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ON THE MINIMUM NUMBER OF PARTIAL WAVES IN COLLISIONS WITH MORE THAN TWO PARTICLES IN THE FINAL STATE

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Summary

An universal inequality is obtained to determine the minimum number of partial waves participating in a reaction with more than two particles in the final state.

1. Introduction

The minimum number of partial waves participating in the high energy collisions L_{min} has been discussed by several authors. Rarita and Schwed[1] showed a method to determine L_{min} for elastic scattering by using the total interaction cross section. Recently, Grishin and Ogievetski[2] proved an inequality, which is very effective to determine the minimum number of partial waves in two body collisions once the total elastic cross section and the differential cross section at some angles are known.

As is well known in the high energy physics that most of the high energy collision processes are multiple production processes with more than two particles in the final state. So a question rises, how to determine the minimume number of partial waves in such collisions ? Such a question has its experimental significance, since L_{min} is related to the minimum interaction radius.

In the present paper, the inequality obtained by Grishin and Ogievetski for two particle reactions is generalized to cases with more than two particles in the final states. In relates the angular distribution of one of the final particles, the total partial cross section with L_{min}, the minimum number of partial waves paticipating the reaction.

In section 2, the choise of independent variables for the description of the three particles final states is discussed in detail.

In section 3 the inequality is obtained for the spinless particle cases $0+0 \to 0+0+0$.

In section 4 it is shown that the inequality obtained in sec. 3 can be extended to the cases $0+\frac{1}{2} \to 0+0+\frac{1}{2}$, $\frac{1}{2}+\frac{1}{2} \to 0+\frac{1}{2}+\frac{1}{2}$ and $\frac{1}{2}+\frac{1}{2} \to 0+0+0$ without any change.

In sec. 5 the inequality is generalized to the case with n particles in the final state.

In sec. 6 some remarks of the application of the inequality are made.

In the appendix, a proof is given to show that the phase volume integral can be separated in terms of a suitable choosen set of independent variables. This result is used in the proof of the inequality.

II. Kinematics of the 3-Particles System

For the description of the states of a 3-particles system, we introduce the following nine quantities instead of the three momentum $\vec{p}_1$, $\vec{p}_2$, $\vec{p}_3$ of the three particles. The first three are the momenttum of the mass center $\vec{\mathcal{P}} = \vec{p}_1 + \vec{p}_2 + \vec{p}_3$. At the center of mass system of the three particles (later on we will call it as the 3-C system for simplicity) $\mathcal{P} = 0$. Next, we choose a vector $\vec{p}_{3c}(|\vec{p}_{3c}|, \vec{\Omega}_{3c})$, the momentum of one of the particles in the 3-C system, which can be indentified experimentally (for instance, the recoil nucleon, the K - meson or the hyperon). In the system where the rest two particles as a whole are at rest (later on, we will call it as the 2-C system), these two particles move in opposite direction, then the last three variables can be chosen as $\vec{\Omega}_{2c}$, the direction of the relative momentum of these two particles in the 2-C system and $\mathfrak{M}_{2c}$, the energy of these two particles in the 2-C system.

Such a choice of independent variables has two advantages. First, the phase volume integration can be separated into two parts

$$\iiint \frac{d\vec{p}_1\, d\vec{p}_2\, d\vec{p}_3}{8\, E_1 E_2 E_3}\, \delta(\vec{p}_1 + \vec{p}_2 + \vec{p}_3 - \vec{p}_i)\, \delta(E_1 + E_2 + E_3 - E_i) = \tag{1}$$

$$= \int d\,\mathfrak{M}_{2c}^2 \int d\vec{\Omega}_{3c} \int d\vec{\Omega}_{2c}\; G(\mathfrak{M}_{3c}^2, \mathfrak{M}_{2c}^2)$$

where $\mathfrak{M}_{3c}$ represents the total energy of the three particles in the 3-C system. $G(\mathfrak{M}_{3c}^2, \mathfrak{M}_{2c}^2)$ is a function independent on the angles. The integration over angles and over energy is then separated. The proof of (1) can be found in the appendix.

Second, starting from the paper of Chou and Shirokov/3/, it can be shown that the total angular momentum of the three particles can be obtained by suming ℓ_{2c}, the relative angular momentum of the two particles in the 2-C system with ℓ_{3c}, the relative angular momentum of the distinguished particle and the rest two particles as a whole in the 3-C system by the usual sum rule.

