ОБЪЕДИНЕННЫЙ ИНСТИТУТ ЯДЕРНЫХ ИССЛЕДОВАНИЙ

Лаборатория теоретической физики

B.V.Medvedev and M.K.Polivanov

D-694

ON THE DEGREES OF GROWTH OF MATRIX ELEMENTS IN THE AXIOMATIC METHOD

ЖЭТФ, 1961, т41, в4, с1130-1141.

Дубна 1961

B.V.Medveded and M.K.Polivanov *

ON THE DEGREES OF GROWTH OF MATRIX ELEMENTS IN THE AXIOMATIC METHOD

Submitted to JETP

* V.A.Steklov Mathematical Institute of the Academy of Sciences, Moskow, USSR

The general physical principles supplemented by the requirement of renormalizability are shown to lead to very strong restrictions on the admissible degrees of growth of matrix elements when imposed on the scattering matrix in the "axiomatic" approach.

1. Introduction.

A great attention was drawn during the last five years to the general structure of the local-theory of quantized fields[1-4]. The main problem to be explored here is the question, how close do the general principles only – these being the (1) Lorentz covariance, (2) unitarity and completness of the set of positive energy states, and (3) locality – define the theory without any dynamical specifications, that are necessarily being made in the theory based on the Hamiltonian formalism.

The system of basic physical principles may be formulated in many different ways. It seems to us most convenient to emphasize the scattering matrix, as it was first proposed by Heisenberg and to formulate the physical principles as requirements imposed on its matrix elements. Some local operators should necessarily be introduced into the theory besides the S-matrix. Indeed, without these we could not distinguish between the different space-time points and formulate the causality condition. To do this, we write down the S-matrix as an expansion in normal products of asymptotic fields

$$S = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int dx_1 \dots dx_n \, \Phi^n(x_1, \dots, x_n) : \phi(x_1) \dots \phi(x_n) : \qquad (1)$$

and then extend it out of the energy-shell, removing the condition

$$(\Box - m^2)\, \phi(x) = 0. \qquad (2)$$

Now, the local Heisenberg operators desired, may be obtained by taking variational derivatives of the extended S-matrix with respect to the ϕ-fields.

Such a system of fundamental principles was proposed by Bogolubov[6] for a theory with adiabatic switching the interaction and – as demonstrated by Bogolubov and Shirkov[7] – in the frame of perturbation theory it leads essentially to the same conclusions as the usual Lagrangian formalism supplemented by the renormalization programme. Later on this system of basic principles was reformulated by Bogolubov and the present authors[4]*, this scheme being especially suited to the derivation of dispersion relations and the

* Hereafter referred to as PTDR.

Kallen-Lehmann spectral representations. The method to develop the quantum field theory based on the system of fundamental physical principles as formulated in PTDR, Sec. 2, and making use of dispersion relations technique we will call later on the dispersion approach. The meaning of dispersion approach to the quantum field theory is not

exhausted by the limited number of exact results achieved thereby, this approach being able to give a new trend to develop the whole theory.

If, in particular, we would try to satisfy all the requirements of dispersion approach with the formal series in powers of some small parameter, then we may find out the solution term by term – as it is customary in the perturbation theory. What we gain here over the usual theory, is that we do not have to resort any more to the physically unsatisfactory procedure of adiabatic switching of the interaction. Furthermore, only the renormalized quantities may be used, and thus we avoid the question about the relations between the "renormalized" and "unrenormalized" quantities, this question being meaningless in the present theory. It was pointed out recently[8] that in this way successive perturbation terms are determined up to a finite number of constants coming out instead of the finite counter terms* of the Hamiltonian method.

* It is not out of place to mention here still another advantage of the dispersion approach. It does especially clarify, whence do the counterterms come and where or do the divergencies of the usual theory arise. Namely the cause is the incorrect procedure of multiplying the functions not sufficiently regular by the θ-functions, this being equivalent in the momentum space to application of Cauchy's integral formula to a function which does not decrease at infinity, without taking the contribution of the great circle into account. (Cf. the discussion in PTDR, Sec. 1 and 4).

The precise number of constants need is governed by the degreed of growth of matrix elements. The interaction Lagrangian being given, the degrees of growth are defined in the well-known manner. It was claimed repeatedly[1,4] that in the dispersion approach the prescription of definite degrees of growth may substitute in some respects the adopting of the specific form of the Lagrangian. On the other hand, it seemed obvious[1,8] that we cannot set those quite arbitrarily.

The aim of this paper is to clear up the extent up to which the differen matrix elements degrees of growth can be precribed arbitrarily. Somewhat surprisingly, the arbitrariness proves to be very small and for the most important class of the "properly renormalizable theories" the degrees of growth need not to be set up by a special postulate, but are almost uniquely determined by the sytem of fundamental principles of PTDR and the transformation properties of the fields.

The investigation which concern here the simplest case of spinless self-interacting field may be performed without explicit use of the perturbation theory.

