ОБЪЕДИНЕННЫЙ ИНСТИТУТ ЯДЕРНЫХ ИССЛЕДОВАНИЙ

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D- 784

PRINCIPLE OF LEVELLING THE STATES AND ITS APPLICATION TO THE CALCULATIONS OF MULTIPLE PRODUCTION

Nucl. Phys., 1962, v37, n3, p425-437.

Дубна 1961

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PRINCIPLE OF LEVELLING THE STATES AND ITS APPLICATION TO THE CALCULATIONS OF MULTIPLE PRODUCTION

For estimating the probability of the multiple production the covariant expression for the phase volume in the momentum space begins recently to be more frequently used. It is applied as an original model (the covariant model[1-6]), and as a component of the calculations by the Chew-Low scheme [7]. For the calculations of the covariant phase volume the recurrent relations are usually used. They are the basis for the tables and graphs[4,8] on which the phase volume is plotted against energy for some reactions. The analytical formulae[1,9-11] are known only for some limite (by the masses) cases, for the production of three particles and for any number of identical particles [8] *. The finite expressions for the phase volume of arbitrary systems of particles are absent.

This paper makes an attempt to fill up the gap. A new principle is formulated of drawing approximate formulae for the phase volumes followed up to the calculation formulae in case of the covariant phase space integrals. In conclusion the fitness of these phase volumes for the calculations of the average particle production multiplicity is shown.

1 The Principle of Levelling the States

This principle is widely used in the calculation of multiple integrals by the Monte-Carlo method and is known as 'importance sampling'. As far as we know, to use it as an analytical method is something new.

Meanwhile, the analysis of the application of the 'importance sampling' at the electronic computers shows that this procedure is most succesful, when it reduces to the (analytical) change of the integration variables decreasing the variation of the integrand.

The speed with which the Monte-Carlo process converges to the magnitude of the integral $S=\int \Phi(\vec{a})\, d\vec{a}$ is given by the formula **

$$N = \frac{\eta}{\epsilon}\,\delta^2, \qquad (1)$$

where

N is the necessary number of random points

$1-\frac{1}{\eta}$ is the reliability of the result

ϵ is the relative accuracy of the result

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* The formula from [16] cannot be used because of a misprint.

** It is assumed here that the points $\vec{a}$ are distributed over the integration region uniformly.

δ^2 is the degree of the variation $\Phi(\vec{a})$

$$\delta^2 = \frac{T - S^2}{S^2} \tag{2}$$

$$T = \int \Phi^2(\vec{a})\, d\vec{a} \tag{3}$$

Hence it is seen that the convergence may be accelerated whatever much by decreasing δ^2; in the limit $\Phi(\vec{a}) \to \text{Const}$ $\delta^2 \to 0$. It is possible to achieve the decreasing of δ^2 by such a choice of the variables which makes the integrand (in new variables) more close to the constant, levells it.

The levelling principle may be also serve for the analytical calculation of the integral S. Let there be found such a change of the variables that in new variables δ^2 is a small number throughout the interval of the change of the parameters on which the integral depends. Then, instead of the formula for S one may make an attempt to draw a formula for the calculation T (3), and S will be then determined from (2):

$$S = \frac{1}{\sqrt{1 + \delta^2}} \sqrt{T} \simeq \sqrt{T} \tag{4}$$

Thus, instead of the integral of the function the levelling makes it possible to calculate the integral of the squared function; it may turn out that the latter problem is simpler (say, if $\Phi(\vec{a})$ is the product of the square roots).

Moreover, formula (2) may be generalized. If $\Phi(\vec{a})$ is close to the constant, then, for some sequence of the operations F the quantity

$$\delta^2 \equiv \frac{\int F(\Phi(\vec{a}))\, d\vec{a} - F(\int \Phi(\vec{a})\, d\vec{a})}{F(\int \Phi(\vec{a})\, d\vec{a})} \tag{5}$$

is close to zero. As F (x) may be chosen the operations x^2, x^4, $\ln x$, etc, and their choice depends on the form of the function Φ.

If $\int F(\Phi(\vec{a}))\, d\vec{a}$ is calculated easier than $\int \Phi(a)\, da$, then

$$\int \Phi(\vec{a})\, d\vec{a} \simeq F^{-1}\left[\frac{\int F(\Phi(\vec{a}))\, d\vec{a}}{1 + \delta^2}\right] \tag{4'}$$

Another method of calculating the integrals of the levelled functions consists in applying to them a theorem of the mean

$$\int \Phi(\vec{\alpha})\, d\vec{\alpha} = \Phi(\tilde{\alpha}) \int d\vec{\alpha} \qquad (6)$$

For the functions which change little over the integration interval, it is easier to find the dependence of the point position $\tilde{\alpha}$ on the integral parameters.

