ОБЪЕДИНЕННЫЙ ИНСТИТУТ ЯДЕРНЫХ ИССЛЕДОВАНИЙ

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ON MEANING OF GAUGE INVARIANCE

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1.1. Electrodynamics is the only field theory, which is completed to some extent and confirmed by experiment. Such a theory is invariant not only under the group of charged field transformations

$$\psi'(x) = \exp[ie\Lambda]\psi(x) \qquad \Lambda = \text{const} \tag{1}$$

(this group is connected with charge conservation) but also under a wider group of gauge transformations

$$\psi'(x) = \exp[ie\Lambda(x)]\psi(x) \tag{2}$$

where the group parameter (phase) $\Lambda(x)$ is an arbitrary function of a space-time point x. Simultaneously the 4-vector of electromagnetic field undergoes the transformation

$$A'_\mu(x) = A_\mu(x) + \frac{\partial\Lambda(x)}{\partial x_\mu} . \tag{3}$$

Up to now the physical meaning of gauge invariance is still rather mysterious. It will be shown below that the gauge invariance can be understood as the condition for 4-vector field A_μ to describe spin 1 only. In the particular case of limited gauges this meaning had been cleared up in our previous paper/18/.

1.2 By analogy to the transition from (1) to (2) one can attempt to suppose an arbitrary dependence on x of the parameters of other known groups also, e.g., of the isotopic group, of groups of transformations connected with the baryon and hyperon-charge conservation laws and so on. Then one obtains new gauge transformations. Recently a series of attempts have been undertaken to raise the invariance under such transformations to the level of principle and to construct a theory of strong and weak interactions by means of this gauge principle. These attempts have been made in papers of Yang and Mills, Lee and Yang, Utiyama, Salam and Ward, Sakurai, Glashow and Gell-Mann and others/1-17/.

1.3. The phylosophy of the papers/1-13/ is based on the belief, that gauge principle necessitates the existence of massless vector fields coupled to corresponding conserved currents. For instance, it is stated that the requirement of the invariance under transformation (2) leads automatically from an initial free lagrangian, say, for the spinor field:

$$L_0(x) = -\bar{\psi}\left(\gamma_\mu \frac{\partial}{\partial x_\mu} + m\right)\psi , \tag{4}$$

to the electrodynamical lagrangian

$$L(x) = -\bar{\psi}(\gamma_\mu \frac{\partial}{\partial x_\mu} + m)\psi + j_\mu A_\mu - \tfrac{1}{4} F_{\mu\nu} F_{\mu\nu} \tag{5}$$

where

$$j_\mu = ie\,\bar{\psi}\gamma_\mu \psi \tag{6}$$

$$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu} \; . \tag{7}$$

Thus, as if with necessity[1-13], the electromagnetic field – the massless vector field $A_\mu(x)$ appears.

1.4. However, this statement is not correct. In fact, gauge invariance will be assured if instead of lagrangian (5), one takes the lagrangian

$$L(x) = -\bar{\psi}(\gamma_\mu \frac{\partial}{\partial x_\mu} + m)\psi + j_\mu \frac{\partial B}{\partial x_\mu} \tag{8}$$

and prescribes to the scalar field $B(x)$ the gauge transformation

$$B'(x) = B(x) + \Lambda(x) \, . \tag{9}$$

Lagrangian (8) has no free part for the field $B(x)$ and therefore it does not lead to the equation of motion for this field.

1.5. But the same situation takes place with lagrangian (5) also. Four-vector A_μ can be decomposed symbolically as*

$$A_\mu = (A_\mu - \Box^{-1}\frac{\partial}{\partial x_\mu}\frac{\partial A_\nu}{\partial x_\nu}) + \Box^{-1}\frac{\partial}{\partial x_\mu}\frac{\partial A_\nu}{\partial x_\nu} \equiv A^1_\mu + A^0_\mu \tag{10}$$

where A^1_μ corresponds to three degrees of freedom and does not change at gauge transformations (3), A^0_μ corresponds to one degree of freedom and takes the whole gauge variation of A_μ

$$A^{0\prime}_\mu = A^0_\mu + \frac{\partial \Lambda}{\partial x_\mu} \quad ; \quad A^{1\prime}_\mu = A^1_\mu \tag{11}$$

A^0_μ and A^1_μ describe spin 0 and 1, respectively, as one can see with the help of the invariant operator of the square of the spin momentum for vector field[19,18]:

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* $\Box^{-1}$ in formulae (10) and (12) should be understood as an integral operator, e.g. $\Box^{-1} f(x) = \int dy\, D_F(x-y)\, f(y)$ where D_F is the causal Green function of D'Alembert operator.

$$(S^2)^{\mu\nu} = 2(\delta_{\mu\nu} - \Box^{-1} \frac{\partial^2}{\partial x_\mu \partial x_\nu}) . \qquad (12)$$

1.6. The free lagrangian for A_μ contains a field tensor $F_{\mu\nu}$ only. This tensor is gauge invariant and consequently does not depend on the part of A_μ which changes at the gauge transformations.

