ОБЪЕДИНЕННЫЙ ИНСТИТУТ ЯДЕРНЫХ ИССЛЕДОВАНИЙ

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A.V. Efremov, D.V. Shirkov

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HIGHEST PARTIAL WAVES IN THE LOW ENERGY APPROXIMATION

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* Институт математики СО АН СССР.

Abstract

The highest partial waves are taken into account in the equations for the low-energy pion-pion scattering obtained by the differential method. It is shown that their influence is small.

The procedure of the introduction of infinite number of partial waves in the low-energy equations is considered and is shown to have no sense.

1. The Formulation of the Differential Method taking as an Example the Scattering of Neutral Mesons

Recently Sarker[1] and Lovelace[2] raised the problem of the correspondnece between the equations for partial waves of the low-energy pion-pion scattering obtained by the differential method[3,4,5] and those obtained by the integral method[6]. In view of the inexactitude of a number of formulations in[1,2] below we shall investigate in detail the problem of taking into account highest partial waves in the differential method and also its correspondence with the Chew-Mandelstam method.

In[3] the equations for pion-pion scattering were obtained from the combination of the forward and the backward dispersion relations, and in[4] the information from the first derivatives with respect to the momentum transfer was also taken into consideration. Therefore, in[3] one took into account the real parts of only S- and P- waves while in[4] one took into account d- and f- waves too.

We shall study the problem of taking into account an ever-increasing number of partial waves including the limiting case of infinite number of these waves.

First, we write down formulae which express the lowest partial waves f_i in terms of the values of the function $f(c)$ and its derivatives at the points $c = \pm 1$. These formulae have a different form depending on the number of harmonics which approximate the function $f(c)$.

In the lowest approximation, restricting ourselves to the s- and p- waves

$$f(c) \simeq f_o + 3c\, f_1$$

we have

$$f_o = \frac{f(1) + f(-1)}{2} \tag{1.1}$$

$$f_1 = \frac{f(1) - f(-1)}{6} \,. \tag{1.2}$$

In the following approximation which takes into account d- and f- waves also

$$f(c) \cong f_o + 3c\, f_1 + 5/2(3c^2 - 1)\, f_2 + 7/2\,(5c^3 - 3c)\, f_3$$

we get

$$f_o = \frac{f(1) + f(-1)}{2} - \frac{f'(1) - f'(-1)}{6}$$

$$f_1 = \frac{f(1) - f(-1)}{5} - \frac{f'(1) + f'(-1)}{30}$$

$$f_2 = \frac{f'(1) - f'(-1)}{30}$$

$$f_3 = \frac{f(1) - f(-1)}{70} + \frac{f'(1) + f'(-1)}{70} .$$

Finally to the limiting case

$$f(c) = \sum_{n=0}^{\infty} (2n + 1) \quad f_n P_n(c)$$

there correspond the formulae

$$f_0 = \tfrac{1}{2} \sum_{n=0}^{\infty} \frac{f^{(n)}(-1) + (-)^n \; f^{(n)}(1)}{(n + 1)!}$$

$$f_1 = -\tfrac{1}{2} \sum_{n=0}^{\infty} \frac{f^{(n)}(-1) - (-)^n \; f^{(n)}(1)}{(n + 2)!}$$

$$f_2 = -\tfrac{1}{4} \sum_{n=0}^{\infty} \frac{n(n+5)}{(n+3)!} \left[f^{(n)}(-1) + (-)^n f^{(n)}(1) \right] \tag{1.3}$$

which express the partial waves of the function $f(c)$ in terms of the infinite set of its derivatives at the points $c = \pm 1$.

The expressions (1.3) can be formally obtained by means of the repeated integration by parts of integrals determining partial waves. The convergence of the series of the type (1. 3) is defined by the singularities of the function $f(c)$ in the complex plane of the variable c. Somewhat further (§ 3) we shall consider this problem for the case of the pion-pion scattering.

