{\beta=1}^{s}\omega_\beta\langle\overset{+}{B}_\beta B_\beta\rangle\right)^{1/3} + (sd_{2,N})^{2/3}, \tag{22}$$

where d_1 and $d_{2,N}$ are simple combinations of the constants M_1, M_2 and M_3 (see (3a-c)) and of the parameters occuring in (1) (see Appendix, (A9a,b)).

On the basis of (18a), (20) and (22) we obtain:

$$\sum_{\alpha=1}^{s}\omega_\alpha\langle\overset{+}{B}_\alpha B_\alpha\rangle \le \left(\frac{sd_1}{N}\right)^{2/3}\left(\sum_{\beta=1}^{s}\omega_\beta\langle\overset{+}{B}_\beta B_\beta\rangle\right)^{1/3} +$$

$$+\left(\frac{sd_{2,N}}{N}\right)^{2/3} + \frac{s\theta}{N}. \tag{23}$$

whence follows in turn *):

$$\sum_{\alpha=1}^{s}\omega_\alpha\langle\overset{+}{B}_\alpha B_\alpha\rangle \le 2\left(\frac{sd_{2,N}}{N}\right)^{2/3} +$$

$$+\frac{2s(\theta+\sqrt{2}\,d_1)}{N} = \varepsilon_N \xrightarrow[N\to\infty]{} 0. \tag{24}$$

So the bound (10a) is proved. We note, that the majoration estimate in (24) $\varepsilon_N \sim N^{-2/3}$ as $N\to\infty$ appears to be stronger than previous one [14], which is of order $N^{-2/5}$.

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*) When passing from (23) to (24) we have made use of the following simple reasoning: the inequality (23) is of the form $b \le a^{2/3}b^{1/3} + d$ and two situations are possible: (i) $a^{2/3}b^{1/3} \ge d$, then $b^{2/3} \le 2a^{2/3}$ and $b \le 2^{3/2}a$; (ii) $a^{2/3}b^{1/3} \le d$, then $b \le 2d$. So in any case $b \le 2^{3/2}a + 2d$, which is equivalent to (24).

On the basis of (21) and (24) one can easily derive some important asymptotical relations for averages. For example, using (3a), (21) and (24), we get:

$$\left|\left\langle \frac{\overset{+}{a}_\alpha}{\sqrt{N}} L_\beta\right\rangle + \frac{\overset{*}{\lambda}_\alpha}{\omega_\alpha}\langle \overset{+}{L}_\alpha L_\beta\rangle\right| = |\langle \overset{+}{B}_\alpha L_\beta\rangle| \le \sqrt{\langle \overset{+}{B}_\alpha B_\alpha\rangle\langle \overset{+}{L}_\beta L_\beta\rangle} \le M_1\left(\frac{\varepsilon_N}{\omega_\alpha}\right)^{1/2} \xrightarrow[N\to\infty]{} 0, \qquad (25a)$$

$$\left|\left\langle \frac{\overset{+}{a}_\alpha a_\beta}{N}\right\rangle + \frac{\lambda_\beta}{\omega_\beta}\left\langle \frac{\overset{+}{a}_\alpha}{\sqrt{N}} L_\beta\right\rangle\right| = \left|\langle \overset{+}{B}_\alpha B_\beta\rangle - \frac{\overset{*}{\lambda}_\alpha}{\omega_\alpha}\langle L_\alpha B_\beta\rangle\right| \le M_1\frac{|\lambda_\alpha|}{\omega_\alpha}\left(\frac{\varepsilon_N}{\omega_\beta}\right)^{1/2} + \frac{\varepsilon_N}{\sqrt{\omega_\alpha\omega_\beta}} \xrightarrow[N\to\infty]{} 0. \qquad (25b)$$

From (25a) and (25b) it follows in particular:

$$\langle \overset{+}{a}_\alpha a_\alpha\rangle = N\frac{|\lambda_\alpha|^2}{\omega_\alpha^2}\langle \overset{+}{L}_\alpha L_\alpha\rangle\left(1+O\left(N^{-1/3}\right)\right). \qquad (25c)$$

The relations of the type (25a-c) and their physical sense are under retailed discussion in ref. [14]. Here we only note, that for a set of concrete models (in particular, for those of (4)-(7)) the averages $\langle \overset{+}{L}_\alpha L_\alpha\rangle$ $(1\le\alpha\le S)$ characterize the long-range order in the L-subsystem (magnetization, polarization, "gap" in the theory of superconductivity, etc.) Then (25c) means, that the phase transitions in the L-subsystem from disordered $(\langle \overset{+}{L}_\alpha L_\alpha\rangle = 0)$ into ordered $(\langle \overset{+}{L}_\alpha L_\alpha\rangle \ne 0)$ state without fail should be accompanied by the compensating macroscopic inflation of the boson modes $\langle \overset{+}{a}_\alpha a_\alpha\rangle \sim N$. Such inflation, which also renders the averages $\langle \overset{+}{B}_\alpha B_\alpha\rangle$ to be small (24), appears advantageous from energetical considerations (see below (26)).