Although the idea of the levelling is very simple, a concrete application of formulae (4) – (6) requires further efforts. The search for the levelling substitutions $\alpha \to \alpha'$, the proof of the smallness δ^2, the principle of the choice of the mean point $\tilde{\alpha}$ depend upon the skill of the calculator.*

However, if applied to the phase volumes the matters become simpler in view of their generally known monotony with respect to the change of one of the volume parameter (the kinetic energy of the system r_n or the mass of one of the particles m_k) for the fixed values of other parameters. Further, in the limiting cases of the ultrarelativistic ($\mu_n \equiv m_1 + \ldots + m_n = 0$) and non-relativistic ($r_n << \mu_n$) particles the phase volume is calculated accurately. Therefore, it is possible:

1) to look for the representation levelling the density of the states $\Phi(\alpha)$ in these limiting cases ;

2) to become convinced in the smallness of δ^2 in the limiting cases as well;

3) to search for the mean point $\tilde{\alpha}$ so that in the limiting cases (6) would be accurately satisfied.

Further, these principles will be applied to the covariant phase volume.

2. Levelling Representations.

The covariant phase volume of n particles with the masses $m_1, \ldots, m_n$, the momenta $p_1, \ldots, p_n$ and with the energies $e_1, \ldots, e_n$, with the total energy M_n is

$$S_n(M_n) = \int \frac{d^3 p_1 \ldots d^3 p_n}{2e_1 \ldots 2e_n} \delta^4(p_1 + \ldots + p_n - P_n),\ P_n^2 = -M_n^2. \qquad (7)$$

It may be transformed into a repeated integral /11/

$$S_n(M_n) = \frac{(2\pi)^{n-1}}{2M_n} \int_0^{\bar{r}_n} d\bar{r}_{n-1} \int_0^{\bar{r}_{n-1}} d\bar{r}_{n-2} \ldots \int_0^{\bar{r}_3} d\bar{r}_2 \cdot \bar{p}_n \ldots \bar{p}_2 , \qquad (8)$$

where $\bar{r}_k$ is the kinetic energy of particles 1, 2, ..., k , in their rest system, $\bar{p}_k$ is the momentum of the

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* Sometimes physical considerations may help the searching of the levelling representations.

particle k in this system *. $\bar{p}_k$ depends upon $\bar{r}_{k-1}, \ldots, \bar{r}_2$, according to the recurrent formulae

$$\bar{p}_k^2 = \frac{[(M_k - m_k)^2 - M_{k-1}^2][(M_k + m_k)^2 - M_{k-1}^2]}{4M_k^2} \tag{9}$$

$$M_k = \bar{r}_k + \mu_k; \quad \bar{r}_1 \equiv 0 . \tag{10}$$

$$\mu_k = m_1 + \ldots + m_k . \tag{11}$$

The integration region in (8) can be transformed in a many-dimensional unit cube in many ways. Two such changes of variables used in /11/ are levelling ones.

A) a method of ordering the energies. Let $\hat{\pi}$ be commutation operator of the numbers $\bar{r}_{n-1}, \ldots, \bar{r}_2$ (forming a certain vector $\vec{r}$) in the order of increasing. Let us determine the function $\Phi = \bar{p}_n \ldots \bar{p}_2$ given on the simplex $\bar{r}_n \geq \ldots \geq \bar{r}_2$ in all the hypercube in the following manner

$$\Phi(\vec{r}) = \Phi(\hat{\pi}\vec{r}) . \tag{12}$$

In other words, the values of the function Φ at the point with arbitrary coordinates ($\lesseqgtr \bar{r}_n$) are equal to its value at the point with the ordered (commutated in order of increasing $\ldots \leq \bar{r}_k \leq \bar{r}_{k+1} \leq \ldots$) energies. Let us make the substitution

$$\bar{r}_k = \bar{r}_n \, a_k . \tag{13}$$

It will lead to

$$S_n(M_n) = \frac{(2\pi)^{n-1}}{2(n-2)!} \frac{r_n^{n-2}}{M_n} \int_0^1 \ldots \int_0^1 da_{n-1} \ldots da_2 \prod_2^n \bar{p}_k \tag{14}$$