Note here that the transition from (1.1) to (1.2) and from (1.2) to (1.3) does not reduce to the addition of terms with highest derivatives but also to the change of coefficients in the already available terms.

The formulas of the type (1.1) - (1.3) will be applied to the scattering amplitude specified by the spectral representation with respect to the momentum or energy variable for a fixed value of the cosine of the scattering angle c. For the neutral meson scattering amplitude the representation is of the form

$$A(\nu,c) = 1/\pi \int_0^\infty \frac{Im\, A(\nu',c)}{\nu'-\nu} + 1/\pi \int_0^\infty \frac{Im\, A(\nu',c_+)}{1+\nu'+\frac{1+c}{2}}\, d\nu' + 1/\pi \int_0^\infty \frac{Im\, A(\nu',c_-)}{1+\nu'+\nu\, \frac{1+c}{2}}\, d\nu' \tag{1.4}$$

Here the first integral describes the physical cut from the first reaction whose squared energy is $s = 4(\nu+1)$, the second one describes the crossing reaction with the squared energy $u = -2\nu(1+c)$ and the third integral - the crossing reaction with the squared energy $t = -2\nu(1-c)$, where

$$c_+ = \frac{2+3\nu' - c(2+\nu')}{\nu'(1+c)} \quad ; \qquad c_- = \frac{2+3\nu'+c(2+\nu')}{\nu'(1-c)} \, . \tag{1.5}$$

From (1.5) it is seen that

$$c_\pm(\nu', c = \pm 1) = \pm 1 \, , \qquad c_\mp(\nu', c = \pm 1) = \infty \, .$$

Therefore, in the limiting cases $c = \pm 1$ the numerator of one of the crossing integrals corresponds to the forward (or backward) scattering, while the second one contains an unphysical infinite cosine.

In the Chew-Mandelstam scheme as well as in the Cini-Fubini representation/8/ the functions $Im\, A(\nu', c_\pm)$ are approximated by the S- waves in the whole interval $-1 \le c \le 1$, i.e. up to infinitly large values of the cosines $c_\pm$. After the integration over c has been made by performing the aforementioned approximation we obtain the Chew-Mandelstam equation for the neutral model (comp./8,9/)

$$A_0(\nu) = 1/\pi \int_0^\infty \frac{d\nu'}{\nu'-\nu}\, Im\, A_0(\nu') - 2/\pi\nu \int_0^\infty d\nu'\, Im\, A_0(\nu')\, \ell n\left(1 - \frac{\nu}{1+\nu+\nu'}\right) . \tag{1.6}$$

Hence it is clear once more that in the Chew-Mandelstam and Cini-Fubini methods one uses the analytical continuation by means of the first term of the Legendre expansion into a region where this series does not exist.

We obtain now an equation for the S- wave by means of the differential approximation. By inserting (1.4) in the first formula of (1.1) and approximating $Im\, A(\nu, \pm 1)$ by the S- wave, we have

$$A_0(\nu) = 1/\pi \int_0^\infty d\nu' \left(\frac{1}{\nu'-\nu} + \frac{1}{1+\nu'+\nu}\right) Im\, A_0(\nu') + a \tag{1.7}$$

where

$$a = 1/\pi \int \frac{d\nu'}{1+\nu'} \quad \frac{Im\, A(\nu', \infty) + Im\, A(\nu', -\infty)}{2} . \tag{1.8}$$

