ОБЪЕДИНЕННЫЙ ИНСТИТУТ ЯДЕРНЫХ ИССЛЕДОВАНИЙ

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ON THE EXTRAPOLATION OF THE EXPERIMENTAL SCATTERING AMPLITUDE TO THE SPECTRAL FUNCTION REGION

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In the last time many authors[1-6] used the conformal mapping approach in various problems connected with analytical properties of the scattering amplitude. An interesting application was done by Lovelace[1] who calculated the spectral function by extrapolating the experimental data for $\pi + N \to \pi + N$ at 5.17 GeV. In the present note we will make some remarks on the convergence of this procedure.

The following properties of the conformal mapping approach can be pointed out:

1) Any function $f(w)$ which is real for $-1 < w < +1$ and analytical in the unit circle of the w-complex plane has automatically the analytic properties required for the amplitude, with the singularities in the correct position. The jump across the cut is obtained directly by calculating $f(w)$ on the unit circle and taking the imaginary part.

2) The Taylor expansion in $w(z)$ converges in every point of the cut z-plane (in which the amplitude is known to be analytical).

3) The conformal mapping which carries the whole cut z-plane into the unit circle leads to the fastest convergence if compared to all other mappings which carry only a certain part of the cut z-plane into the unit circle (see Appendix of[2]).

From points 2 and 3 it follows that the comformal mapping technique is a powerful tool for analytical continuation of the amplitude from $-1 \leq z = \cos \theta \leq 1$ to any point of the cut plane. However when studying the w-expansion on the cut (boundary of the unit circle) there arise problems of convergence.

One can, in principle, overcome them - and by the same enhance the convergence in every point - by mapping conformally into the unit circle the cut z-plane together with some additional region of the nonphysical Riemann sheets. But, at high energies, one encounters two difficulties:

Since the position of the resonance poles is not known, there exists the danger that some of them could be taken into the interior of the unit circle reducing by this the convergence of the expansion in all directions (see the dotted circle of Fig. 1.). Secondly, the great number of branching points of the spectral function at high energies considerably complicates the practicle construction of the conformal transformation (the cut transforms into the complicated curve of Fig. 1). At 5.17 GeV there are fifteen branching points lying in the strip $4 \leq t \leq 16$, at:

$t_0 = 4.043$	$t_1 = 4.33$	$t_2 = 4.54$	$t_3 = 5.19$	$t_4 = 5.42$
$t_5 = 6.03$	$t_6 = 6.47$	$t_7 = 7.13$	$t_8 = 8.60$	$t_9 = 8.68$
$t_{10} = 9.27$	$t_{11} = 10.59$	$t_{12} = 10.86$	$t_{13} = 12.95$	$t_{14} = 13.35$

In the following we will study the convergence on the cut without appealing to the properties of the amplitude on the nonphysical Riemann sheets, but using the fact that the spectral function has no singularities worse than $\frac{1}{\sqrt{t_1 - t}}$. One can feel the analogy with the well-known Chew method /7/ of obtaining the coupling constant by extrapolation to the pole: in his method an essential point was the fact that he had some apriori information as to the power of the pole denominator.

For sake of simplicity, we shall treat the case with cuts $(-\infty, -a)$ and (a, ∞) and require $w(z=0)=0$, i.e.

$$w(z) = \frac{\sqrt{a+z} - \sqrt{a-z}}{\sqrt{a+z} + \sqrt{a-z}}$$

but the results remain unchanged also in the general case. The dispersion relation in the w-plane is

$$A_1(w) = \frac{1}{\pi} \int_C K(w', w) \frac{\sigma(w')}{w'-w} dw'$$

where $K(w', w) = \frac{1-w'^2}{1+w'^2} \frac{1+w^2}{1-ww'}$, C is the unit circle and $\sigma(w')$ is the limit value of A_1 on C, which is supposed to vanish at $w = \pm i$ (corresponding to $t = \pm\infty$). As $K(w' w)$ has no pole inside C and $K(w, w) = 1$, for $|w| < 1$, we have directly $A = 1/\pi \int_C \frac{\sigma(w')}{w'-w} dw'$.

On C, we define
$$\sigma_{eff}^{N} = \frac{1}{2i} \sum_{0}^{N} c_n e^{in\phi}$$

where c_n is $2i$-times the Fourier coefficient $f_n = 1/2\pi \int_0^{2\pi} \sigma(\phi) e^{-in\phi} d\phi$.

As $A_1(w) = \sum_0^\infty c_n w^n$, using the Schwartz lemma one can easily show that for physical values of w

$$\left| 1/\pi \int_C \frac{\sigma(w') - \sigma_{eff}^N(w')}{w'-w} d w' \right| = \left| \sum_{N+1}^\infty c_n w^n \right| \leq |w| \mathcal{E}_N; \quad \mathcal{E}_N = \sum_N^\infty c_{n+1} w^n \Big|_{z=1} \approx \frac{w^N(z=1)}{1-w(z=1)}$$

($z=1$ is the physical point at which the expansion converges most slowly).

This is equivalent with the trivial fact that the 'effective spectral function' gives a good approximation for $A_1(w)$ in the physical region of the original reaction. However, if we want to use the obtained information for the low-energy region of the crossing reaction, as σ does no more appear under the integral over t, we need the convergence of $\sum c_n w^n$ directly on the boundary $w = e^{i\phi}$.