In fact,

$$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu} = \frac{\partial A^1_\nu}{\partial x_\mu} - \frac{\partial A^1_\mu}{\partial x_\nu} . \qquad (13)$$

Therefore, lagrangian (5) gives the equation of motion not for the field A_μ on the whole, but for the gauge independent part A^1_μ only.

The gauge dependent part A^0_μ enters into an interaction term only and lagrangian (5) does not give any equation for it. The very fact that A^0_μ changed by quite an arbitrary function $\frac{\partial \Lambda}{\partial x_\mu}$ means that A^0_μ cannot obey any equation. The situation is exactly the same as for the field B in the case of lagrangian (8).

Therefore, lagrangian (5) which does not give the equation for A^0_μ is not a bit better than lagrangian (8) which does not give the equation of motion for scalar field B. Moreover, three components of A_μ for which (5) gives equations have nothing to do with gauge invariance as they fail to undergo gauge transformations.

2. Thus, we are led to the following conclusions on the gauge principle:

2.1. The gauge principle does not lead to electrodynamical lagrangian (5), as having a dynamical manifestation part A^1_μ is introduced quite arbitrarily. Therefore references on 'minimality' are out of place.

2.2. This principle causes, as we have shown above, the appearance of some scalar field only though it were B or $\Box^{-1} \frac{\partial A_\nu}{\partial x_\nu}$ which have no dynamical manifestation. The 'minimal' lagrangian which followed directly from gauge principle is lagrangian (8).

2.3. The scalar field B under consideration does not obey any equation and does not lead to any dynamical manifestation. It is quite fictitions and harmless. It can be removed by the canonical point transformation of field variables*

$$\psi'(x) = \exp[\, ie\, B(x)\,]\, \psi(x) . \qquad (14)$$

In fact, lagrangian (8) describes the free field ψ, but at the same time the so desired /1-13/ arbitrariness of phases at different space time points ('locality') have been achieved.

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* The similar transformation in a similar situation has been applied in /18/. General questions concerning such transformations have been discussed in /20/.

2.4. Such an invariance can be introduced into any theory if one switches on the field B by means of the transformation inverse to (14).

3. As for the meaning and the significance of the gauge invariance the above consideration allows us to say the following:

3.1. Though it is impossible to deduce the existence of a vector field A_μ from the gauge invariance, one can postulate the existence of such a field. After this, of course, it is possible to switch on the interaction with this postulated field in a gauge invariant manner, e.g. as in lagrangian (5). Then four-vector A_μ will describe spin 1 only.

3.2. Actually, gauge invariance plays the role similar supplementary conditions in the higher spin theory: it limits the number of degrees of freedom*. This limitation is realized in such a way, that zero-spin part turns out to be quite arbitrary instead of being excluded. And really, just to this part quite arbitrary function $\frac{\partial \Lambda}{\partial x_\mu}$ is added at gauge transformations. Hence, it cannot be and in fact it does not follow any equation for this part from lagrangian. It enters only in the interaction term.

3.3. The absence of the equation of motion does not permit the interpretation of the corresponding field variable in a mechanical language of conjugate coordinates and moments, and, therefore it is impossible to carry out conventional quantization**. This explains the difficulty of the quantization of Maxwell's equations – it is impossible to obtain commutation relations for components of the vector-potential A_μ . But at the same time it seems natural that commutation relations can be obtained for gauge invariant (independent of $\overset{\circ}{A}_\mu$) quanities /21/. Long ago this has been made by Heisenberg and Pauli for a field tensor $F_{\mu\nu}$ /22/.

3.4. As the spin 1 field A^1_μ is gauge independent then, apparently, the gauge invariance is not connected with problems of the mass of these quanta and of the universality of the electric charge.

4. Many authors implied that a Lagrangian density L (x) is gauge invariant strictly and that Λ (x) is quite an arbitrary function. Then, as we have seen, there is no equation of motion for the gauge dependent component. Such an equations can be postulated if one considers Lagrangian density which is invariant only up to a four-divergence and a gauge functions Λ (x) which are limited by an appropriate equation,

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* In addition, one of Maxwell equations plays the role of the ordinary supplementary condition. This reduces the number of degrees of freedom corresponding A_μ to 2.

** Another quantization method – method Fermi – till now containes some vagueness (an indefinite metric, a meaning of a supplementary condition).