2. Coupled equations for matrix elements.

The system of basic principles, as formulated in PTDR permits to derive in numerous ways sets

coupled equations connecting the generalized vertices with different numbers of external lines, i.e. the matrix elements corresponding to different numbers of initial and final particles. Since the causality condition is used thereat we have – as just mentioned – to introduce some local Heisenberg operators besides the scattering matrix. Two such operators at least must be introduced which have the meaning of the first and second variational derivatives of the S-matrix or – to be precise they are the " radiative operators" (Cf. PTDR) of the first and second order. Then we may formulate the theory in such a manner that all external lines except respectively one or two would be " real" lines on the energy shell*.

So, we shall deal with the on the energy shell matrix elements.

$$J(p_1, \dots, p_\ell ; q_1, \dots, q_s) = \langle p_1, \dots, p_\ell \mid J \mid q_1, \dots, q_s \rangle \qquad (3)$$

and

$$J(x \mid p_1, \dots, p_\ell ; q_1, \dots, q_s) = \langle p_1, \dots, p_\ell \mid J(x) \mid q_1, \dots, q_s \rangle \qquad (4)$$

of the two Hermitian operators J and J(x). The first operator J is here just the Heisenberg current taken at the point x=0 to eliminate trivial dependence on the coordinate x:

$$J = j(0); \quad j(x) = i \frac{\delta S}{\delta \phi(x)} S^+. \qquad (5)$$

By translation invariance, its matrix elements are connected with those $j(x \mid \dots p \dots ; \dots q \dots)$ of the operator j(x) as follows:

$$J(p_1, \dots, p_\ell ; q_1, \dots, q_s) = j(x \mid p_1, \dots, p_\ell ; q_1, \dots, q_s) e^{-i(\Sigma p_i - \Sigma q_j)x}. \qquad (6)$$

The second operator J(x) is the retarded radiative operator of the second order :

$$J(x) = - \frac{\delta j(-x/2)}{\delta \phi(x/2)} \quad ; \quad J^+(x) = J(x) \qquad (7)$$

the trivial coordinate dependence being again eliminated.

---

* The authors are indebted to N.N.Bogolubov who drew their attention to the convenience of such a treatment.

Its matrix elements coincide with the functions $F^{ret}_{\alpha\omega}$, being used in PTDR. Note, that the arguments of the matrix elements eq. (3), include in fact one out of the energy shell momentum

$$P = \Sigma p_i - \Sigma q_j \neq 0 \qquad (8)$$

and these of matrix elements eq. /4/ – two such momenta that given by eq. /8/ and that corresponding to explicitly written down co-ordinate x.

One may see that irrespectivd of the causality condition, the matrix elements eqs /3/ and /4/ are connected by the relations*:

* The operator $P\left(\frac{q_1}{q_2, \ldots, q_s}\right)$ in eqs. /9/ is the symmetrising operator defined in /7/, Sec. 18.

$$J(p, p_1, \ldots, p_\ell; q_1, \ldots, q_s) = P\left(\frac{q_1}{q_2, \ldots, q_s}\right)\delta(p - q_1)\, J(p_1, \ldots, p_\ell; q_2, \ldots, q_s) - \frac{1}{(2\pi)^{3/2}\sqrt{2p^0}} \int dx\, J(x|p_1, \ldots, p_\ell; q_1, \ldots, q_s)\, e^{i\left(p + \frac{\sum p_j - \sum q_j}{2}\right)x}; \tag{9,1}$$

$$J(p_1, \ldots, p_\ell; q, q_1, \ldots, q_s) = P\left(\frac{p_1}{p_2, \ldots, p_\ell}\right)\delta(p_1 - q)\, J(p_2, \ldots, p_\ell; q_1, \ldots, q_s) - \frac{1}{(2\pi)^{3/2} 2p^0} \int dx\, J(x|p_1, \ldots, p_\ell; q_1, \ldots, q_s)\, e^{-i\left(q + \frac{p_1 - q_1}{2}\right)x} \tag{9.2}$$

and

$$J(x) - J(-x) = i\{j(x/2)\, j(-x/2) - j(-x/2)\, j(x/2)\}. \tag{10}$$

The causality condition provides an additional restriction on the operator J(x):

$$J(x) = 0 \quad \text{for} \quad x \leq 0. \tag{11}$$

Thus, we obtained a set of equations for the matrix elements of operators J and J(x). One may think that this system is sufficient to determine these operators. At any rate, we could show that this is really the case (Cf. /8/) in the frame of the perturbation theory.

To eliminate J (x) from the system obtained, we rewrite the Eq. (10) in terms of matrix elements:

$$J(x|p_1, \ldots, p_\ell; q_1, \ldots, q_s) - J(-x|p_1, \ldots, p_\ell; q_1, \ldots, q_s) = \tag{12}$$

$$= i \sum_{\nu=0}^{\infty} \frac{1}{\nu!} \int d k_\ell \ldots d k_\nu\, J(p_1, \ldots, p_\ell; k_1, \ldots, k_\nu)\, J(k_1, \ldots, k_\nu; q_1, \ldots, q_s)$$

$$\left(e^{i\left(\frac{\sum p_i + \sum q_j}{2} - \sum k_\nu\right)x} - e^{-i\left(\frac{\sum p_i + \sum q_j}{2} - \sum k_\nu\right)x}\right).$$