B) a method of covariant levelling consists in the two-staged change of variables

$$\bar{r}_k = \bar{r}_{k+1} \sqrt{\chi_k} \qquad (k = n-1, \ldots, 2) \tag{15}$$

$$a_k = k\,\chi_k^{k-1} - (k-1)\,\chi_k^k \tag{16}$$

Now (8) turns into

$$S_n(M_n) = \frac{(\pi/2)^{n-1}}{(n-2)!\,(n-1)!} \frac{r_n^{2n-3}}{M_n} \int_0^1 \ldots \int_0^1 da_{n-1} \ldots da_2 \prod_2^n \frac{\bar{p}_k \, 2\bar{r}_k}{\bar{r}_k^2 - \bar{r}_{k-1}^2} = \tag{17}$$

* If compared with /11/ the order of the enumeration is inverse.

$$= \frac{\pi^{n-1}}{(n-2)!\,(n-1)!}\ \frac{r_n^{n-2}}{M_n}\int_0^1\!\!\cdots\!\int_0^1 d\alpha_{n-1}\cdots d\alpha_2 \prod_2^{n-1} \frac{1}{\chi_k^{\frac{k-1}{2}}\,(1-\chi_k)} \prod_2^n \bar{p}_k \qquad (18)$$

To prove the levelling properties of the substitutions A and B we calculate δ^2 (see (2)) for the limiting cases α ($\mu_n = 0$) and β($r_n << \mu_n$) . In these cases (14) and (17) are divided into the product of the one-dimentional integrals what makes the calculation easier. The expressions for S_n are already known /1,9,11/, there remains only to calculate T (see (3)).

A) Here

$$T_n^{(A)}(M_n) = \int \Phi^2\, d\vec{\alpha} = \left[\frac{(2\pi)^{n-1}\ r_n^{n-2}}{2M_n\,(n-2)!}\right]^2 \int_0^1\!\!\cdots\!\int_0^1 d\alpha_{n-1}\ \cdots d\alpha_2 \prod_2^n \bar{p}_k^2 = \qquad (19)$$

$$= \left[\frac{(2\pi)^{n-1}\ r_n^{n-2}}{2M_n}\right]^2 \frac{1}{(n-2)!}\int_0^1 d\alpha_{n-1}\int_0^{\alpha_{n-1}} d\alpha_{n-2}\ \cdots\int_0^{\alpha_3} d\alpha_2 \prod_2^n \bar{p}_k^{-2}\ ;$$

$$T_n^{(A)}(M_n) = \frac{(2\pi)^2}{n-2}\ \frac{r_n^{n-2}}{M_n^2}\int_0^{r_n} d\bar{r}_{n-1}\ \frac{\bar{p}_n^2\, M_{n-1}^2}{\bar{r}_{n-1}^{n-3}}\ T_{n-1}^{(A)}(M_{n-1})\ . \qquad (20)$$

By a direct calculation from (19) and by the induction method from (20) it is possible to get

A α) for $\mu_n = 0$

$$T_n = \frac{2^{3n-5}\ \pi^{2n-2}\ r_n^{4n-8}}{(n-2)!\,(3n-4)!\,(3n-2)} \qquad (21)$$

Taking into account that

$$S_n\ (\mu_n = 0) = \frac{(\pi/2)^{n-1}}{(n-2)!\ (n-1)!}\ r_n^{2n-4}\ , \qquad (22)$$

we obtain

$$\delta_n^2 = \frac{2^{5n-7}\ (n-2)!\ (n-1)!^2}{(3n-4)!\,(3n-2)} - 1\ ; \quad \delta_3^2 = .22;\ \delta_4^2 = .96;\ldots;\ \delta_{10}^2 = 3.1;\ldots \qquad (23)$$

A β) for $r_n << \mu_n$

$$\bar{p}_k^2 = \frac{2m_k\ \mu_{k-1}}{\mu_k}\ (\bar{r}_k - \bar{r}_{k-1})\ , \qquad (24)$$

so that

$$T_n = \frac{(2^3\pi^2)^{n-1}}{4\,(n-2)!(2n-3)!}\;\frac{\prod_1^n m_k\; r_n^{3n-5}}{\mu_n^3} \tag{25}$$

Comparing with

$$S^2_{n\,(r_n \ll \mu_n)} = \frac{(2\pi)^{3(n-1)}}{2^{2n}\,\Gamma^2(3/2(n-1))}\;\frac{\prod_1^n m_k\; r_n^{3n-5}}{\mu_n^3}\,, \tag{26}$$

we get

$$\delta_n^2 = \frac{(4/\pi)^{n-1}\,\Gamma^2(3/2(n-1))}{(n-2)!(2n-3)!} - 1;\quad \delta_3^2 = .08,\ \delta_5^2 = .24;\ \ldots \delta_{10}^2 = .79;\ \cdots \tag{27}$$

The monotony of δ^2 with respect to the change of the parameters follows from physical considerations (see also Table 1). Therefore, for any M_n , m_1, ..., m_n $\quad .08 \le \delta_3^2 \le .22; .21 \le \delta_4^2 \le .96; \ldots$ so that for n = 3,4 , the representation (A) we have is a levelling one.