4. Free energy.

In this section we derive the upper and lower bounds in (12).

a) The upper bound. Note first of all, that the Hamiltonian (1) can be rewritten in the form:

$$H = \tilde{H} + N\sum_{\alpha=1}^{S}\omega_\alpha \overset{+}{B}_\alpha B_\alpha, \qquad (26)$$

where $\tilde{H}$ is defined by (11). Let us now introduce the auxiliary Hamiltonian

$$H_\rho = \tilde{H} + \rho N\sum_{\alpha=1}^{S}\omega_\alpha \overset{+}{B}_\alpha B_\alpha, \qquad (27)$$

where ρ is real positive parameter, $0<\rho<1$. Putting in the inequalities (15) $\mathcal{U}_1 = H$ (1) and $\mathcal{U}_2 = H_\rho$ (27), we obtain:

$$f[H] - f[H_\rho] \le (1-\rho)\sum_{\alpha=1}^{S}\omega_\alpha\langle \overset{+}{B}_\alpha B_\alpha\rangle_{H_\rho}. \qquad (28)$$

On the other hand, one can obtain the Hamiltonian H_ρ (27) through the following transformation of the parameters $\varkappa_\alpha$, ω_α and λ_α, $\overset{*}{\lambda}_\alpha$ in the Hamiltonian H (1):

$$\begin{aligned} &\varkappa_\alpha \to \varkappa_\alpha + (1-\rho)|\lambda_\alpha|^2/\omega_\alpha, \\ &\omega_\alpha \to \rho\omega_\alpha, \\ &\lambda_\alpha \to \rho\lambda_\alpha, \quad \overset{*}{\lambda}_\alpha \to \rho\overset{*}{\lambda}_\alpha . \end{aligned} \qquad (29)$$

Since the operators $\overset{+}{B}_\alpha$, B_α (10b) are invariant under the transformations (29), we can estimate the right-hand side of the inequality (28), performing (29) on both sides of the inequality (24). By doing so, and taking into consideration, that (see (A9a,b))

$$d_1 \to d_1^{transf} \le \rho d_1 \le d_1,\ d_{2,N} \to d_{2,N}^{transf} \le \sqrt{\rho}\, d_{2,N},$$

we get:

$$\sum_{\alpha=1}^{S} \omega_\alpha \langle \overset{+}{B}_\alpha B_\alpha \rangle_{H_\rho} \le 2\left(\frac{S d_{2,N}}{N\rho}\right)^{2/3} + \frac{2S(\theta+\sqrt{2}d_1)}{N}. \quad (30)$$

Let us now leave the bound (30) for the time being and make use of Theorem 2. Putting in (15) $\mathcal{U}_1 = H_\rho$ and $\mathcal{U}_2 = \tilde{H} + \rho H_B$, where

$$H_B = N \sum_{\alpha=1}^{S} \omega_\alpha \overset{+}{a}_\alpha a_\alpha, \quad (31)$$

we find:

$$f[H_\rho] - f[\tilde{H} + \rho H_B] \le$$
$$\le \rho \sum_{\alpha=1}^{S} \left\langle \frac{|\lambda_\alpha|^2}{\omega_\alpha} \overset{+}{L}_\alpha L_\alpha + \lambda_\alpha \frac{\overset{+}{a}_\alpha}{\sqrt{N}} L_\alpha + \overset{*}{\lambda}_\alpha \frac{a_\alpha}{\sqrt{N}} \overset{+}{L}_\alpha \right\rangle_{\tilde{H}+\rho H_B}. \quad (32)$$