In other words, we need the convergence of the Fourier expansion

$$\sigma = \sum_{n=-\infty}^{+\infty} f_n e^{in\phi}$$

where $f_n = \frac{1}{2i} c_n$ for $n > 0$ and $f_n = -f_{-n}$. This formula follows from the fact that only the antisymmetrical part of σ is important in (1) because

$$\frac{K(e^{i\phi}, w)}{e^{i\phi} - w} \quad \frac{\partial e^{i\phi}}{\partial \phi} = tg\,\phi \quad \frac{1 + w^2}{(e^{i\phi} - w)(e^{-i\phi} - w)}$$

which is an odd function. Furthemore, the spectral function is defined as $\frac{A_1(z + i\epsilon) - A_1(z - i\epsilon)}{2i}$

A singularity of the spectral function having the form $\frac{1}{\sqrt{a - z}}$, turns into $\frac{a\sqrt{w^2 + 1}}{\sqrt{(aw - a)^2 - (a^2 - a^2)}}$

which is Fourier-expandable with the exception of the case $a = a$, which produces a pole. This pole leads to the divergence of the series on the boundary, but it can be removed by shifting the $w(z)$ -cut to another point $a' < a$, or - better - by expanding $\sqrt{a - z}\, A_1$ rather than A_1*. The higher singularities are positive powers of the root and do not cause any difficulties (we are indebted to Prof. Ter-Martirosyan for this remark). Thus, we may conclude that the expansion of $\sqrt{a - z}\, A_1$ into $w^n(z)$ can be used for calculating the spectral function, which is now represented by a convergent Fourier series (divided by $\sqrt{a - z}$). If we have an analogous situation for the left-hand cut, then $\sigma(z) = \frac{1/2i}{\sqrt{a^2 - z^2}} \sum_{n=-\infty}^{+\infty} c_n e^{in\phi}$, where now $c_{-n} = +c_n$.

Returning to the paper of C. Lovelace we would like to remark that the function $e^{-b\eta^2}$ has not the correct behaviour in the interval $0 < t < 4$ where all derivatives of A_1 must be positive**. Secondly, we note that the possible zeros of A_1 might reduce considerably the convergence radius of the expansion $\ln A_1 = \sum a_n \eta^n$ which is used in /1/, even if the experimental fit is good. However both difficulties can easily be avoided by another fitting of the experimental data, which fulfills the above mentioned requirements.

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* The analogy with the Chew approach seems to be deeper than it was expected

** We thank Prof. I.Ya. Pomeranchuk who pointed out this fact to us.

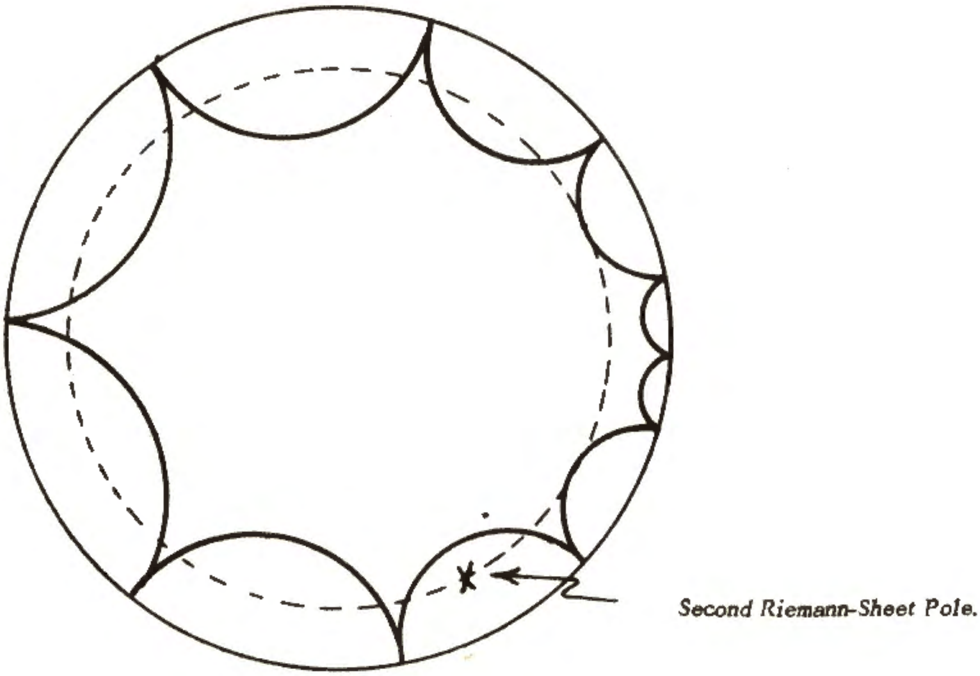

Fig. 1.

References

1. C.Lovelace " Diffraction Scattering and Mandelstam Representation", Preprint, Imperial College, London, Sept. 1961
2. S.Ciulli and J.Fischer, Nucl.Phys. 24, 465 (1961), Dubna prepr. D-615, June 1960.
3. S.Ciulli and J.Fischer. "Integral Equations for $\pi\pi$ Scattering and Convergence Problems of the Amplitude Expansion", Dubna preprint, February 1961, and ЖЭТФ , 41, 1 (7) 256.
4. D.Greenberger and B.Margolis. Phys.Rev.Lett 6, 6 , 310 (1961).
5. W.R.Frazer. " Application of Conformal Mapping to the Phenomenological Representation of Scattering Amplitude", Preprint Univer. of California, 1961.
6. J.E.Bowcock and J.C.Stoddart. " Pion-Pion Scattering and Nuclear Form Factors". Manchester University, preprint 1961).
7. G.F.Chew. Phys.Rev. 112, 1380 (1958).

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