Let us show that for the solution of Eq. (1.7) to exist, it is necessary to put $a = 0$. For this purpose by making in (1.7) one subtraction, we reduce it to the form (2.5)/7/. Repeating then the reasoning of § 2/7/, we find that $Im\, A(\infty) = 0$ from where the account of the unitarity condition

$$Im\, A_0(\nu) = \sqrt{\frac{\nu}{\nu+1}} \; | A_0(\nu) |^2 \qquad \nu \geqslant 0 \tag{1.9}$$

it follows that

$$Re\, A_0\ (\infty) = 0 .$$

In other words Eq. (1.7) has a solution only for

$$a = 0 . \tag{1.10}$$

Let us notice now that the quantity a is a high-energy contribution. Indeed, for example, the third integral in (1.4) corresponds to the part of the line $c = 1 - \epsilon = const$ from $-\infty$ to the point $a(\epsilon)$ with the coordinates $t = 4$, $\nu = -2/\epsilon$ which goes to infinity for $\epsilon \to 0$. The first term of the right hand side of (1.8), being the limit of this integral for $c = 1$, represents, therefore, a high-energy contribution. The same concerns the second integral in (1.4) for $c = -1$ and the second term in (1.8). Thus, a is a contribution from the high-energy region which lies beyond the threshold of any state with finite mass. It is clear therefore, that the quantity a can be safely neglected since we omitted all the intermediate states starting with the four-meson one.

We pass now to the second approximation described by the formulae (1.2). By approximating $Im\, A(\nu, \pm 1)$ by the S- waves and neglecting high-energy contributions of the kind of $a' + \beta\nu$, we obtain from the first equation (1.2).

$$A_0(\nu) = 1/\pi \int_0^\infty d\nu' Im\, A_0(\nu') \left[\frac{1}{\nu'-\nu} + \frac{1}{\nu'+\nu+1} \left(1 - \frac{\nu}{6(\nu'+\nu+1)} \right) \right] . \tag{1.11}$$

It is interesting to investigate the asymptotics (1.11) coincide with that of Eq. (1.7) for $a = 0$

$$Re\ A_0(\nu) \approx \frac{\pi b}{\ln \nu} \quad ; \qquad b = ½$$

since

$$\nu \frac{\partial}{\partial \nu} \frac{1}{\ln \nu} = - \frac{1}{\ln^2 \nu} .$$

Hence it follows that the account of the. d- wave in the real part of the scattering amplitude changes only slightly the logarithmic branch of the neutral model solution (1.7).

We now turn to the limiting case (1.3). By omitting the power series in ν with 'high-energy coefficients 'we obtain for the s- waves

$$A_0(\nu) = 1/\pi \int_0^\infty \frac{d\nu'}{\nu'-\nu} \; Im\, A_0(\nu') \; +$$

$$+ 1/\pi \int_0^\infty d\nu' \, Im\, A_0(\nu') \; 2/\nu \sum_{n=1}^{\infty} 1/n \left(\frac{\nu}{2(1+\nu'+\nu)} \right)^n .$$

The sum in the crossing integral can be combined into

$$A_0(\nu) = 1/\pi \int_0^\infty \frac{d\nu'}{\nu' - \nu} \; Im\, A_0(\nu') - 2/\pi\nu \int_0^\infty d\nu' \, Im\, A_0(\nu') \; \ell n \left(1 - \frac{\nu}{2(\nu'+\nu+1)}\right). \qquad (1.13)$$

It is not difficult to see that Eq. (1.13) allows the logarithmic asymptotics (1.12).

It is important to note that Eq. (1.13) differs essentially from the Chew-Mandelstam Eq. (1.6) which possesses a logarithmic asymptotics of the form (1.12) for $b = 1/3$.

From the aforeasaid follows the disprovment of Sarker's statement /1/. His conclusion that equation of the type (1.7) (1.11) can be obtained from (1.6) by expanding the logarithm is based on an insufficiently accurate study of the numerical coefficients of the corresponding series.