B) For this choice of variables

$$T_n^{(B)} = \frac{\pi^{2(n-1)}\;r_n^{2(n-2)}}{[(n-2)!\;(n-1)!\;M_n]^2}\int_0^1\!\!\ldots\!\int_0^1 d\alpha_{n-1}\ldots d\alpha_2 \prod_2^{n-1}\frac{1}{\chi_k^{k-1}(1-\chi)^2_k}\prod_2^n \bar{p}_k^{\,2} = \tag{28}$$

$$= \frac{\pi^{2(n-1)}\;r_n^{2(n-2)}}{(n-2)!\;(n-1)!\;M_n^2}\int_0^1\!\!\ldots\!\int_0^1 d\chi_{n-1}\ldots d\chi_2 \prod_2^{n-1}\frac{1}{\chi_k(1-\chi_k)}\prod_2^n \bar{p}_k^{\,2}\,, \tag{29}$$

or

$$T_n^{(B)}(\bar{r}_n) = \frac{2\pi^2}{(n-2)\,(n-1)}\,\frac{\bar{r}_n^{\,2n-2}}{M_n^2}\int_0^{\bar{r}_n} d\bar{r}_{n-1}\;\frac{M_{n-1}^2\;\bar{p}_n^2}{r_{n-1}^{2n-5}\;(\bar{r}_n^2 - \bar{r}_{n-1}^2)}\;T_{n-1}(\bar{r}_{n-1}). \tag{30}$$

Hence, for $\mu_n = 0$ follows

$$B\ \alpha) \qquad T = S^2\,,\quad \delta^2 \equiv 0. \tag{31}$$

This is also seen from (17) where the factors at $\mu_n = 0$ are all equal to 1.

$B\ \beta)$ $r_n << \mu_n$ Now (see (24) and (15))

$$p_k^2 = \frac{2m_k}{\mu_k}\,\mu_{k-1}\,\bar{r}_n\,\chi_{n-1}^{1/2}\ldots\chi_k^{1/2}\,(1-\chi_{k-1}^{1/2})\,, \tag{32}$$

and

$$T_n = \frac{(2\pi^2)^{n-1}\ r_n^{3n-5}}{(n-2)!\ (n-1)!}\ \frac{\prod_1^n m_k}{\mu_n^3}\ \prod_1^n \int_0^1 d\chi_k\ \frac{\chi_k^{\frac{k-3}{2}}}{1+\chi_k^{1/2}} \qquad (33)$$

The integrals $u_k = \frac{1}{2}\int_0^1 d\chi_k \frac{\chi_k^{\frac{k-3}{2}}}{1+\chi_k^{1/2}}$ are calculated by the formula $u_k = \frac{1}{k-2} - u_{k-1}$, $u_2 = \ln 2$, so that we have .

$$\delta_n^2 = \frac{2^n\ \Gamma^2(3/2(n-1))}{\pi^{n-1}\ (n-2)!(n-1)!}\ \prod_2^{n-1} u_k - 1;\quad \delta_3^2 = .07;\ \delta_4^2 = .13;\ \ldots\ \delta_{10}^2 = 1.13;\ \ldots$$

So, the representation B leads to comparatively small δ^2 even for n = 10.

3. Calculation Formulae

Being aware of the levelling representations, one can proceed to the deduction of approximate formulae for the phase volume. We have already seen that the approximate constancy of the density of the states $\Phi(\vec{\alpha})$ in the $\vec{\alpha}$ – representation may be used in two ways: 1) by reducing according to (4) the calculation of the phase volume to that of the integral of the square of the state density and 2) by applying the theorem of the mean.