Now, we have to explioit the causality condition (11) imposed on $J(x)$. This condition demands $J(-x|\ \ldots p \ldots; \ldots q \ldots)$ to vanish for $x \leq 0$. To this end we may formally multiply Eq. (12) by the function $\theta(x^\circ)$;

$$J(x|p_1,\ldots,p_\ell;q_1,\ldots,q_s) = i\sum_\nu \frac{1}{\nu!}\int d\underline{k}_1 \ldots d\underline{k}_\nu$$

$$J(p_1,\ldots,p_\ell;k_1,\ldots,k_\nu)\,J(k_1,\ldots,k_\nu;q_1,\ldots,q_s)\cdot \tag{13}$$

$$\theta(x^\circ)\left(e^{i\left(\frac{\Sigma p+\Sigma q}{2}-\Sigma k\right)x} - e^{-i\left(\frac{\Sigma p+\Sigma q}{2}-\Sigma k\right)x}\right)$$

If the functions entering Eq. (12) are not sufficiently regular, the well-known divergences can certainly arise in a straightforward calculations. The dispersion relations theory tells that to avoid them some preliminary substractions have to be drawn. As a consequence, the right-hand-side of Eq(13) is to be suplemented with appropriate arbitrary polynomials in the momentum space. We shall understand the Eq.(13) and the forthcoming formulae, just in this sense without writting down the polynomials explicitly.

With this agreement we substitute (13) into eq. (9) (e.g. into the first one), thus arriving at an infinite set of coupled equations;

$$J(p,p_1,\ldots,p_\ell;q_1,\ldots,q_s) = P\left(\frac{q_1}{q_2,\ldots,q_s}\right)\delta(\underline{p}-\underline{q}_1)$$

$$J(p_1,\ldots,p_\ell;q_2,\ldots,q_s) - \frac{(2\pi)^{3/2}}{\sqrt{2p^0}}\sum_{\nu=0}^{\infty}\frac{1}{\nu!}\int d\underline{k}_1\ldots d\underline{k}_\nu \tag{14}$$

$$J(p_1,\ldots,p_\ell;k_1,\ldots,k_\nu)\,J(k_1,\ldots,k_\nu;q_1,\ldots,q_s)$$

$$\left(\frac{\delta(\underline{p}+\Sigma\underline{p}_j-\Sigma\underline{k}_\nu)}{\Sigma k^\circ_\nu-\Sigma p^\circ_i-p^\circ-i\epsilon} - \frac{\delta(\underline{p}-\Sigma\underline{q}_j+\Sigma\underline{k}_\nu)}{-\Sigma k^\circ+\Sigma q^\circ_j-p^\circ-i\epsilon}\right)$$

and at a similar set which may be derived from eq. /9.2/. Both of them have to determine the matrix elements of J.

It is convenient to deal not with the matrix elements $J(\ldots)$ themselves, but with the invariant matrix elements;

$$I(p_1,\ldots,p_\ell;q_1,\ldots,q_s)=\sqrt{2p^\circ_1\ldots 2p^\circ_\ell\ 2q^\circ_1\ldots 2q^\circ_s}\,J(p_1,\ldots,p_\ell;q_1,\ldots,q_s) \tag{15}$$

normalized in the common way. (In order to relate our notation to the usual one, remind that the number of external lines of the corresponding generalized graph exceed by one the number of the arguments of our functions, that are explicitly written down. This is because we make use of translation invariance.) In terms of the matrix elements $I(\ldots p \ldots;\ldots q \ldots)$ the basic equations read

$$I(p_1,\ldots,p_\ell;q_1,\ldots,q_s) = P\left(\frac{p_1}{p_2,\ldots,p_\ell}\right)\delta(p_1-q)\sqrt{2p_1^\circ 2q^\circ}\; I(p_2,\ldots,p_\ell;q_1,\ldots,q_s)$$

$$-(2\pi)^{3/2}\sum_\nu \frac{1}{\nu!}\int \frac{dk_1\ldots dk_\nu}{2k_1^\circ\ldots 2k_\nu^\circ}\, I(p_1,\ldots,p_\ell;k_1,\ldots,k_\nu)\, I(k_1,\ldots,k_\nu;q_1,\ldots,q_s) \qquad (16.1)$$

$$\left(\frac{\delta(p+\Sigma p_i-\Sigma k_\nu)}{\Sigma k_\nu^\circ-\Sigma p_i^\circ-p^\circ-i\epsilon} - \frac{\delta(p-\Sigma q_i+\Sigma k_\nu)}{-\Sigma k_\nu^\circ+\Sigma q_j^\circ-p^\circ-i\epsilon}\right).$$

The lower limit in the sum over ν is determined here in the following way: First of all, it is easily seen that only the connected graphs contribute to the current J; hence, the $\nu = 0$ does not enter. Further on due to the vacuum and one-particle states stability conditions, PTDR I, (6), we have to put

$$I(-;-) = I(-;q) = I(p;-) = 0. \qquad (17)$$

Then, if any of the numbers l or s is equal to zero the sum begins at $\nu = 2$.