For example, such a Lagrangian density is

$$L(x) = -\bar{\psi}(\gamma_\mu \frac{\partial}{\partial x_\mu} + m)\psi + j_\mu \frac{\partial B}{\partial x_\mu} - \frac{1}{2}\frac{\partial B}{\partial x_\mu}\frac{\partial B}{\partial x_\mu} - \frac{M^2}{2} BB. \qquad (15)$$

It is invariant up to four-divergence under transformations

$$\psi'(x) = \exp[\, ie\,\Lambda(x)\,]\psi(x), \quad B'(x) = B(x) + \frac{1}{M^2}\Box\,\Lambda(x) \qquad (16)$$

provided

$$\frac{\partial}{\partial x_\mu}(\Box - M^2)\,\Lambda(x) = 0. \qquad (17)$$

Correspondingly, it is possible to switch on gauge invariant interaction with the massive neutral vector field*[18]

$$L(x) = -\bar{\psi}(\gamma_\mu \frac{\partial}{\partial x_\mu} + m)\psi + j_\mu A_\mu - \frac{1}{2}\frac{\partial A_\nu}{\partial x_\mu}\frac{\partial A_\nu}{\partial x_\mu} - \frac{M^2}{2} A_\nu A_\nu. \qquad (18)$$

This density is invariant again up to four-divergence under transformation

$$\psi'(x) = \exp[\, ie\,\Lambda(x)\,]\,\psi(x), \quad A'_\mu(x) = A_\mu(x) + \frac{\partial\Lambda(x)}{\partial x_\mu} \qquad (19)$$

provided that the same condition (17) is satisfied. The scalar field B and the part of A_μ with zero spin $\frac{1}{M^2}\frac{\partial}{\partial x_\mu}\frac{\partial A_\nu}{\partial x_\nu}$[18] obey free Klein-Gordon equation. Now they can be quantized but being free they have no dynamical manifestations. They can be excluded by means of the contact transformation of field variables[18,20] or the unitary Dyson transformation[24].

In detail the gauge invariant theory of the massive neutral vector field has been considered in our paper[18].

5. As to an Yang-Mills isotopic gauge invariance here we note only that one can guarantee it without introducing vector field. For instance, an isotopic gauge invariant Lagrangian is

$$L(x) = -\bar{\psi}(\gamma_\mu \frac{\partial}{\partial x_\mu} + m)\psi + \bar{\psi}\exp(-ig\tau_a b_a)\gamma_\mu\left[\frac{\partial}{\partial x_\mu}\exp(ig\tau_a b_a)\right]\psi \qquad (20)$$

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* Recently the hypothesis of the existence of neutral vector mesons has been intensively discussed. See I.Kobsarev's and L.Okun's paper[23] and the papers quoted there.

where $b_a(x)$ are three scalar in ordinary space function. When ψ transforms according to a law

$$\psi'(x) = \exp [\, ig\, \tau_a \lambda_a(x) \,]\, \psi(x)$$

transformation of b_a is defined not uniquely. For example, we quote two possibilities (written in implicit form)

$$\exp [\, ig\, \tau_a\, b'_a(x) \,] = \exp [\, ig\, \tau_a\, b_a(x) \,] \cdot \exp [-ig\, \tau_a \lambda_a(x) \,]$$

or

$$\exp [\, ig\, \tau_a\, b'_a(x) \,] = \exp [\, ig\, \tau_a \lambda_a(0) \,] \cdot \exp [\, ig \tau_a\, b_a(x) \,] \cdot \exp [-ig \tau_a\, \lambda_a(x) \,]$$

The introduced field $b_a(x)$ has as many components as there are gauge functions $\lambda_a(x)$. Yang-Mills vector field contains more components, than needed*. Our field does not lead to any dynamical manifestations as it can be excluded by means of the point transformation of field variables $\psi \to \exp [\, ig\, \tau_a\, b_a \,]\, \psi$ [18,19].

Ikeda and Miyachi [5], Salam and Ward [11] and Glashow and Gell-Mann [13] considered vector fields, which seem necessary for generalized Yang-Mills invariances to be guaranteed. It is easy to see, that here one can also do without introducing vector fields, without superfluous components. It is quite enough to use scalar fields which have no dynamical manifestations. Corresponding Lagrangians will be an evident generalization of Lagrangian (20).

These examples show once more the impossibility of deducing vector fields from the gauge principle.

It is worthwhile to note that an investigation of meaning of the gravitational gauge invariance and a criticism of a corresponding gauge principle [4,17] may be made similarly.

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* Of course, all statements of Section 3 are valid for the Yang-Mills vector field also.

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