In virtue of the independence of the Hilbert spaces $\mathcal{H}_L$ and $\mathcal{H}_B$ on which the Hamiltonians $\tilde{H}$ (11) and H_B (31) are definite (see Sec.2), we have *)

$$f[\tilde{H} + \rho H_B] = f[\tilde{H}] + f[\rho H_B] \quad (33)$$

and

$$\langle \overset{+}{a}_\alpha L_\alpha \rangle_{\tilde{H}+\rho H_B} = \langle \overset{+}{a} \rangle_{\rho H_B} \langle L_\alpha \rangle_{\tilde{H}} = 0, \quad (34a)$$

$$\langle a_\alpha \overset{+}{L}_\alpha \rangle_{\tilde{H}+\rho H_B} = \langle a \rangle_{\rho H_B} \langle \overset{+}{L}_\alpha \rangle_{\tilde{H}} = 0. \quad (34b)$$

The right-hand side equalities in (34a,b) are valid due to the gauge invariance of the Hamiltonian H_B (31) with respect to the transformations $a_\alpha \to a_\alpha \exp(i\varphi_\alpha)$, $\overset{+}{a}_\alpha \to \overset{+}{a}_\alpha \exp(-i\varphi_\alpha)$. We note next, that the free energy $f[\rho H_B]$ for the free boson gas is defined by a well-known formular and proves to be negative:

$$f[\rho H_B] = \frac{\theta}{N} \sum_{\alpha=1}^{S} \ln\left(1 - e^{-\rho\frac{\omega_\alpha}{\theta}}\right) \le 0. \quad (35)$$

On the basis of (3a) and (32)-(35) we obtain:

$$f[H_\rho] - f[\tilde{H}] \le \rho M_1^2 \sum_{\alpha=1}^{S} \frac{|\lambda_\alpha|^2}{\omega_\alpha}. \quad (36)$$

From (28), (30) and (36) follows in turn:

$$f[H] - f[\tilde{H}] \le \rho M_1^2 \sum_{\alpha=1}^{S} \frac{|\lambda_\alpha|^2}{\omega_\alpha} +$$
$$+ 2\left(\frac{S d_{2,N}}{\rho N}\right)^{2/3} + \frac{2S(\theta+\sqrt{2}d_1)}{N}. \quad (37)$$

Since the left-hand side of this inequality is independent of ρ, the value ρ ($0 < \rho < 1$) is our choice. In order to obtain the best majoration bound (as $N \to \infty$) we choose in (37) $\rho = N^{-2/5}$ and finely get:

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*) Owing to the independence of the spaces $\mathcal{H}_L$ and $\mathcal{H}_B$, for arbitrary operator $\mathcal{U}_{L\otimes B} = \mathcal{U}_L \otimes \mathcal{U}_B \in \mathcal{H}_L \otimes \mathcal{H}_B$, where $\mathcal{U}_L \in \mathcal{H}_L$ and $\mathcal{U}_B \in \mathcal{H}_B$, we have

$$\mathrm{Tr}_{L\otimes B}\left(\mathcal{U}_{L\otimes B}\, e^{-\frac{\tilde{H}+\rho H_B}{\theta}}\right) = \mathrm{Tr}_L\left(\mathcal{U}_L e^{-\frac{\tilde{H}}{\theta}}\right) \mathrm{Tr}_B\left(\mathcal{U}_B e^{-\rho\frac{H_B}{\theta}}\right),$$

whence (33) and (34,left) follows. Here in the subscripts $L\otimes B$, L, B mean, that "Trace" should be taken over the spaces $\mathcal{H}_L \otimes \mathcal{H}_B$, $\mathcal{H}_L$ and $\mathcal{H}_B$, respectively. The free energies (see (16a)) in (33) should be considered as $f[\tilde{H}+\rho H_B]_{B\otimes L}$, $f[\tilde{H}]_L$, $f[\rho H_B]_B$.

$$f[H]-f[\tilde{H}]\leq$$

$$\leq \frac{M_1^2 \sum_{\alpha=1}^{s}\left(|\lambda_\alpha|^2/\omega_\alpha\right)+2\left(s d_{2,N}\right)^{2/3}}{N^{2/5}}+ \tag{38}$$

$$+\frac{2s\left(\theta+\sqrt{2}\,d_1\right)}{N}=\eta_N \xrightarrow[N\to\infty]{} 0\,.$$

Thus the upper bound in (12) is completely proved.