Quite similarly the eq. (9.2) gives us the second half of the set of equations;

$$I(p_1,\ldots,p_\ell;q,q_1,\ldots,q_s) = P\left(\frac{p_1}{p_2,\ldots,p_\ell}\right)\delta(p_1-q)\sqrt{2p_1^\circ 2\,q^\circ}\; I(p_2,\ldots,p_\ell;q_1,\ldots,q_s)$$

$$-(2\pi)^{3/2}\sum_\nu \frac{1}{\nu!}\int \frac{dk_1\ldots dk_\nu}{2k_1^\circ\ldots 2k_\nu^\circ}\, I(p_1,\ldots,p_\ell;k_1,\ldots,k_\nu)\, I(k_1,\ldots,k_\nu;q_1,\ldots,q_s) \qquad (16.2)$$

$$\left(\frac{\delta(-q+\Sigma p_i-\Sigma k_\nu)}{\Sigma k_\nu^\circ-\Sigma p_i^\circ+q^\circ-i\epsilon} - \frac{\delta(-q-\Sigma q_i+\Sigma k_\nu)}{-\Sigma k_\nu^\circ+\Sigma q_j^\circ+q^\circ-i\epsilon}\right).$$

Let us remember once more the polynomials not written down explicitly we should remember to keep in mind in the r.h.s -es of eqs. (16), as we spoke about in connection with eq. (13).

The matrix elements with two on the energy shell legs do not enter the system (16), cf. (17). Hence, to include the self-energy-parts we have to go out of the energy shell. For sake of this we turn to the matrix elements (4) of the operator $J(x)$ and define their four-dimensional Fourier-transforms as

$$\bar{I}(k|p_1,\ldots,p_\ell;q_1,\ldots,q_a)=\frac{1}{\sqrt{2p_1^0\ldots 2p_\ell^0\,2q_1^0\ldots 2q_a^0}}$$

$$\int dx\, e^{ikx}\, J(x|p_1,\ldots,p_\ell;q_1,\ldots,q_a) \qquad (18)$$

(the vector k here is not subject to the condition $k^2=m^2$!). According to eq. (13), we can express these Fourier-transforms in terms of $I(\ldots)$ as

$$\bar{I}(k|p_1,\ldots,p_\ell;q_1,\ldots,q_a)=(2\pi)^3\sum_{\nu=0}^{\infty}\frac{1}{\nu!}\int\frac{dk_1\ldots dk_\nu}{2k_1^0\ldots 2k_\nu^0}$$

$$\left\{-\frac{\delta\left(k-\frac{\sum p_i+\sum q_j}{2}+\sum k_\nu\right)}{-k^0+\frac{\sum p_i^0+\sum q_j^0}{2}-\sum k_\nu^0-i\varepsilon}+\frac{\delta\left(k+\frac{\sum p_i+\sum q_j}{2}-\sum k_\nu\right)}{-k^0-\frac{\sum p_i^0+\sum q_j^0}{2}+\sum k_\nu^0-i\varepsilon}\right\} \qquad (19)$$

$$I(p_1,\ldots,p_\ell;k_1,\ldots,k_\nu)\;\bar{I}(k_1,\ldots,k_\nu;q_1,\ldots,q_a).$$

The expression (19) allows, in particular, to find the self-energy parts, all other matrix elements on the energy shell being known.

We may call eq. (19) as formulae "to leave the energy shell". The inverse relations may be written, as well which read:

$$I(p,p_1,\ldots,p_\ell;q_1,\ldots,q_a)=P\left(\frac{q_1}{q_2,\ldots,q_a}\right)\sqrt{2p^0 2q_1^0}\;I(p_1,\ldots,p_\ell;q_2,\ldots,q_a)-$$

$$-\frac{1}{(2)^{3/2}}\bar{I}\left(p+\frac{\sum p-\sum q}{2}\,|\,p_1,\ldots,p_\ell;\,q_1,\ldots,q_a\right) \qquad (20)$$

with $p^2=m^2$.

Without entering the problem of solving the equations (16)*, these being in effect the relativistic analogue to Low equations, we will use them to estimate the admissible degrees of growth of the matrix elements of the operator J.

---

* Two kinds of obstacles come into play when trying to solve such a set of equations. Firstly, the set expresses the lower matrix elements (i.e. the matrix elements with lower number of arguments) in terms of the higher ones. Hence, as to the ways of the exact solution, they are quite obscure. If on the other hand some sort of approximation is to be used, the overdeterminacy of the system manifests itself. Indeed the causality condition leads not only to the

set (18) but to the numerous other infinite sets[9] of similar nature. Now, would we get an exact solution of Eqs.(16) this should obey automatically all other sets of equations. But an approximation which is good for the one set may turn out to be very bad for another one.

3. Degrees of growth of matrix elements.

The dependence of matrix elements on the various moments can be very complicated and we do not pretend to examine it in detail. We are raising a much more simple problem, that concerning only the total degree of growth at a uniform extension of all the momenta, just like one proceeds in the perturbation theory (Cf.e.g.[7], Sec. 26) to estimate the degree of growth of any graph.