Let us consider the influence of the highest waves in the imaginary part of the scattering amplitude on the s- wave. With this aim we repeat the arguments by taking into consideration (with the aid of the formulas (1.2) with s- and d- waves in the real as well as in the imaginary parts of the scattering amplitude. Pefroming the calculations with omitting the 'high-energy terms' and the differentiation formulas

$$\left.\frac{\partial c_+}{\partial c}\right|_{c=1} = \left.\frac{\partial c_-}{\partial c}\right|_{c=-1} = -\frac{1+\nu'}{\nu'} \qquad (1.14)$$

we obtain the system of equations for s- and d- waves

$$A_0(\nu) = 1/\pi \int_0^\infty \frac{Im\ A_0(\nu')}{\nu' - \nu} d\nu' + 1/\pi \int_0^\infty \frac{d\nu'}{1+\nu'+\nu} \left(1 + \frac{\nu}{6(\nu'+\nu+1)}\right) \left(Im\ A_0(\nu') + 5 Im\ A_2(\nu')\right) +$$

$$+ \frac{10}{3\pi} \int_0^\infty \frac{d\nu'}{1+\nu'+\nu} \frac{1+\nu'}{\nu'} Im\ A_2(\nu') \qquad (1.15)$$

$$A_2(\nu) = 1/\pi \int_0^\infty \frac{Im\ A_2(\nu')\, d\nu'}{\nu' - \nu} - 1/\pi \int_0^\infty \frac{d\nu'(1+\nu')}{\nu'(1+\nu'+\nu)} Im\ A_2(\nu') -$$

$$- \frac{\nu}{30\pi} \int_0^\infty \frac{d\nu'}{(1+\nu'+\nu)^2} \left(Im\ A_0(\nu') + 5\, Im\ A_2(\nu')\right). \qquad (1.16)$$

From (1.18) it follows that the logarithmic asymptotics of the function A_2 is determined by a crossing integral which contains A_0 and is of the form

$$Re\ A_2(\nu) \approx -\frac{1}{\ell n^2 \nu} \quad ; \quad Im\ A_2(\nu) \approx \frac{1}{\ell n^4\ \nu} \quad . \qquad (1.17)$$

Due to this fact the term containing Im A_2 in the crossing integral (1.17) for large ν behaves as $\ln^{-3}\nu$ and does not change the asymptotics (1.12) of the s-wave. Hence, two important conclusions follow:

a) When taking into account highest partial waves the approximations in the real and imaginary parts of the scattering amplitude should be consistent. So, the approximation which follows (1.11) is an approximation when together $Re\,A_4$ with one takes into account Im A_2 too. Eq. (1.13) does not therefore improve the accuracy in comparison with (1.11).

b) The logarithmic asymptotics (1.12) does not change when taking into account highest waves in the real as well as in the imaginary part of the scattering amplitude.

As it will be clear from below the conclusion (a) is a special property of the neutral model and is due to the absence of the p- wave. In the following it will be shown that in the case of the scattering of charged mesons the coefficient of the logarithmic asymptotics changes, however, this change is insignificant.

2. Scattering of Charged Pions

We turn to the real case of the scattering of charged pions. The formulae (1.1), (1.2) will be applied to the functions

$$A^{\circ} = 3A + B + C, \quad A^{1} = B - C, \quad A^{2} = B + C,$$

specified by the representations

$$\begin{bmatrix} A \\ B \\ C \end{bmatrix}(\nu, c) = 1/\pi \int \frac{d\nu'}{\nu' - \nu} \, Im \begin{bmatrix} A \\ B \\ C \end{bmatrix}(\nu' c) + 1/\pi \int \frac{d\nu'}{1 + \nu' + \nu \frac{1+c}{2}} \, Im \begin{bmatrix} C \\ B \\ A \end{bmatrix}(\nu', c_{+}) +$$

$$+ 1/\pi \int \frac{d\nu'}{1 + \nu' + \nu \frac{1-c}{2}} \, Im \begin{bmatrix} B \\ A \\ C \end{bmatrix}(\nu', c_{-}) . \tag{2.1}$$

The cosines of the crossing reactions c_{+} and c_{-} are determined in (1.5).