III. Interaction of Case $0 + 0 \to 0 + 0 + 0$.

Consider first the simplest case where all particles are spinless. When the variables was chosen as pointed out in sec. 2, the general form of the Feynmann amplitude of the process is

$$F = \sum_{\substack{LM\ell_{2c}\ell_{3c}\\ \mu_{2c}\mu_{3c}}} R_L^{\ell_{2c},\ell_{3c}}\, Y_{LM}^*(\vec{\Omega}_i)\, C_{L,M}^{\ell_{2c},\mu_{2c};\,\ell_{3c},\mu_{3c}}\, Y_{\ell_{2c},\mu_{2c}}(\vec{\Omega}_{2c})\, Y_{\ell_{3c},\mu_{3c}}(\vec{\Omega}_{3c}) \tag{2}$$

where $R_L^{\ell_{2c},\ell_{3c}}$ is a function depend on L, the total angular momentum of the three particles system, ℓ_{2c} and ℓ_{3c}, the angular momentum mentioned at the end of sec. 2. The arguments of $R_L^{\ell_{2c},\ell_{3c}}$ are invariants. $C_{L,M}^{\ell_{2c},\mu_{2c};\ell_{3c},\mu_{3c}}$ is the usual Clebsch-Gordan coefficient. $Y_{\ell,\mu}$ is the spherical harmonic function. $\vec{\Omega}_i$ is the direction of the incident particle in the $3-c$ system.

Choosing the z axis along the direction $\vec{\Omega}_{3c}$, then (2) becomes

$$F = \sum_{LM\ell_{2c}\ell_{3c}} R_L^{\ell_{2c},\ell_{3c}} \sqrt{\frac{2\ell_{3c}+1}{4\pi}}\, C_{L,M}^{\ell_{2c},M;\ell_{3c},0}\, Y_{\ell_{2c},M}(\vec{\Omega}_{2c})\, Y_{L,M}^{*}(\vec{\Omega}_i). \qquad (3)$$

Then, the angular distribution of the identified particle is

$$\sigma(\theta) = \int |F|^2\, G(\mathfrak{M}_{3c}^2, \mathfrak{M}_{2c}^2)\, d\vec{\Omega}_{2c}\, d\mathfrak{M}_{2c}^2 =$$

$$= \sum_{M\ell_{2c}} \int G(\mathfrak{M}_{3c}^2, \mathfrak{M}_{2c}^2)\, d\mathfrak{M}_{2c}^2 \left| \sum_{L\ell_{3c}} R_L^{\ell_{2c},\ell_{3c}} \sqrt{\frac{2\ell_{3c}+1}{4\pi}}\, C_{L,M}^{\ell_{2c},M;\ell_{3c},0}\, Y_{L,M}^{*}(\vec{\Omega}_i)\right|^2 \qquad (4)$$

and the cross section for this channel (total partial cross section) is

$$\sigma_3 = \int \sigma(\theta)\, d\vec{\Omega} \cdot$$

$$= \sum_{LM\ell_{2c}} \int d\mathfrak{M}_{2c}^2\, G(\mathfrak{M}_{3c}^2, \mathfrak{M}_{2c}^2) \left| \sum_{\ell_{3c}} \sqrt{\frac{2\ell_{3c}+1}{4\pi}}\, R_L^{\ell_{2c},\ell_{3c}}\, C_{L,M}^{\ell_{2c},M;\ell_{3c},0} \right|^2 \qquad (5)$$

If it is assumed that the summation over L in expressions (4) and (5) can be confined by a finite number of partial waves L_{min}, then, using the Cauchy's inequality $\sum_i |a_i b_i|^2 \leq \sum_i |a_i|^2 \sum_i |b_i|^2$ from (4) and (5) we obtain

$$\sigma(\theta) \leq \sum_{M\ell_{2c}} \int \sum_L \left| \sum_{\ell_{3c}} R_L^{\ell_{2c},\ell_{3c}} \sqrt{\frac{2\ell_{3c}+1}{4\pi}}\, C_{L,M}^{\ell_{2c},M;\ell_{3c},0} \right|^2 \sum_L \left| Y_{LM}(\theta) \right|^2 G(\mathfrak{M}_{3c}^2, \mathfrak{M}_{2c}^2)\, d\mathfrak{M}_{2c}^2 \leq$$