The lower bound: We shall derive the lower bound in (12) following the scheme of ref. [14]. Putting in (15) $\mathcal{U}_1=H$ and $\mathcal{U}_2=\tilde{H}-\Delta_\rho+\rho H_B$, where

$$\Delta_\rho = N\frac{\rho}{1-\rho}\sum_{\alpha=1}^{s}\frac{|\lambda_\alpha|^2}{\omega_\alpha}\overset{+}{L}_\alpha L_\alpha\,,\quad 0<\rho<1,$$

we find:

$$0\leq f[H]-f[\tilde{H}-\Delta_\rho+\rho H_B]\,. \tag{39}$$

On the other hand

$$f[\tilde{H}-\Delta_\rho+\rho H_B]=f[\tilde{H}-\Delta_\rho]+f[\rho H_B]\,, \tag{40}$$

where (see (35))

$$f[\rho H_B]\geqslant-\sum_{\alpha=1}^{s}\left(\frac{\theta}{N}\ln\frac{\theta}{\rho\omega_\alpha}+\frac{\rho\omega_\alpha}{N}\right). \tag{41}$$

Next, putting in (15) $\mathcal{U}_1=\tilde{H}-\Delta_\rho$ and $\mathcal{U}_2=\tilde{H}$, we get:

$$-\frac{\rho}{1-\rho}M_1^2\sum_{\alpha=1}^{s}\frac{|\lambda_\alpha|^2}{\omega_\alpha}\leq f[\tilde{H}-\Delta_\rho]-f[\tilde{H}] \tag{42}$$

On the basis of (39)-(42) one can easily obtain the inequality:

$$-\sum_{\alpha=1}^{s}\left(\frac{\rho}{1-\rho}M_1^2\frac{|\lambda_\alpha|^2}{\omega_\alpha}+\frac{\theta}{N}\ln\frac{\theta}{\rho\omega_\alpha}+\frac{\rho\omega_\alpha}{N}\right)\leq \tag{43}$$

$$\leq f[H]-f[\tilde{H}]\,,$$

where the right-hand side does not depend on ρ. Choosing in the left-hand side $\rho=1/N$, we finely obtain:

$$-\zeta_N\leq f[H]-f[\tilde{H}]\,, \tag{44a}$$

where

$$\zeta_N=\frac{s\theta\ln N}{N}+\frac{M_1^2}{N-1}\sum_{\alpha=1}^{s}\frac{|\lambda_\alpha|^2}{\omega_\alpha}+\sum_{\alpha=1}^{s}\frac{\theta\ln\frac{\theta}{\omega_\alpha}+\omega_\alpha}{N}\,, \tag{44b}$$

$\zeta_N\sim(\ln N)/N$ as $N\to\infty$. The bound (44) completes the proof of the range of inequalities (12).

APPENDIX

Here we derive the bound (22). Taking into account that $a_\alpha^{\#}$ commute with $T, L_\beta^{\#}$ ($1\leq\alpha,\beta\leq s$) (see Sec.1) and making use of (2), one can easily calculate the operator R_α (18b):

$$R_\alpha=\omega_\alpha B_\alpha+\frac{\lambda_\alpha}{\omega_\alpha}\left(X_\alpha+Y_\alpha+Z_\alpha\right), \tag{A1}$$

where

$$X_\alpha=\sum_{\beta=1}^{s}\lambda_\beta\overset{+}{B}_\beta\, N[L_\alpha;L_\beta]\,, \tag{A2a}$$

$$Y_\alpha=\sum_{\beta=1}^{s}\overset{*}{\lambda}_\beta B_\beta\, N[L_\alpha;\overset{+}{L}_\beta]\,, \tag{A2b}$$

$$Z_\alpha=-\sum_{\beta=1}^{s}\frac{|\lambda_\beta|^2}{\omega_\beta}N\left(L_\beta[L_\alpha;\overset{+}{L}_\beta]+\right. \tag{A2c}$$

$$\left.+\overset{+}{L}_\beta[L_\alpha;L_\beta]\right)+[L_\alpha;T]\,.$$

Making use of (21), (A1), and elementary inequality $2|xy| \leq$ $\leq x^2+y^2$, we get:

$$\langle R_\alpha \overset{+}{R}_\alpha \rangle \leq 4\omega_\alpha^2 \langle \overset{+}{B}_\alpha B_\alpha \rangle + \frac{4|\lambda_\alpha|^2}{\omega_\alpha^2}\left(\langle X_\alpha \overset{+}{X}_\alpha \rangle + \langle Y_\alpha \overset{+}{Y}_\alpha \rangle + \langle Z_\alpha \overset{+}{Z}_\alpha \rangle\right). \quad \text{(A3)}$$