The most simple equations for s- and p- waves (see/3,5/) can be obtained from (2.1) by means of the formulae (1.1). Restricting ourselves in the amplitudes A^{J} only to s- and p- waves

$$A^{\circ}(\nu, c) \cong A^{\circ}_{o}(\nu) \equiv A_{o}(\nu) \qquad A^{1}(\nu, c) \cong 3cA^{1}_{1}(\nu) \equiv 3cA_{1}(\nu)$$

$$A^{2}(\nu, c) \cong A^{2}_{o}(\nu) \equiv A_{2}(\nu) \tag{2.2}$$

with account of the inverse relations

$$A(\nu, \pm 1) = \frac{A_o - A_2}{3} \qquad B(\nu, \pm 1) = \frac{A_2 \pm 3A_1}{2}; \qquad C(\nu, \pm 1) = \frac{A_2 \mp 3A_1}{2}$$

we get successively from (2.2) omitting high-energy constants of the type (1.8)

$$A(\nu, 1) = A(\nu, -1) = 1/\pi \int_0^{\infty} \frac{d\nu'}{\nu' - \nu} \frac{Im\, A_o(\nu') - Im\, A_2(\nu')}{3} +$$

$$+ 1/\pi \int_0^{\infty} \frac{d\nu'}{1 + \nu + \nu'} \frac{Im\, A_2(\nu') - 3 Im\, A_1(\nu')}{2} .$$

$$B(\nu, 1) = 1/\pi \int_0^{\infty} \frac{d\nu'}{\nu' - \nu} \frac{Im\, A_2(\nu') + 3 Im\, A_1(\nu')}{2} +$$

$$+ 1/\pi \int_0^{\infty} \frac{d\nu'}{\nu' + \nu + 1} \frac{Im\, A_2(\nu') + 3\, Im\, A_1(\nu')}{2} = C(\nu, -1) \tag{2.3}$$

$$B(\nu,-1) = C(\nu, 1) = 1/\pi \int_0^\infty \frac{d\nu'}{\nu'-\nu} \frac{Im\, A_2(\nu') - 3\, Im\, A_1(\nu')}{2} + \quad (2.3)$$

$$= 1/\pi \int_0^\infty \frac{d\nu'}{1+\nu'+\nu} \frac{Im\, A_0(\nu') - Im\, A_2(\nu')}{3} .$$

Going on to the partial waves we obtain the equations

$$A_i(\nu) = 1/\pi \int_0^\infty \frac{Im\, A_i(\nu')}{\nu'-\nu} d\nu' + 1/\pi \int_0^\infty \frac{f_i(\nu')}{1+\nu+\nu'} d\nu' \quad (2.4)$$

where $f_i(\nu) = Im\, A_i(\nu) + \ell_i \phi(\nu)$

$$\phi(\nu) = 2\, Im\, A_0(\nu) + 9\, Im\, A_1(\nu) - 5\, Im\, A_2(\nu) \quad (2.5)$$

$$\ell_0 = -1/3 \qquad \ell_1 = -1/18 \qquad \ell_2 = 1/6$$

with an additional threshold condition on the p- wave

$$A_1(0) = 0, \quad (2.6)$$

following from the crossing symmetry property

$$B(s, u, t) = C(s, t, u).$$

Eqs. (2.4) have been studied in detail in /5/. There, in particular, it has been established the existence of the logarithmic branch of solutions with the asymptotic behaviour (see also /2/)

$$A_i(\nu) \approx \frac{d_i}{\ln \nu}; \quad d_0 = 2{,}13; \quad d_1 = 0{,}118; \quad d_2 = 0{,}640. \quad (2.7)$$

We go on now to the next approximation which takes into account d- and f- waves in the real part of the amplitudes.