$$(6)$$

$$\leq \int \sum_{LM\ell_{2c}} \left| \sum_{\ell_{3c}} R_L^{\ell_{2c},\ell_{3c}} \sqrt{\frac{2\ell_{3c}+1}{4\pi}}\, C_{L,M}^{\ell_{2c},M;\ell_{3c},0} \right|^2 \cdot \sum_{L,M} \left| Y_{LM}(\theta) \right|^2 G(\mathfrak{M}_{3c}^2, \mathfrak{M}_{2c}^2)\, d\mathfrak{M}_{2c}^2 =$$

$$= \sigma_3 \sum_{L=0}^{L_{min}} \frac{2L+1}{4\pi}.$$

Therefore, the inequality required is

$$\frac{4\pi\sigma(\theta)}{\sigma_3} \leqslant (L_{min}+1)^2. \qquad (7)$$

It is to be pointed out that after the summation over M is carried out, the right side of (6) is independent of θ. This differs from the two particle case, where has no summation over M, and then the right side of the inequality does depend on θ /2/.

IV. Interaction of the Case $0+\frac{1}{2} \longrightarrow 0+0+\frac{1}{2}$

Consider the case $0+\frac{1}{2} \longrightarrow 0+0+\frac{1}{2}$, the Feynmann amplitude for this process is

$$F = \sum_{\substack{JLL'M \\ \ell_{2c}\ell_{3c}\mu_{2c}\mu_{3c}}} R^{\ell_{2c},\ell_{3c}}_{JLL'} C^{L',M-\beta;\frac{1}{2},\beta}_{JM} C^{\ell_{3c},\mu_{3c};\ell_{2c},\mu_{2c}}_{L',M-\beta} C^{L,M-\alpha;\frac{1}{2},\alpha}_{JM} Y^{*}_{L,M}(\vec{\Omega}_i)\cdot \qquad (8)$$

$$\cdot Y_{\ell_{2c},\mu_{2c}}(\vec{\Omega}_{2c}) Y_{\ell_{3c},\mu_{3c}}(\vec{\Omega}_{3c})$$

where J is the total angular momentum of the three particles system in the $3-C$ system, L and L' are the total orbital angular momentum of the initial and final states in the $3-C$ system respectively, α and β are the spin orientation of the spin particle in the initial and the final state, respectively, the other notations are the same as in sec. 3.

As in the above section, we choose the z - axis along the direction $\vec{\Omega}_{3c}$, so we have

$$F = \sum_{\substack{JLL'M \\ \ell_{2c}\ell_{3c}}} R^{\ell_{2c},\ell_{3c}}_{JLL'} C^{L',M-\beta;\frac{1}{2},\beta}_{J,M} C^{\ell_{3c},0;\ell_{2c},M-\beta}_{L',M-\beta} C^{L,M-\alpha;\frac{1}{2},\alpha}_{J,M}\cdot \qquad (9)$$

$$\cdot \sqrt{\frac{2\ell_{3c}+1}{4\pi}}\, Y_{\ell_{2c},M-\beta}(\vec{\Omega}_{2c}) Y^{*}_{L,M}(\vec{\Omega}_i)$$

the angular distribution of the identified particle is

$$\sigma(\theta) = \sum_{\alpha\beta} \int G(\mathfrak{M}^2_{3c},\mathfrak{M}^2_{2c})\, d\mathfrak{M}^2_{2c}\, d\vec{\Omega}_{2c}\, |F|^2 =$$

$$= \sum_{\substack{\alpha\beta \\ M\,\ell_{2c}}} \int d\mathfrak{M}_{2c}^2\, G(\mathfrak{M}_{3c}^2, \mathfrak{M}_{2c}^2) \Big| \sum_{\substack{LL' \\ J\ell_{3c}}} R_{JLL'}^{\ell_{2c},\ell_{3c}} C_{J,M}^{L',M-\beta;\frac{1}{2},\beta} C_{L',M-\beta}^{\ell_{3c},0;\ell_{2c},M-\beta} C_{J,M}^{L,M-\alpha;\frac{1}{2},\alpha} \cdot \sqrt{\frac{2\ell_{3c}+1}{4\pi}}\, Y_{L,M}^{*}(\theta) \Big|^2 \qquad (10)$$

the total partial cross section is

$$\sigma_3 = \int \sigma(\theta)\, d\vec{\Omega} = \sum_{\substack{\alpha\beta \\ LM\ell_{2c}}} \int d\mathfrak{M}_{2c}^2\, G(\mathfrak{M}_{3c}^2, \mathfrak{M}_{2c}^2) \Big| \sum_{JL'\ell_{3c}} R_{JLL'}^{\ell_{2c},\ell_{3c}} C_{J,M}^{L',M-\beta;\frac{1}{2},\beta} C_{L',M-\beta}^{\ell_{3c},0;\ell_{2c},M-\beta} \cdot C_{J,M}^{L,M-\alpha;\frac{1}{2},\alpha} \sqrt{\frac{2\ell_{3c}+1}{4\pi}} \Big|^2 . \qquad (11)$$