Namely, we demand a finite growth index to exist for any matrix element with l,s momenta of each kind. This is defined as a minimum integer $\omega(l,s)$ such that when all the momenta are extended uniformly

$$p_1 = \xi_1 P; \ldots; \quad p_l = \xi_l P; \quad q_1 = \eta_1 P; \ldots; \quad q_s = \eta_s P; \quad P \to \infty \qquad (21)$$

the matrix element $I(p_1, \ldots, p_l; q_1, \ldots, q_s)$ increases slower than $P^{\omega(l,s)+\alpha}$ for any $\alpha > 0$. The theory will be called renormalizable* if it satisfies this condition. Only the renormalizable theories shall

* The concept of renormalizability so defined is somewhat wider than the usual one, the latter requiring further the number of matrix elements with nonnegative degrees of growth to be finite. Yet it will be seen from what follows that the difference between the two classes so defined makes an empty set.

be considered henceforth. Now, we will look for the restrictions, if any, the eqs. (16) impose on the possible choice of the numbers $\omega(l,s)$.

Leaving out the disconnected graphs' contribution that is of no interest to us, the right hand side of eq. (16) is a (infinite) sum of nonlinear terms of the similar structure, such as

$$\int \frac{dk_1 \ldots dk_\nu}{2k_1^\circ \ldots 2k_\nu^\circ} I(p_1, \ldots, p_l; k_1, \ldots, k_\nu) I(k_1, \ldots, k_\nu; q_1, \ldots, q_s) \cdot$$

$$\cdot \left\{ \frac{\delta(p + \Sigma p_j - \Sigma k_\nu)}{\Sigma k_\nu^\circ - \Sigma p_j^\circ - p^\circ - i\epsilon} - \frac{\delta(p - \Sigma q_j + \Sigma k_\nu)}{-\Sigma k_\nu^\circ + \Sigma q_j^\circ - p^\circ - i\epsilon} \right\}.$$

(beside the out of interest contribution from the disconnected graphs). Each of the terms involves 3ν integrations over the components of momenta k , ν factors k° in the denominator and one three-dimensional δ-function, divided by the one-dimensional energy denominator. Furthemore, the integrand includes a product of matrix elements $I(\ldots)$ with the numbers l, ν and ν, s. It is obvious that the exact dependence of

the latter upon the momenta is unknown. Hence, we can not, evidently, find out the momentum dependence of the whole integral. Nevertheless, if we believe in the natural assumption that the main contribution in the integration over $k_1, \ldots, k_\nu$ comes from the high-momenta region, then it is sufficient to know only the high-momenta asymptotic behaviour of the matrix elements contained in the integrand this is, just given by the growth indiced $\omega(\ell, \nu)$ or $\omega(\nu, s)$. Now, to examine the behaviour of a typical integral at high momenta it remains then but to count directly the degrees of the momenta involved, just as one does in the perturbation theory. We obtain thereby

$$3\nu - \nu - 3 = 1 + \omega(1,\nu) + \omega(\nu, s) = \omega(1,\nu) + \omega(\nu,s) + 2\nu - 4 . \qquad (22)$$

In order to estimate the total degree of growth of the right hand side, note that it would seem quite unnatural to imagine the maximal degrees of momenta of the different terms of the sum compensating one another without any physical reason. Such a compensation might be effected only by the existence of some group, as in the well-known example of electrodynamics where the compensation is due to the gauge group.

For the sake of simplicity the theory will be assumed not to permit any group*. Then it follows as a consequence of Eqs. (16) that the degree of growth of the matrix element in the left hand side should be at any rate not less than that, of the any term in the right hand side eq. (22). Thus, we arrive at the two sets of inequalities

---

* A special investigation is necessary for the case when such a group exists.

$$\omega(\ell+1, s) \geq \omega(1,\nu) + \omega(\nu, s) + 2\nu - 4 \quad ; \qquad (23.1)$$

$$\omega(\ell, s+1) \geq \omega(1,\nu) + \omega(\nu, s) + 2\nu - 4 . \qquad (23.2)$$

They are to be valid for all the $1, \nu, s$, obeying the conditions

$$\nu \geq 1 \; ; \quad \ell + s \geq 1 \; ; \quad \nu + \ell \geq 2 \; ; \quad \nu + s \geq 2 . \qquad (24)$$

It the same manner the eq. (19) allows to estimate the degrees of growth of the out of the energy shell matrix elements $\bar{I}(\ldots)$:

$$\omega(1|\ell, s) \geq \omega(\ell,\nu) + \omega(\nu, s) + 2\nu - 4 . \qquad (25)$$

It is easily seen that the degrees of growth should not depend upon the numbers ℓ, s separately but only their sum (i.e. upon the whole number of legs of the graph). Thus

$$\omega(\ell, s) = \Omega(\ell + s). \tag{26}$$

This reduces the two sets (23) to a single one.

$$\Omega(\ell + s + 1) \geq \Omega(\ell + \nu) + (s + \nu) + 2\nu - 4, \tag{27}$$

which holds under the same conditions (24). It is seen, that replacing all the inequalities by equalities the set (27) admits a particular solution

$$\Omega_o(n) = 3 - n. \tag{28}$$

Hence, it is convenient to seek the general solution in the form of sum of this particular solution and some unknown function N (n) :

$$\Omega(n) = \Omega_o(n) + N(n) = 3 - n + N(n). \tag{29}$$

Thereby the set takes the form

$$N(\ell + s + 1) \geq N(\ell + \nu) + N(s + \nu). \tag{30}$$

Of course, one may not regard the reasoning, that led us to the set (30), as a proof in the strict mathematical sense. A mathematician would rather call this heuristic suggestions, saying perhaps in addition that he could invent controversial examples. Without trying to complete here such a rigorous proof, the following consideration will be mentioned. The matrix elements in the integrands in the right hand side of eqs. (16) involve the counter terms. Since the counter terms are polynomials with respect to momenta, the whole integrand becomes explicitly know and the integral - its counter term part being concerned - may be investigated by elementary means. Anyhow, there is really no need in such a treatment for it is essentially the problem already examined in details in the R-operation theory, cf. /7/ Sec. 26.