Calculating the derivatives by means of the relations (1.16) and

$$\frac{\partial\, Im\, A(\nu,c)}{\partial c} = 0, \quad \frac{\partial\, Im\, B(\nu,c)}{\partial c} = \frac{\partial\, Im\, C(\nu,c)}{\partial c} = 3/2\ Im\, A_1(\nu)$$

we find

$$A'(\nu, 1) = A'(\nu,-1) = I_1(\nu) - \nu/2\pi \int_0^\infty \frac{d\nu'}{(1+\nu'+\nu)^2} \frac{Im\, A_2(\nu') - 3\, Im\, A_1(\nu')}{2}$$

$$B'(\nu,1) = -C'(\nu,-1) = -I_1(\nu) - \nu/2\pi \int_0^\infty \frac{d\nu'}{(1+\nu+\nu')^2} \frac{Im\, A_2(\nu') + 3\, Im\, A_1(\nu')}{2}$$

$$C'(\nu,1) = -B'(\nu,-1) = \nu/2\pi \int_0^\infty \frac{d\nu'}{(1+\nu'+\nu)^2} \frac{Im\, A_0 - Im\, A_2(\nu')}{3}, \qquad (2.8)$$

where the following notation is used

$$I_1(\nu) = 3/2\pi \int_0^\infty \frac{d\nu'}{1+\nu+\nu'} \frac{\nu'+1}{\nu'} Im\, A_1(\nu'). \qquad (2.9)$$

By inserting (2.3) and (2.8) into (1.2), we obtain

$$A_0(\nu) = 1/\pi \int_0^\infty \frac{d\nu'}{\nu'-\nu} Im\, A_0(\nu') + 1/\pi \int_0^\infty \frac{d\nu'}{(1+\nu'+\nu)} \left(1 + \frac{\nu}{6(\nu'+\nu+1)}\right) f_0(\nu') - 2/3\, I_1(\nu)$$

$$A_1(\nu) = 1/\pi \int_0^\infty \frac{d\nu'}{\nu'-\nu} Im\, A_1(\nu') + 6/5\pi \int_0^\infty \frac{d\nu'}{(1+\nu'+\nu)} \left(1 + \frac{\nu}{12(1+\nu+\nu')}\right) f_1(\nu') + 1/15\, I_1(\nu) \qquad (2.10)$$

$$A_2(\nu) = 1/\pi \int_0^\infty \frac{d\nu'}{\nu'-\nu} Im\, A_2(\nu') + 1/\pi \int_0^\infty \frac{d\nu'}{1+\nu'+\nu} \left(1 + \frac{\nu}{6(\nu'+\nu+1)}\right) f_2(\nu') + 1/3\, I_1(\nu).$$

The function f_i entering here are determined in (2.5).

Eqs. (2.10) are analogous to Eqs. (21)-(23) from paper /4/. However, there is one essential difference. The fact is that Eqs. (21)-(23) contain terms of the type

$$\int_0^\infty \frac{d\nu'}{\nu'} Im\, A_1(\nu'), \qquad (2.11)$$

which do not depend on ν and do not vanish in the interval of large ν's. Eqs. (2.1)-(2.3) can not therefore be satisfied by the logarithmic asymptotics and, consequently, have no solutions. This remark does not concern the subtracted Eqs. (25) -(27) /4/, which are therefore not equivalent to the non-subtracted Eqs. (21) - (23).

The presence of the terms (2.11) in the Ho, Chang and Zollner equations is explained by the fact that these authors have started not from dispersion relations with the fixed cosine c of the type (1.4), but from dispersion relations with the fixed t, which for $t \neq 0$ contain unphysical low-energy contributions from regions in which the cosine of the scattering angle changes in the limits $1 \leq / c / < \infty$. We investigate the logarithmic asymptotics of the system (2.10).