1 method. From (4), (19), (20), (28)-(30) we get

$$S_n \simeq (1+\delta_n^2)^{-1/2}\ \frac{(2\pi)^{n-1}\ r_n^{\frac{n-2}{2}}}{2M_n\sqrt{(n-2)!}}\left[\int_0^{r_n} d\bar{r}_{n-1} \int_0^{\bar{r}_{n-1}} d\bar{r}_{n-2} \ldots \int_0^{\bar{r}_3} d\bar{r}_2\ \bar{p}_n^{\,2} \ldots \bar{p}_2^{\,2}\right]^{1/2},\ n=3,4 \qquad (35)$$

$$S_n \simeq (1+\delta_n^2)^{-1/2}\ \frac{\pi^{n-1}\ r_n^{n-3/2}}{M_n[2(n-2)!\ (n-1)!]^{1/2}}\left[\int_0^{r_n} d\bar{r}_{n-1} \ldots \int_0^{\bar{r}_3} d\bar{r}_2 \prod_2^n \frac{2\bar{r}_k\ \bar{p}_k^{\,2}}{\bar{r}_k^{\,2} - \bar{r}_{k-1}^{\,2}}\right]^{1/2}; \qquad (36)$$

$$(n = 3 - 10)$$

The accuracy of (35)-(36) is rather high. So, even for n = 20 (36) gives the inaccuracy of not more than 25%.

$$S_{20} = (1.0 \pm 0.5)\ T_{20}^{1/2},$$

especially if we take for δ^2 the expression

$$\delta^2 = (\mu_n / M_n)^{1/2}\ \delta^2_{r_n \ll \mu_n} \qquad \text{(see (34)).}$$

It follows from (9), (10) that $\bar{p}_k^2$ is the rational function $\bar{r}_{k-1}, \ldots, \bar{r}_2$. Therefore, the integrals in (35)-(36) are expressed in elementary functions.

So, for n =3, it follows from (35)

$$S_3 = (.93 \pm .03)\, T_3^{1/2}; \quad T_3 = \frac{\pi^4 r_3}{4M_3^4}\Big[\frac{a_3^{7/2} - a_2^{7/2}}{7} - \frac{a_3^{5/2} - a_2^{5/2}}{5}\sum_1^4 a_i + \frac{a_3^{3/2} - a_2^{3/2}}{3}\sum_{i>j}^4 a_i a_j - a_1 a_4\sqrt{(a_2 a_3)}\,(\sqrt{a_3} - \sqrt{a_2})\,(\sqrt{(a_2 a_3)}\sum_1^4 a_i^{-1} + 1)\Big], \tag{37}$$

$$a_{1,2} = (m_1 \pm m_2)^2; \quad a_{3,4} = (M_3 \pm m_3)^2 .$$

An exact formula for three arbitrary particles is much more cumbersome. /11/

From (36) one can get a simple formula for the phase volume of n−1 ultra-high energy (m_k=0) particles and one arbitrary particle with the mass $m_n = m$ (cf. with (B.3) in /11/). Having substituted into (36) $\bar{p}_{n-1}^2 = \frac{(\bar{r}_n^2 - \bar{r}_{n-1}^2)\,[(2m + \bar{r}_n)^2 - \bar{r}_{n-1}^2]}{(2M_n)^2}$ and $T_{n-1} = S_{n-1}^2$, we get

$$S_n(M_n) = \frac{(\pi/2)^{n-1}\, m^{2n-3}}{\sqrt{1+\delta_n^2}\,(n-1)!\,(n-2)!\,M_n^{2n-4}}\sqrt{[4mM_n(n-1) + \bar{r}_n^2]} \tag{38}$$

2. The second method. The theorem of the mean in the combination with the representation B will give an approximate formula

$$S_n \simeq \frac{\pi^{n-1}\, r_n^{n-2}}{(n-2)!\,(n-1)!\,M_n}\prod_2^{n-1}\frac{1}{\tilde{\chi}_k^{\frac{k-1}{2}}(1-\tilde{\chi}_k)}\prod_2^n \tilde{\bar{p}}_k \tag{39}$$

where $\tilde{\chi}_k$, $\tilde{\bar{p}}_k$ (k=2, ..., n-1) must be taken at a certain intermediate point. To find it we require that (39) would be fulfilled exactly in the non-relativistic case for all n (in another limiting case (39) is fulfilled automatically). This leads (see (25) and (32)) to the equation for $\tilde{\chi}_2, \ldots, \tilde{\chi}_{n-1}$

$$\frac{2^{\frac{n-3}{2}}\,\pi^{(3/2)(n-1)}}{\Gamma(3/2\,(n-1))}\Big(\frac{m_1 \ldots m_n}{\mu_n^3}\Big)^{1/2} r_n^{\frac{3n-5}{2}} = \frac{2^{\frac{n-1}{2}}\,\pi^{n-1}\, r_n^{\frac{3n-5}{2}}}{(n-2)!\,(n-1)!}\Big(\frac{m_1 \ldots m_n}{\mu_n^3}\Big)^{1/2} \times \tag{40}$$

$$\times \int_0^1 \ldots \int_0^1 d\alpha_{n-1} \ldots d\alpha_2 \prod_2^{n-1}\frac{\sqrt{1-\sqrt{\chi_k}}}{\chi_k^{\frac{k-1}{4}}(1-\chi_k)},$$

hence

$$\prod_{2}^{n-1} \int_{0}^{1} d\alpha_k \frac{\sqrt{1-\sqrt{\chi_k}}}{\chi_k^{\frac{k-1}{4}}(1-\chi_k)} = \frac{\pi^{\frac{n-1}{2}}(n-2)!(n-1)!}{2\Gamma(3/2(n-1))} \tag{41}$$