Now, we are going to solve the set (30).

Let us prove, first of all, that all the N(n) are non positive. For n odd, it suffices to put in eq.(30) $\nu = s + 1 \geq 2$, l being arbitrary. The l.h.s. cancels then with the first term in the r.h.s. and we obtain that

$$0 \geq N(2s+1) \qquad \text{for} \qquad s \geq 1. \tag{31}$$

To establish the same for the n even we choose $s = \ell \geq 1$, $\nu \geq 1$, this being permitted by the conditions (24).

Then

$$2N(\ell + \nu) \leq N(2\ell + 1) \leq 0. \tag{32}$$

Here we made use of eq. (31) to obtain the last inequality. Now,

$$\Omega(n) \leq \Omega_0(n) = 3-n \qquad \text{for energy} \qquad n \geq 2. \tag{33}$$

Thus we see that the particular solution (28) gives the greatest possible growth indices of the matrix elements $I(p_1, \ldots, p_\ell; q_1, \ldots, q_s)$. In particular, it follows therefrom that there can be but a finite number of matrix elements with the non-negative growth indices. Hence, our definition of a renormalizable theory turns out to coincide with the usual one.

As a second step, we have to demonstrate that there is an upper limit for the admissible growth indices as well as the lower one. Let us denote the argument of the l.h.s. of eq. (30) by a single letter n and choose for the ν the lowest possible value $\nu = 1$. Then,

$$N(n) \geq N(n-s) + N(s+1), \tag{34}$$

where n and s are restricted (cf. eq. (24)) by the conditions

$$n \geq s+2; \qquad s \geq 1. \tag{35}$$

Now, we can once again apply the same reasoning (with a new n) to the first term in the r.h.s. of eq.(34). Carrying out this process k times we obtain a lower limit for $N(n)$:

$$N(n) \geq N(n-ks) + kN(s+1). \tag{36}$$

In accordance with eq. (35) K must obey the condition

$$k < \frac{n-2}{s}. \tag{37}$$

If we choose in eq. (36) $s = 1$ and k to have the highest possible value $k = n-2$, one obtains that for all $n \geq 2$

$$N(n) \geq N(2) + (n-2)N(2) = (n-1)N(2). \tag{38}$$

The eq. (38) gives a strict estimate in the sense that the lower limit ascertained in this way may be really reached. Indeed, if we substitute for all $N(n)$ in the basic set (30) their least values allowed by eq. (38), the set would reduce to the condition $\nu \geq 1$, which holds always in view of eq. (24).

The conditions just obtained, eq. (32) and (38), do not exhaust all the restrictions on the admissible $N(n)$. Namely, if for any $n = n_0 > 2$ $N(n)$ exceeds its lowest value given by eq. (38), this provides further restrictions on the $N(n)$ with $n > n_0$. One may derive them from the eq. (36), by putting in $s > 1$. We shall not do it.

Finally, eq. (25) allows to estimate from below the growth index of the graph with two out of the energy shell legs

$$\tilde{N}(1) = \tilde{\Omega}(1) - 2 = \tilde{\omega}(1\,|-;-) - 2 \geq 2N(\nu)\,;\ \nu \geq 2\,. \qquad (38a)$$

There are no limitations from above in this case.

4. Properly renormalizable theories.

In the preceding section we treated the theory to be renormalizable and this made it possible to extract just a system of inequalities (23). To do this, we appealed to the fact that the drowth index of the l.h.s. of eq. (16) may not exceed those of each term in r.h.s., i.e. it has to be more or equal to the maximum of the r.h.s. indices. It would be like to go further and to replace the inequalities by equalities. However, we need a new assumption to be made for sake of this, for as emphasized above there are counter terms in the r.h.s. of eq. (16) besides the integral terms written down explicitly.

We shall call the theory properly renormalizable, if the degrees of polynomials to be added to T-products do not exceed the growth indices of the corresponding T-products themselves. Assuming now a properly renormalizable theory, the condition eq. (23) can be indeed rewritten in the stronger form:

$$\omega(\ell+1, s) = \operatorname*{Max}_{\nu}\,(\omega(\ell,\nu) + \omega(\nu, s) + 2\nu - 4)\,;$$
$$\omega(\ell, s+1) = \operatorname*{Max}_{\nu}\,(\omega(\ell,\nu) + \omega(\nu,s) + 2\nu - 4)\,. \qquad (39)$$

The strong condition upon the N(n) reads;

$$N(\ell + s + 1) = \operatorname{Max}\,(N(\ell+\nu) + N(s+\nu))\,, \qquad (40)$$

the maximum being taken over all the arguments admitted by eq. (24).