Assuming
$$A_i(\nu) \approx \frac{\pi}{c_i \ln \nu}, \tag{2.12}$$

we get from (2.10) the system of equations for the coefficients
$$\pi d_i - d_i^2 = \sum_k \sigma_{ik} d_k^2, \tag{2.13}$$

where
$$\sigma_{ik} = \begin{pmatrix} 1/3 & -4 & 5/3 \\ -2/15 & 7/10 & 1/3 \\ 1/3 & 2 & 1/6 \end{pmatrix}. \tag{2.14}$$

This system has a single non-trivial solution
$$d_0 = 2{,}13, \qquad d_1 = -0{,}137, \qquad d_2 = 0{,}653. \tag{2.15}$$

which recently was found by Lovelace/2/. A remarkable property of this solution is its proximity to (2.7). From the comparison of numerical coefficients it is seen, that the logarithmic asymptotics turn out to be rather stable with respect to taking into account of d and f - waves. It is interesting to consider the influence of the real parts of highest waves on the asymptotics. We study at once the limiting case of taking into account all waves by using the formulae (1.3). In the equations for partial waves written down below only terms which contribute to the logarithmic asymptotics are retained

$$A_0(\nu) \approx 1/\pi \int_0^\infty \frac{d\nu'}{\nu' - \nu} \, Im\, A_0(\nu') + 1/\pi \int_0^\infty \frac{d\nu'}{1+\nu+\nu'} \, f_0(\nu') - 4(2\ln 2 - 1) I_1(\nu)$$

$$A_1(\nu) \approx 1/\pi \int_0^\infty \frac{d\nu'}{\nu' - \nu} \, Im\, A_1(\nu') + 3/2\pi \int_0^\infty \frac{d\nu'}{1+\nu+\nu'} \, f_1(\nu') + (3 - 4\ln 2) I_1(\nu)$$

(2.16)

$$A_2(\nu) \approx 1/\pi \int_0^\infty \frac{d\nu'}{\nu'-\nu} \; Im\; A_2(\nu') + 1/\pi \int_0^\infty \frac{d\nu'}{1+\nu+\nu'} \; f_2(\nu') + 2(2\ell n 2 - 1)\, I_1(\nu)\,. \qquad /2.16/$$

To these equations there corresponds the matrix

$$\sigma_{ik} = \begin{pmatrix} 1/3 & -3(4\ell n\, 2-1) & 5/3 \\ -1/6 & 21/4(1-\frac{24}{21}\ell n\, 2) & 5/12 \\ 1/3 & 3/4(4\ell n 2-1) & 1/6 \end{pmatrix} \qquad (2.17)$$

and, respectively, the asymptotic coefficients

$$d_0 = 2{,}15\,, \qquad d_1 = -0{,}167\,, \qquad d_2 = 0{,}667\,. \qquad (2.18)$$

Comparing the numerical coefficients of (2.18) with those of (2.15) and (2.7) we see that the logarithmic asymtotics of the partial wave equations obtained by differential method reduces very rapidly to its limit (2.18). In considering more accurately the real part of the scattering amplitude we see also that the equations of the differential method do not turn into the corresponding Chew-Mandelstam equations.

3. Problem of the Account of High-Energy Partial Waves

We discuss the significance of the results obtained. In constructing the low-energy scheme we have to deal with two approximations. It is, on the one hand, an elastic approximation in the unitarity condition. On the other hand, it is a restriction to lowest partial scattering waves. The second approximation touches upon the real as well as imaginary parts of the scattering amplitude; it may therefore be performed in different ways. As has been shown in § 1, the approximation in Re A and Im A should be 'self-consistent'. The account of Re A_4 makes no sence, without introducing Im A_2.

The Chew-Mandelstam equations[6] are an example of the unstable scheme. In these equations approximations in Re A were not made while Im A is approximated by the s- and p- waves. The Chew-Mandelstam equations do not allow, therefore, to introduce into Im A even d- and f- waves, and, apparently[2,10] have no solutions.

It is appropriate to raise the question of the improvement of the equations such as (2.4), (2.10), owing to taking into account more and more high partial waves in the scattering amplitude by means of formulae such as (1.3). Here the temptation can arise to take into account an infinite number of partial waves, or what is equivalent, an infinite number of terms in the sums (1.3) for the real and imaginary parts of the scattering amplitudes and to obtain in such a way 'exact' equations which do not contain highest partial waves neglected.