From (9) and (10), using the Cauchy inequality, we obtain

$$\sigma(\theta) \leq \sum_{\substack{\alpha\beta \\ LM\ell_{3c}}} \int d\mathfrak{M}_{2c}^2\, G(\mathfrak{M}_{3c}^2, \mathfrak{M}_{2c}^2) \Big| \sum_{JL'\ell_{3c}} R_{JLL'}^{\ell_{2c},\ell_{3c}} \sqrt{\frac{2\ell_{3c}+1}{4\pi}}\, C_{J,M}^{L',M-\beta;\frac{1}{2},\beta} \cdot C_{J,M}^{L,M-\alpha;\frac{1}{2},\alpha} C_{L',M-\beta}^{\ell_{3c},0;\ell_{2c},M-\beta} \Big|^2 \sum_{L,M} \left| Y_{L,M}(\theta) \right|^2 = \sigma_3 \sum_{L=0}^{L_{min}} \frac{2L+1}{4\pi}$$

or (12)

$$\frac{4\pi\,\sigma(\theta)}{\sigma_3} \leq (L_{min} + 1)^2 . \qquad (13)$$

Therefore, the inequality required is just the same as in the spinless case. Moreover, it can easily be shown in a similar way that the result for the cases $\frac{1}{2}+\frac{1}{2} \to 0+0+0$, $0+\frac{1}{2} \to \frac{1}{2}+\frac{1}{2}+\frac{1}{2}$, $0+0 \to \frac{1}{2}+\frac{1}{2}+0$ is also the same as in the spinless case.

V. Interaction with n Particles in the Final State

As an example, we consider first the case $0+0 \to \overbrace{0+0+\cdots+0}^{n}$, the variables for such case can be chosen as follows: $\vec{\mathcal{P}}$ the momentum of the mass center of the n particles. In the center of mass system of the n particles (n-c system) $\vec{\mathcal{P}} = 0$. Next, we choose $\vec{P}_{3c}\ (|\vec{P}_{3c}|, \vec{\Omega}_{3c})$, the momentum of the identified particle from the n particles in the 3-c system. In the system where the rest $n-1$ particles as a whole are at rest (the $(n-1)$-c system), the next variables can be chosen as $\mathfrak{M}_{(n-1)c}$ the total energy of these $n-1$ particles in the $(n-1)$-c system and $\vec{\Omega}_{(n-1)c}$, the unit vector along the direction of $\vec{P}_{(n-1)c}$, the relative momentum of one of these $n-1$ particles in the $(n-1)$-c system. Continuing such a consideration, we can choose the other independent variables as $\mathfrak{M}_{(n-2)c}, \vec{\Omega}_{(n-2)c}; \cdots; \mathfrak{M}_{2c}, \vec{\Omega}_{2c}$.

The advantages of such a choice of independent variables are the same as mentioned in sec. 2. The phase space integration is now separated in the following form:

$$\int \frac{d\vec{p}_1 \cdots d\vec{p}_n}{2^n E_1 \cdots E_n} = \int d\mathfrak{M}_{2c}^2 \, d\mathfrak{M}_{3c}^2 \cdots d\mathfrak{M}_{(n-1)c}^2 \, G(\mathfrak{M}_{nc}^2, \cdots, \mathfrak{M}_{2c}^2) \int d\vec{\Omega}_{2c} \, d\vec{\Omega}_{3c} \cdots d\vec{\Omega}_{nc} \qquad (14)$$

G is a function only depends on energies but not on angles.