Let us try to solve this set of inequalities.

We rewrite it in the form

$$N(n_1) = \operatorname{Max}\,(N(n_2) + N(n_3))\,. \qquad (41)$$

The maximum is to be taken over all n_2, n_3 obeying the conditions

$$n_2 + n_3 = n_1 - 1 + 2\nu\,;\quad n_2 > \nu \geq 1\,;\quad n_3 > \nu \geq 1\,. \qquad (42)$$

Now, in order to solve eq. (41) we arrange the sets of pairs of $N(n)$ over which the maximum is to be taken into two tables, the one for the even, the other for the odd n_1. The rows are numbered there by the sums $n_1 + n_2$, and the columns – by the differences $n_1 - n_2$. The sums in the cells of tables represent all possible divisions of $n_1 - \ell + 2$ into the sum of numbers n_2 and n_3, admitted by eq. (42).

Table 1

	②					
5	3+2	④				
7	4+3;	5+2	⑥			
9	5+4;	6+3;	7+2	⑧		
11	6+5;	7+4;	8+3;	9+2	⑩	
13	7+6;	8+5;	9+4;	10+3;	11+2	⑫
15	8+7;	9+6;	10+5;	11+4;	12+3;	13+2
17	9+8;	10+7;	11+6;	12+5;	13+4;	14+3; 15+2

Table 2

		③				
4	2+2					
6	3+3;	4+2	⑤			
8	4+4;	5+3;	6+2	⑦		
10	5+5;	6+4;	7+3;	8+2	⑨	
12	6+6;	7+5;	8+4;	9+3;	10+2	⑪
14	7+7;	8+6;	9+5;	10+4;	11+3;	12+2

Evidently they present the sums under the maximum sign in eq. (41) as well. Does any combination of n_2 and n_3 obey the inequalities (42), all the combinations down the column should clearly do, for the displacement down the column means merely the increase of ν. Putting now $\nu = 1$, the owe possible value, one sees that by the first of the restrictions (42) — only the rows with $n_2 + n_3 \geq 1 + n_1$ will enter the domain over which the maximum for any n_1 is taken in (41). As to the number of columns, it is to be determined from the condition $(n_2 - n_3) \leq (n_2 + n_3) - 4$, this also coming from (42). These domains are marked on the tables 1,2 with the corresponding n_1 encircled.

Now, the tables 1 and 2 visualize at once that

$$N(2) \leq N(4) . \tag{43}$$

Really, the domain over which the maximum for N(2) is taken is seen to enter completely into that for N(4).

Further on, the matter becomes more involved, since not only a new column (the third one in the table 2) is added but also the element 2 + 2 is striked out, when passing from N(3) to N(5). However, in virtue of eq. (43), N(2) + N(2) $\leq$ N(2) + N(4), while the combination 4 + 2 enters the both domains under consideration. Thus, we can enlist a new inequality

$$N(3) \leq N(5) . \tag{44}$$

One may see that the situation remains the same in every further step. Namely, the transition from n to n-2 leads to the partial reduction of the domain over which the maximum is taken. But once the chains of inequalities of the type (43) – (44) are established for any $k \leq n-1$ this reduction is of no importance. Thus, the complete induction is possible, and we come to infinite chains:

$$N(2) \leq N(4) \leq N(6) \leq \ldots \leq N(2k) \leq \ldots \tag{45.1}$$

and

$$N(3) \leq N(5) \leq N(7) \leq \ldots \leq N(2k+1) \leq \ldots . \tag{45.2}$$

On the other hand, all the N(n) are restricted from above by the condition (33). Therefore, both the non-decreasing sequences of integers (34) must reach their upper limits, i.e. the

$$\underset{k \geq 1}{\text{Max}}\, N(2k) = -a \leq 0 \quad \text{and} \quad \underset{k \geq 1}{\text{Max}}\, N(2k+1) = -b \leq 0 \tag{46}$$

must exist.

Now, take account that the domain (42) includes arbitrary large numbers n_2 and n_3 for every n_1. Then we conclude that instead of eq. (41) the equations

$$\begin{aligned} N(2k) &= \text{Max}\{-a-b\} = -a-b ; \\ N(2k+1) &= \text{Max}\{(-a-a), (-b-b)\} \end{aligned} \tag{47}$$

may be written. Hence, N(2k) and N(2k+1) do not depend upon k. But then

$$N(2k) = -a = -a-b , \quad \text{i.e.} \quad b = 0$$

and

$$N(2k+1) = -b = \text{Max}(-2a, -2b) , \text{ i.e. } -a \leq 0 .$$

Thus the solution of the set (40), has the form

$$N(2k) = -\alpha \; ; \; N(2k+1) = 0 \; , \qquad (48)$$

α being an arbitrary non-negative integer. In terms of more physical quantities the solution obtained means that the general form for the admissible growth indices of the matrix elements $I(\ldots)$ in a properly renormalizable theory is:

$$\Omega(2k) = 3 - 2k - \alpha \; ; \quad \Omega(2k+1) = 2 - 2k \; ; \quad \alpha \geq 0; \quad \alpha \in N. \qquad (49)$$