Let $\chi_k = \tilde{\chi}_k$ be chosen for $k = 2,3, \ldots, n-2$ so that

$$\prod_{2}^{n-2} \frac{(1-\sqrt{\tilde{\chi}_k})^{\frac{1}{2}}}{\tilde{\chi}_k^{\frac{k-1}{4}}(1-\tilde{\chi}_k)} = \prod_{2}^{n-2} \int_{0}^{1} d\alpha_k \frac{(1-\sqrt{\chi_k})^{\frac{1}{2}}}{\chi_k^{\frac{k-1}{4}}(1-\chi_k)} = \frac{\pi^{\frac{n-2}{2}}(n-3)!(n-2)!}{2\Gamma(3/2(n-2))}$$

then for $\chi_{n-1} = \tilde{\chi}_{n-1}$ the equation is obtained by the theorem of the mean

$$\frac{(1-\chi_{n-1})^{\frac{1}{2}}}{\chi_{n-1}^{\frac{n-2}{4}}(1-\chi_{n-1})} = \frac{\sqrt{\pi}(n-2)(n-1)\Gamma(3/2(n-2))}{\Gamma(3/2(n-1))} \tag{42}$$

It is necessary to choice the least root of (42). The roots (42) $\chi_{n-1} = \tilde{\chi}_{n-1}$ should then be substituted into (39). For $n \leq 10$, the quantities $\tilde{\chi}_{n-1}^{\frac{1}{2}}$ and the coefficients

$$C_n = \frac{\pi^{n-1}}{(n-2)!(n-1)!} \prod_{2}^{n-1} \frac{1}{\tilde{\chi}_k^{\frac{k-1}{2}}(1-\tilde{\chi}_k)} \tag{43}$$

are listed in the Table 2. Finally, we have the formula applicable for any masses and energies

$$S_n = C_n \frac{r_n^{n-2}}{M_n} \bar{p}_n \cdots \bar{p}_2 \tag{44}$$

$$\bar{p}_k^2 = \bar{e}_k^2 - m_k^2 ; \quad \bar{e}_k = \frac{M_k^2 + m_k^2 - M_{k-1}^2}{2M_k} ; \quad M_{k-1} = \bar{r}_{k-1} + \mu_{k-1} ; \tag{45}$$

$$\bar{r}_{k-1} = \bar{r}_k \tilde{\chi}_{k-1}^{\frac{1}{2}} ; \quad \mu_{k-1} = m_1 + \ldots + m_{k-1} .$$

This formula for different m_k up to $n = 10$ was compared with accurate values of S_n calculated by the tables from /8/. For identical particles (see the Table 3) its error is not high than 3% (by the way, for this case in /8/ a more convenient formula (10') is found). For non-identical (by their mass) particles the order of their enumeration affects the magnitude of the error. The best order of enumeration is symmetrical (see the Table 3). When the mass of only one particle is different from the rest ones (e.g., in the system $N\underbrace{\pi \ldots \pi}_{n}$), then the number $[n/2] + 1$ has to be ascribed to it; if as in the system $2N\underbrace{\pi \ldots \pi}_{n}$ two particles have the same mass, and the other ones have a different mass, then the mean numbers $[n/2]+1$, $[n/2]+2$ are also ascribed to these two particles; when there are three kinds of particles (just as in

$K\Sigma\pi\ldots\pi$) the best enumeration is $\pi\pi\pi\Sigma K\pi\pi\pi$. For 3 –4 particles the accuracy is little dependent on the order of enumeration. Two opposite orders (e.g., $3\pi 4N 3\pi$ and $2N 6\pi 2N$) lead to the errors of the opposite sign so that the mean geometric value gives an increased accuracy. Up to n = 10 it is possible by using such methods to gain the accuracy not lower than 6%.

If $r_n \ll \mu_n$ the calculation of $\bar{p}_k$ by (45) may be followed by the loss in accuracy. In this case $\bar{p}_k$ is better to be calculated by (24).

Note, that by (44) the distribution by the effective mass of group of particles can be expressed in a closed form (but not in terms of an integral as usual).