Such an 'improvement' , however, has no sence for two different reasons.

The first reason is that in the region of not too low energies where highest waves can turn out to be important, contributions from inelastic processes which were neglected by us play an essential role. If we assume, however, that for some reason the contributions from inelastic processes are small in their absolute magnitude, or what is equivalent, we consider the model in which the inelastic processes are forbidden. Even in this case the indicated 'improvement' can not be made because of the presence of spectral functions. We explain this fact taking as an example the s- wave. We notice that the first formula of (1. 3) may be considered as a result of integration over c of the two Taylor series for the function f (c) in points c = + 1 and c = -1 . In this case the Taylor series at the point c = –1 is integrated in the interval (–1,0) and the series at the point c = 1 in the interval (0,1). Thus, in order that the sums (1.3) may exist it is necessary that the region of the convergence of the two Taylor's series cover entirely the physical interval (–1,1).

This requirement is fulfilled for any $\nu > 0$ for the function specified by the spectral representation of the kind (1.4), provided that the numerators of integrals have no singularities with respect to c (i.e. that the spectral functions are absent), and provided that the polinoms with respect to ν with high energy coefficients of the type (1.8) are omitted. These two conditions lead to the fact that, for example, the second integral in (1.4) is expanded in the Taylor's, series only near the point c = + 1 the singularities with respect to c being specified by its numerator.

The situation changes essentially in taking into account spectral functions. Then the analiticity region is defined by the Lehmann ellipsis and, as is easily seen, the series (1.3) for the real part of the amplitude diverges in the region $\nu > 2$.

However, for this conditions too, the finite sum of terms from (1.3) can give a good approximation. Integrating by parts N times we obtain for the s- wave the following expression

$$f_0 = \tfrac{1}{2} \int_{-1}^{+1} dc\, f(c) = \tfrac{1}{2} \sum_{n=0}^{N-1} \frac{f^{(n)}(-1) + (-)^n f^{(n)}(-1)}{(n+1)!} + \frac{(-1)^N}{2\, N!} \int_{-1}^{+1} f^{(N)}(c)\, c^N\, dc\, , \tag{3.1}$$

Assuming that in the energy region, we are interested highest partial waves starting with f_m are small we may omit in (3.1) a remaining term for $N = m$ and get expressions of the type (1.1), (1.2).

Hence, it follows that the scheme which takes into account a small number of lowest partial waves can give a good approximation in the low-energy region. Taking into account of an infinite number of terms in the sums (1.3), on the one hand does not improve really the accuracy because of the presence of inelastic processes and, on the other hand, it is impossible mathematically because of the presence of spectral functions. In this case the series (1.3) should be considered as asymptotics.

To illustrate this thesis we consider one more scheme of the succesive account of partial waves. We shall expand the second integral in (1.4) near the point $c = +1$ and the third one - near the point $c = -1$ and use these expansions along the hole interval $(+1, -1)$, instead of omitting high-energy coefficients. In this case the imaginary parts are approximated by the s- and p- waves (we imply the scattering of charged mesons). If we shall go on to an infinite number of terms then non- subtracted equations will not exist since the imaginary parts of crossing integrals have a pole with respect to c at the point $c = +1$ or $c = \beta - 1$ due to the presence of the p- wave. One subtraction leads exactly to the Chew-Mandelstam equations and to all mathematical complications/10/, connected with them.

The difficulty connected with spectral functions can be partially removed by taking into account the elastic two - particle part of spectral functions. So, we arrive at the program of 'the strip approximation' of Chew and Frautschi/11/. However, unlike these authors, we expect that the account of the spectral functions in the elastic strips changes only slightly the low-energy approximation. On the contrary, the behaviour of the scattering amplitude in the high-energy region and small momentum transfers may turn out to be completely determined by the low-energy scattering properties. Such a perspective seems to be especially probable in the light of the recent results of the paper/12/.

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