It remains but to write down the condition on the growth index of the two legs graph, this is:

$$\tilde{N}(1) = 2 \max_{\nu \geq 2} N(\nu) = 0 \; ; \; \tilde{\Omega}(1) = N(1) + 2 = 2 \; . \qquad (50)$$

5. Discussion.

The most important conclusion, we obtained in the preceding sections, is that the admissible growth indices are limited from above. Accordingly, the non-negative indices may be prescribed only to the matrix elements with three or four legs ($n = 2,3$), to say nothing of the self-energy part, not entering the set (16). Thus, these are the only matrix elements that can be accompanied by counter terms (the latter cannot have negative indices being polynomials in momenta). Therefore one see the dynamical priciple to be almost superfluous in the framework of the dispersion approach – the admissible interactions are determined up to a small number of constants by the transformation properties of the fields only. In the case just considered that of the zero-spin particles there are only two such constants – the constant counter term of the four-vertex ($n = 3$), and that of the three-vertex (the linear counter term, formally permissible here, is prohibited by Lorentz covariance).

Especially strong are the restrictions, that occur in a properly renormalizable theory. It might seem that a condition, coinciding at first sight with that of proper renormalizability, is always imposed in the usual perturbation treating as well when we choose for the counter terms in momentum representation the polynomials of the minimal possible degree (Cf. /7/, Sec. 26; otherwise it would be impossible to carry through the "R-operation"). Yet the following point shows an essential difference. In the usual approach one deals in fact with counter terms (in a broad sense) of the two kinds. Besides the proper counter terms (the renormalization constants) when difining the products of singular functions, there are also the "charges" under consideration, these latter coming from the initial Lagrangian. The degrees of corresponding polynomials are not determined by the minimum requirement but are adopted ad hoc when specifying the theory.

In our approach we treat all the counter terms in a unified manner; both the charges and the renormalization constants come out on the equal footing. Both of them play the role of the inhomogenities as some kind of boundary conditions for the basic set (16); there are reasons[10] to think that were they absent, this set would permit only the trivial zero solutions. This will surely be the case if we admit any expansion in terms of a small parameter in spirit of /8/.

Now, in this philosophy of minimal growth all the logical scheme of the theory comes to be especially beautiful: were all the counter terms absent, the set (16) would have but the trivial solutions; we gain non-zero solutions when introducing counter terms; but in doing so we do not bring any alien elements into the set; but merely exploit the arbitrariness inherently implied by its singular character. Were the set of the type (16) regular, there would be no intrinsic reasons to add counter terms, and we should come to a unique – the zero – solution, cf. /11/

Thus, the "minimal growth" requirement is to be imposed as well on the counter terms usually included in the initial Lagrangian. Naturally, it may give rise to further restrictions on the class of admissible theories – the class of properly renormalizable theories is narrower than that of the theories renormalizable in the conventional sense. In particular, the theory of self-interacting – via 3-vertex only – scalar field (the Hurst -Thirring field) drops out of the former class. Indeed, turning to the 3-vertex in this theory, one finds the simplest three-leg graph to have the growth index -2, and the more involved ones – still lower. Hence, from the point of view of the minimal growth requirement the 3-vertex should not be accompanied by counter terms, i.e. the chrge should vanish*. It is of interest, that a properly renormalized theory with two

* Compare this observation with the opinion expressed /12/ this theory be a self-contradictory one.

kinds of scalar interactions – the 3-fold and the 4-fold ones – presenting simultaneously is allowed. Indeed, whereas the growth index of 3-vertex (2) may be taken either equal to zero or negative (due to the presence of a in eq. (49)), that of four-vertex must be necesserarily taken equal to zero; i.e. the 4-fold interaction must inevoidably participate in a theory of zero-spin particles.

6. Acknowledgements.

The authors are indebted to N.N.Bogolubov, V.S. Vladimirov and I.F.Ginzburg for valuable discussions and helpful comments.

References

1. H.Lehmann, K.Symanzik und W.Zimmermann, Nuovo Cimento, 1, 205 (1955).
2. R.Haag, Dan.Mat.Fys.Medd. 29, N 12 (1955).
3. A.S.Wightman, Colloque International en Lille, June 1957.
4. Н.Н. Боголюбов, Б.В. Медведев и М.К. Поливанов. Вопросы теории дисперсионных соотношений, М. Физматгиз /1958/.
5. W.Heisenberg, Zs.f.Phys. 120, 513; 673 (1943).
6. Н.Н. Боголюбов. Изв. АН СССР, Сер.физ. 19, 237 /1955/.
7. Н.Н. Боголюбов и Д.В. Ширков. Введение в теорию квантованных полей, М. Гостехиздат /1957/.
8. Б.В. Медведев, ДАН СССР, 135, 1087 /1960/.
9. Б.В. Медведев, ЖЭТФ, № 3 /1961/.
10. Б.В. Медведев и М.К. Поливанов. ДАН СССР / to be published /.
11. В.Л. Бонч-Бруевич и Б.В. Медведев. ЖЭТФ, 22, 425 /1952/.
12 Gordon Baym, Phys.Rev. 117, 886 (1960).

Received by Publishing Department
on November 9, 1961.