If out of n particles, ν have the masses $m_{11}, \ldots, m_{1\nu}$ and the effective mass m_{eff} , and the rest $n-\nu$ particles have the masses $m_{21} \ldots , m_{2,n-\nu}$, and if the phase volume of any k particles with the masses $m_1, \ldots, m_k$ and the energy M are to be denoted by $S_k (M; m_1 \ldots m_k)$, then from (2.20) [11 2] follows

$$\frac{dS_n}{dm_{eff.}} = 2m_{eff}\ S_\nu(m_{eff}; m_{11}..m_{1\nu})\, S_{n-\nu+1}(M_n; m_{eff}\ m_{21}\ldots m_{2,n-\nu}) \qquad (46)$$

4. Comparison of the co-variant model with experiment

The formulae of the present paper and the tables of paper /8/ allow to compare the predictions of the co-variant model with the experiment. It has been already compared with the experimental data on pp-scattering /2/ and, in the main, on the pair annihilation /3,4,13,14/. In these papers it has been emphasized that this model reflects satisfactorily different mean characteristics of the multiple production.

It remains, however, obscure how to choose the only fitting parameter of the model k (see /2/) or λ (see below) for pp-interaction.

The expression for the statistical weight of the reaction may be put as in /6/ either in the form

$$W_n = \left(\frac{(4\pi/3)(\lambda/m_\pi)^3}{8\pi^3} \right)^{n-1} f_{T,S} \prod_1^n 2m_k\ S_n(M_n) \qquad (47)$$

or

$$W_n = \left(\frac{(4\pi/3)(\lambda/m_\pi)^3}{8\pi^3}\ \frac{m_a + m_b}{M_n} \right)^{n-1} f_{T,SI} \prod^n 2m_k\ S_n(M_n)\,. \qquad (48)$$

Here m_π is the meson mass, a and b are two colliding particles, $f_{T,S}$ gives the number of isotopic and spin states. (48) is written so that it would coincide with the Fermi model in the non-relativistic case. For the annihilation of $N\bar{N}$ at rest there is no difference between (47) and (48). In /2/ formula (47) was used (without $\prod 2m_k$).

To compare these two ways of introducing the parameter λ the data on inelastic pp-scattering at 9 BeV [15] were taken. The average multiplicity is here $\bar{n}_s = 3.34 \pm 0.6$. The calculation of the probability of 1–9 meson production (by [8] and (44)) has shown that such $\bar{n}_s$ is obtained if in (47) λ is taken to be .75 or in (48) to be 1. For these two values of λ the calculation was then made for the energy range from 1,5 up to 23 BeV (Table 4). In another Table 5 the distribution in the number of prongs is compared with experiment (a list of experimental data was kindly communicated by I.M.Gramenitski whom the author expresses his gratitude).

It follows from the tables 4,5 that (47) for $\lambda = .75$ and (48) for $\lambda = 1$ are both in good agreement with experiment. If one considers that $\lambda = 1$ is an exact number, but not a fitting parameter, one can prefer (48). Then the final calculation formula is as follows

$$W_n(M_n) = \left(\frac{m_a + m_b}{6\pi^2 \, m_\pi^3 \, M_n}\right)^{n-1} f_{T,S} \prod_1^n 2 m_k \; S_n(M_n) \qquad (49)$$

Further, it follows from the same tables that the covariant model agrees with the experimental data just as well as the Fermi model. Since the calculation formulae for the first of them are simpler the covariant phase volumes can be recommended for estimating the average multiplicities or mean momenta [6], as well as for the calculations by the Chew-Low scheme.

5. Some words about the calculations of the Fermi model

To check the assertion about the monotony S_n we have made the calculations of the statistical weight by the Monte-Carlo method (using B). Some results are presented in Table 1. The Fermi model was calculated in this way together with the covariant model. As these calculations have shown : 1) The quantities δ^2 up to the multiplicities n = 16 - 20 are of the same order and small enough in both models; 2) Therefore, (see table 6) the calculation of the Fermi weight S_n by the Monte-Carlo method up to n = 20 with an accuracy of 5-10% takes 5-10 minutes at the electronic computer M-20 ($N = 10^3$). This time should be compared with 10^3 hours necessary for the calculation of S_n for n = 12 - 14 by the method used at CERN [18].

The author is grateful to L. Isajeva, V.Khlaponina, and V.Komolova for making necessary calculations.

Table 1.

Linear dependence of δ^2 on $(\mu_n/M_n)^{1/2}$ at r_n = const for the system of 5 identical particles.