Using the above mentioned set of independent variables the Feynmann amplitude for this process is

$$F_n = \sum_{\substack{L, M, L_{3c} \cdots L_{(n-1)c} \\ \ell_{2c} \cdots \ell_{nc}, \mu_{2c} \cdots \mu_{nc} \\ M_{3c} \cdots M_{(n-1)c}}} R^{\ell_{2c}, \ell_{3c}, \cdots, \ell_{nc}}_{L, L_{3c}, \cdots L_{(n-1)c}} \, C^{\ell_{2c}, \mu_{2c}; \ell_{3c}, \mu_{3c}}_{L_{3c}, M_{3c}} \, C^{L_{3c} M_{3c}; \ell_{4c} \mu_{4c}}_{L_{4c}, M_{4c}} \cdots \cdot C^{L_{(n-1)c}, M_{(n-1)c}; \ell_{nc}, \mu_{nc}}_{L, M} \cdot Y^{*}_{LM}(\vec{\Omega}_i) Y_{\ell_{2c}\mu_{2c}}(\vec{\Omega}_{2c}) \cdots Y_{\ell_{nc}\mu_{nc}}(\vec{\Omega}_{nc}) \qquad (15)$$

where L and M are the total angular momentum of the n particles system in the n-c system and their z components respectively; ℓ_{ic} and μ_{ic} are the relative angular momentum with respect to $\vec{p}_{ic}$ in the i-c system and their z components respectively; L_{ic} and M_{ic} are the total angular momentum and their z components in the i-c system.

Choosing the z axis along the $\vec{\Omega}_{nc}$ direction, then the angular distribution of the identified particle is

$$\sigma_n(\theta) = \int |F_n|^2 \, G(\mathfrak{M}_{nc}^2, \cdots, \mathfrak{M}_{2c}^2) \, d\mathfrak{M}_{2c}^2 \cdots d\mathfrak{M}_{(n-1)c}^2 \, d\vec{\Omega}_{2c} \cdots d\vec{\Omega}_{(n-1)c} \qquad (16)$$

and the total partial cross section is

$$\sigma_n = \int \sigma_n(\theta) \, d\vec{\Omega} \qquad (17)$$

as in the above sections, using the Cauchy's inequiality, from (16) and (17), we obtain

$$\frac{4\pi \, \sigma_n(\theta)}{\sigma_n} \leq (L_{min} + 1)^2 \qquad (18)$$

which is again just the same as that in the spinless case.

For the general cases in which some of the particles are spin particles, it can be shown in a similar way that the result is also the same as in the spinless case.

VI. Discussion

We have obtained an universal inequality (18), which is useful to determine L_{min} and for the high energy analysis. In order to use this inequality, it is necessary to measure the total partial cross section of a channel of reaction with definite number of particles in the final state and the angular distribution of an identified particle (K - meson, hyperon, recoil nucleon or anti-baryon) in the final state with respect to the incident direction in the center of mass system.

Since the right side of (18) does not depend on θ, the inequality (18) is in fact

$$\frac{4\pi \,[\sigma_n(\theta)]_{max}}{\sigma_n} \leqslant (L_{min}+1)^2. \tag{19}$$

It should be pointed out that if we choose the orientation of the z axis on the direction $\vec{\Omega}_i$ then between the angular distribution of an identified particle from the particles in the $i-c$ system

$$\sigma_n(\theta_j) = \int |F_n|^2\, G(\mathfrak{M}^2_{nc}, \cdots \mathfrak{M}^2_{2c})\, d\mathfrak{M}^2_{2c} \cdots d\mathfrak{M}^2_{nc} \cdot$$
$$\cdot\, d\vec{\Omega}_{2c} \cdots d\vec{\Omega}_{(j-1)c}\, d\vec{\Omega}_{(j+1)c} \cdots d\vec{\Omega}_{nc} \tag{20}$$

and the total partial cross section there is an inequality

$$\frac{4\pi\, \sigma_n(\theta_j)}{\sigma_n} \leqslant [\,(\ell_j)_{min}+1\,]^2. \tag{21}$$

From inequality (21) one can determine the minimum number of partial waves, participating in the subsystem $j-c$.

The authors are indebted to Chou Kuang Chao and V.I. Ogievetski for stimulating discussions.