On the basis of (3c), (21), (A2a) and Caushy inequality we obtain in turn:

$$\langle X_\alpha \overset{+}{X}_\alpha \rangle \leq M_3^2 \left(\sum_{\beta=1}^{s} |\lambda_\beta| \sqrt{\langle \overset{+}{B}_\beta B_\beta \rangle}\right)^2 \leq$$

$$\leq M_3^2 \left(\sum_{\beta=1}^{s} \frac{|\lambda_\beta|^2}{\omega_\beta}\right)\left(\sum_{\beta=1}^{s} \omega_\beta \langle \overset{+}{B}_\beta B_\beta \rangle\right) \leq$$

$$\leq M_3^2 P_s \left(\sum_{\beta=1}^{s} \omega_\beta \langle \overset{+}{B}_\beta B_\beta \rangle\right), \quad \text{(A4)}$$

where

$$P_s = \sum_{\beta=1}^{s} \frac{|\lambda_\beta|^2}{\omega_\beta}. \quad \text{(A5)}$$

By analogous way one can find:

$$\langle Y_\alpha \overset{+}{Y}_\alpha \rangle \leq M_3^2 P_s \sum_{\alpha=1}^{s} \langle B_\alpha \overset{+}{B}_\alpha \rangle =$$

$$= M_3^2 P_s \sum_{\alpha=1}^{s} \omega_\alpha \langle \overset{+}{B}_\alpha B_\alpha + [B_\alpha ; \overset{+}{B}_\alpha] \rangle \leq$$

$$\leq M_3^2 P_s \left[\sum_{\alpha=1}^{s} \omega_\alpha \langle \overset{+}{B}_\alpha B_\alpha \rangle + \frac{1}{N}\left(M_3 P_s + \sum_{\alpha=1}^{s} \omega_\alpha\right)\right]. \quad \text{(A6)}$$

Besides, in virtue of the additional conditions (3a-c) we have:

$$\langle Z_\alpha \overset{+}{Z}_\alpha \rangle \leq \| Z_\alpha \|^2 \leq \left(M_2 + 2M_1 M_3 P_s\right)^2. \quad \text{(A7)}$$

Substituting the bounds (A4)-(A7) into (A3), we obtain the bound for $\langle R_\alpha \overset{+}{R}_\alpha \rangle$. Quite the same bound can be derived for the average $\langle \overset{+}{R}_\alpha R_\alpha \rangle$. So one can easily find as a result:

$$\sum_{\alpha=1}^{s} \frac{\omega_\alpha}{2} \langle R_\alpha \overset{+}{R}_\alpha + \overset{+}{R}_\alpha R_\alpha \rangle \leq 4 \sum_{\alpha=1}^{s} \omega_\alpha^3 \langle \overset{+}{B}_\alpha B_\alpha \rangle + 8 (M_3 P_s)^2 \sum_{\beta=1}^{s} \omega_\beta \langle \overset{+}{B}_\beta B_\beta \rangle + 4 P_s (M_2 + 2M_1 M_3 P_s)^2 + \frac{(2M_3 P_s)^2 (M_3 P_s + \sum_{\beta=1}^{s} \omega_\beta)}{N} \leq d_1^2 \sum_{\alpha=1}^{s} \omega_\alpha \langle \overset{+}{B}_\alpha B_\alpha \rangle + d_{2,N}^2 \quad \text{(A8)}$$

where

$$(d_1)^2 = 4\left[2(M_3 P_s)^2 + \sum_{\beta=1}^{s} \omega_\beta^2\right], \quad \text{(A9a)}$$

$$(d_{2,N})^2 = 4 P_s (M_2 + 2M_1 M_3 P_s)^2 + \frac{1}{N}(2M_3 P_s)^2 \left(M_3 P_s + \sum_{\beta=1}^{s} \omega_\beta\right), \quad \text{(A9b)}$$

for P_s see (A5). On the basis of (A8), making use of the Hölder inequality and of the elementary inequality $(x+y)^{1/3} \leq$

$X^{1/3}+y^{1/3}$, we finally obtain the bound:

$$\sum_{\alpha=1}^{s}\left[\frac{\omega_\alpha}{2}\langle R_\alpha \overset{+}{R}_\alpha + \overset{+}{R}_\alpha R_\alpha\rangle\right]^{1/3} \leq$$

$$\leq (sd_1)^{2/3}\left(\sum_{\beta=1}^{s}\omega_\beta\langle \overset{+}{B}_\beta B_\beta\rangle\right)^{1/3} + (sd_{2,N})^{2/3}. \quad \text{(A10)}$$

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