$z_n=(\mu_n/M_n)^{1/2}$	0	0,57	0,71	0,85	0,91	1
numerical experiment	0	0,21	0,26	0,29	0,31	0,35
$\delta^2=\delta^2_{max} z_n$	0	0,20	0,25	0,30	0,32	0,35

Table 2.

The constants of formulae (44) , (45).

k	$\sqrt{\tilde{\chi}_{k-1}}$
2	0,0000
3	0,3420
4	5191
5	6170
6	0,6805
7	7253
8	7590
9	7851
10	0,8060

n	C_n
3	16,34
4	43,45
5	78,20
6	106,7
7	117,4
8	108,4
9	86,21
10	60,27

Table 3.

Errors in formula (44) (in %)

The order of particle enumeration n, n-1, ... 2,1	$\sqrt{\mu_n}/M_n$				
	0,2	0,4	0,6	0,8	0,95
3π	- 1	- 1	- 2	- 1	0
4π	- 1	- 2	- 2	- 2	- 1
6π	0	- 1	- 2	- 2	0
8π	1	1	- 1	- 2	0
10π	3	3	0	- 1	0
$3\pi d 3\pi$	- 5	- 6	- 4	- 2	- 1
$4\pi d 4\pi$	- 5	- 5	- 3	- 2	1
$4\pi 2K 4\pi$	- 3	- 4	- 3	- 2	0
$3\pi 2K 4\pi$	- 2	- 2	0	0	0
$K\Sigma\pi$	- 2	- 3	- 1	0	0
$\pi\Sigma K\pi$	- 2	- 3	- 2	0	0
$3\pi K\Sigma 3\pi$	- 1	- 3	- 4	- 6	- 5

Table 4.

The calculation of $\bar{n}_s$ for p-p -interaction at different energies by the covariant model.

(1) – by (47)

(2) – by (48)

A list of experimental data – from /15/.

energy T_p (BeV)	(1) $\bar{n}_s$	(2) $\bar{n}_s$	(exp.) $\bar{n}_s$	(2) $\bar{n}_\pi$	(exp.) $\bar{n}_\pi$
1,5	2,12	2,13	2,12	1,34	1,2
2,75			$2,36 \pm 0,06$		1,8
3,0	2,36	2,43	$2,37 \pm 0,07$	1,8	1,7
4,2	2,59	2,67	2,9	2,14	
6,2	2,94	3,00	$3,14 \pm 0,1$	2,62	$2,8 \pm 0,4$
6,2				– " –	$2,27 \pm 0,27$
8,7			3		
8,7 and 9,0	3,3	3,3	$3,34 \pm 0,06$	3,1	3,14
2 3	4,3	4,0	$4,1 \pm 0,6$	4,1	

Table 5.

The comparison of the distribution in the number of prongs n for p-p - and p-n -interaction at 9 BeV /15/ with the covariant (49), Fermi's /12/ and Chernavski's model /17/.

n	2	4	6	8	$\bar{n}_s$
[12]	32,8%	58,5	8,6	0,1	3,53
[17]	35	58,9	6,0	0,1	3,46
(49)	41,2	52,6	6,2	0,03	3,32
[15]	45,4 ± 2,8	43,6 ± 2,8	9,5 ± 1,3	1,5 ± 0,5	3,34 ± 0,06
n	1	3	5	7	$\bar{n}_s$
[12]	14,5%	59,4	25,0	1,1	3,25
[17]	18,4	65,2	15,7	0,7	2,96
(49)	21,5	60,5	17,5	0,5	2,94
[15]	32,3 ± 3	49,7 ± 3,5	13,5 ± 1,9	4,2 ± 1,1	2,81 ± 0,08

Table 6.

The calculation of S_n for n ultra-relativistic particles by the Fermi model. $N = 10^3$. The time of calculation is 5-10 minutes. k_n is the scale factor.

n	$k_n S_n$	exact value.
5	$0{,}216 \cdot 10^{-3}$	$0{,}218 \cdot 10^{-3}$
7	$0{,}864 \cdot 10^{-8}$	$0{,}878 \cdot 10^{-8}$
10	$0{,}687 \cdot 10^{-16}$	$0{,}690 \cdot 10^{-16}$
12	$0{,}505 \cdot 10^{-10}$	$0{,}504 \cdot 10^{-10}$
14	$0{,}102 \cdot 10^{-12}$	$0{,}104 \cdot 10^{-12}$
16	$0{,}151 \cdot 10^{-15}$	$0{,}165 \cdot 10^{-15}$
20	$0{,}131 \pm 0{,}007$	

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Received by Publishing Department on August 9, 1961.