Appendix*

The phase volume integration for the n particles case is

$$I = \int \frac{d\vec{p}_1 \cdots d\vec{p}_n}{2^n E_1 \cdots E_n} \delta(\vec{p}_1 + \vec{p}_2 + \cdots + \vec{p}_n - \vec{p}_i)\, \delta(E_1 + E_2 + \cdots + E_n - E_i) \tag{A.1}$$

(A.1) can be written down in an obviously invariant form:

$$I = \int d^4p_1 \cdots d^4p_n\, \delta(p_1^2 + m_1^2) \cdots \delta(p_n^2 + m_n^2)\, \delta^4(p_1 + \cdots + p_n - p_i). \tag{A.2}$$

Introducing the following transformation

$$\begin{aligned} p_1 + p_2 &= k_{2c} \\ p_1 - p_2 &= 2q_{2c} \end{aligned} \tag{A.3}$$

then (A.2) becomes

$$\begin{aligned} I = \int d^4k_{2c}\, d^4q_{2c}\, d^4p_3 \cdots d^4p_n\, \delta\left(\left(\frac{k_{2c}}{2} + q_{2c}\right)^2 + m_1^2\right) \delta\left(\left(\frac{k_{2c}}{2} - q_{2c}\right)^2 + m_2^2\right) \cdot \\ \cdot\, \delta(p_3^2 + m_3^2) \cdots \delta(p_n^2 + m_n^2)\, \delta^4(k_{2c} + p_3 + \cdots + p_n - p_i) \end{aligned} \tag{A.4}$$

noticing that

$$d^4q_{2c} = \frac{1}{2}\sqrt{q_{2c}^2 + q_{2c0}^2}\; dq_{2c0}\, dq_{2c}^2\, d\vec{\Omega}_{2c} \tag{A.5}$$

and we can carry out the integration over d^4q_{2c} in the $2-c$ system, where

$$\begin{aligned} k_{2c}^2 &= -k_{2c0}^2 \\ (k_{2c} \cdot q_{2c}) &= -k_{2c0}\, q_{2c0} \end{aligned} \tag{A.6}$$

then (A.4) becomes

$$\begin{aligned} I = \frac{1}{2}\int d\mathfrak{M}_{2c}^2\, d\vec{\Omega}_{2c}\, d^4k_{2c}\, d^4p_3 \cdots d^4p_n \left[1 - 2\frac{m_1^2 + m_2^2}{\mathfrak{M}_{2c}^2} + \left(\frac{m_1^2 - m_2^2}{\mathfrak{M}_{2c}^2}\right)^2\right]^{\frac{1}{2}} \cdot \\ \cdot\, \delta(k_{2c}^2 + \mathfrak{M}_{2c}^2)\, \delta(p_3^2 + m_3^2) \cdots \delta(p_n^2 + m_n^2)\, \delta^4(k_{2c} + p_3 + \cdots + p_n - p_i) \end{aligned} \tag{A.7}$$

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* After this work is finished, Shirokov M.I. kindly informed us that a similar method of treating the phase space integration was proposed before by Kopylov (cf. JETF, 39, 1091 (1960))

where

$$\frac{1}{4}\mathfrak{M}_{2c}^{2} = q_{2c}^{2} + \frac{m_1^2 + m_2^2}{2}. \tag{A.8}$$

It is not difficult to see that $\mathfrak{M}_{2c}$ is just the total energy of the two particles in the 2-c system. Continuing the above procedure, putting

$$\begin{aligned} k_{ic} + p_{i+1} &= k_{(i+1)c} \\ k_{ic} - p_{i+1} &= 2q_{(i+1)c} \end{aligned} \tag{A.9}$$

and integrating $d^4 q_{(i+1)c}$ in the $(i+1)$-c system finnally we obtain the phase volume integration

$$I = \int G(\mathfrak{M}_{2c}^2, \cdots \mathfrak{M}_{nc}^2)\, d\vec{\Omega}_{2c} \cdots d\vec{\Omega}_{nc}\, d\mathfrak{M}_{2c}^2 \cdots d\mathfrak{M}_{(n-1)c}^2 \tag{A.10}$$

where

$$\begin{aligned} \frac{1}{4}\mathfrak{M}_{ic}^{2} &= q_{ic}^{2} + \frac{\mathfrak{M}_{(i-1)c}^2 + m_i^2}{2} \qquad i = 2, 3, \cdots n \\ \mathfrak{M}_{1c} &= m_1 \end{aligned} \tag{A.11}$$

and

$$G(\mathfrak{M}_{nc}^2, \cdots, \mathfrak{M}_{2c}^2) = \left(\frac{1}{2}\right)^{n-1} \prod_{i=2}^{n} \left[1 - 2\frac{\mathfrak{M}_{(i-1)c}^2 + m_i^2}{\mathfrak{M}_{ic}^2} + \left(\frac{\mathfrak{M}_{(i-1)c}^2 - m_i^2}{\mathfrak{M}_{ic}^2}\right)^2\right]^{\frac{1}{2}}. \tag{A.12}$$

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