ОБЪЕДИНЕННЫЙ ИНСТИТУТ ЯДЕРНЫХ ИССЛЕДОВАНИЙ

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V.I. Ogievetski, I.V. Polubarinov

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Abstract

It has been shown that theory of the neutral vector field with the non-zero rest mass may be formulated in a gauge-invariant form without introducing auxiliary fields. The gauge invariance in such a theory has a trivial physical meaning: the zero spin quanta described by a four-vector A_μ interact with nothing. Only the quanta with spin 1 interact.

1. Introduction

There is a widespread opinion[1-10] that in contrast to electrodynamics the theory of the neutral vector field $A_\mu(x)$ with the non-zero rest mass cannot be formulated in a gauge invariant form without resorting to auxiliary fields. This is considered as a serious obstacle in the recent attempts to draw an analogy between the baryonic and electric charges or between the hypercharge and electric charge by introducing the corresponding vector fields (Lee and Yang, Sakurai and others)[5,6,7,9,10].

In the usual formulation of the theory of a neutral vector field the equation

$$\Box A_\mu - \frac{\partial^2 A_\nu}{\partial x_\mu \partial x_\nu} - m^2 A_\mu = -j_\mu \tag{1}$$

is used (see, e.g.,[11,12]). It is also equivalent to the equation

$$(\Box - m^2) A_\mu = -j_\mu \tag{2}$$

with a supplementary condition

$$\frac{\partial A_\mu}{\partial x_\mu} = 0 . \tag{3}$$

Neither Eq. (1), nor Eq.(2) with the supplementary condition (3) are gauge-invariant.

A necessity in the supplementary condition (3) is usually explained by a desire to eliminate spin 0 and to quarantee the positive definiteness of the energy.

Note, that at the cost of the introduction of a certain auxiliary scalar field $B(x)$ besides the four-vector $A_\mu(x)$, Stueckelberg[3] succeeded in constructing the gauge invariant formalism of the vector field with a supplementary condition[1,4,8,11]. However, the meaning of the gauge invariance in such a theory is considerably veiled.

It will be shown below that the theory of the neutral vector field with the non-zero rest mass may be formulated in a gauge-invariant form without resorting to any auxiliary fields. To do this, it is necessary to give up a supplementary condition at all. In the theory under consideration $A_\mu(x)$ obeys only the gauge invariant equation (2) (§ 2). It turns out that only a part of $A_\mu(x)$ with spin 0 is subject to the gauge transformations. From the physical point of view the gauge invariance implies here that the zero spin quanta of the vector field do not interact with other fields and with each other. (§ 3). Therefore, the supplementary condition turns out to be superfluous for the elimination of zero spin. Nor is it necessary for the energy to be positively definite. (§ 3). Such a theory is wholly equivalent to the conventional theory of a neutral massive vector field based on Eqs. (1), or (2), (3). (§ 4).

It may be said that in the case under consideration the gauge invariance plays the same role as the supplementary conditions do in the theory of higher spins. In contrast to usual supplementary conditions the gauge invariance does not exclude the undesirable spin quanta, but makes it harmless.

2. Basic Equations

The Lagrangian density describing the neutral vector field A_μ interacting with the spinor field ψ * is chosen, as is done very often,/12/ to be

$$\mathcal{L}(x) = -\frac{1}{2}\frac{\partial A_\nu}{\partial x_\mu}\frac{\partial A_\nu}{\partial x_\mu} - \frac{m^2}{2}A_\mu A_\mu + j_\mu A_\mu - \bar{\psi}\left(\gamma_\mu\frac{\partial}{\partial x_\mu} + M\right)\psi \qquad (4)$$

$$/\ j_\mu = ig\bar{\psi}\gamma_\mu\psi\ /.$$

The Lagrangian and the equations of motion which follow from it

$$(\Box - m^2)A_\mu = -j_\mu \qquad (5)$$

$$\left(\gamma_\mu\frac{\partial}{\partial x_\mu} + M\right)\psi = ig\gamma_\mu\psi A_\mu \qquad (6)$$

as well as equal-time ($x_0 = y_0$) commutation relations analogous to electrodynamic ones**

$$\{\psi(x), \psi(y)\} = 0 \qquad \{\psi(x), \bar{\psi}(y)\} = \gamma_4\delta(\vec{x}-\vec{y})$$

$$[A_\mu(x), A_\nu(y)] = 0 \qquad \left[A_\mu(x), \frac{\partial A_\nu}{\partial y_4}(y)\right] = \delta_{\mu\nu}\delta(\vec{x}-\vec{y}) \qquad (7)$$

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* Similarly one can describe the interaction with several fields, as well as the interaction of type one due to the anomalous magnetic moment.

** The equal-time commutation relations may be chosen in this form since the supplementary condition on the field operators A_μ is not imposed.

$$[\psi(x), A_\nu(y)] = 0 \qquad\qquad [\psi(x), \frac{\partial A_\nu}{\partial y_4}(y)] = 0 \tag{7}$$

are invariant* with respect to the gauge transformations

$$\psi'(x) = \exp\left[ig\Lambda(x)\right]\psi(x) \tag{8}$$

$$A'_\mu(x) = A_\mu(x) + \frac{\partial \Lambda(x)}{\partial x_\mu} \tag{9}$$

with an arbitrary $\Lambda(x)$ satisfying the equation

$$\frac{\partial}{\partial x_\mu}(\Box - m^2)\Lambda = 0. \tag{10}$$

So, the gauge invariance takes place for $m \neq 0$ just as it holds for $m=0$ in (4), (5), and (10) (quantum electrodynamics).

Note, that in contrast to /5,6,7,9/ , where $\Lambda(x)$ were assumed to be quite arbitrary, in the transformations (8), (9) $\Lambda(x)$ are restricted here by condition (10). The restriction of $\Lambda(x)$ take place also in the quantum electrodynamics in contrast to the classical one.

From the gauge invariance of the Lagrangian or the equation follows the conservation law**

$$\frac{\partial}{\partial x_\mu}\left[-j_\mu\Lambda - \frac{\partial A_\nu}{\partial x_\mu}\frac{\partial \Lambda}{\partial x_\nu} + A_\nu\frac{\partial^2\Lambda}{\partial x_\mu \partial x_\nu}\right] = 0. \tag{11}$$

This equality must be fulfilled for any $\Lambda(x)$ satisfying Eq. (10). In particular, for Λ = const we get

$$\frac{\partial j_\mu}{\partial x_\mu} = 0 \tag{12}$$

what is the law of current conservation.

By differentiating (5) and taking into account (12), we obtain

$$(\Box - m^2)\frac{\partial A_\mu}{\partial x_\mu} = 0 \tag{13}$$

which we shall be in need of farther.

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* The Lagrangian density (4) is invariant up to an unessential divergence. (see Appendix 1).

** The derivation of this conservation law and the discussion of the operator of the transformation of the state vectors see in Appendix 1.

3. Spin 0 Quanta do not Interact

The four-vector $A_\mu(x)$ is used in order to describe particles with spin 1. In the framework of a homogeneous Lerentz group there are no quantities describing only spin 1. In consistence with this, $A_\mu(x)$ describes the quanta with spin 0, besides the quanta with spin 1: $A_\mu(x)$ may be decomposed into two parts

$$A_\mu = \left(A_\mu - \frac{1}{m^2}\frac{\partial}{\partial x_\mu}\frac{\partial A_\nu}{\partial x_\nu}\right) + \frac{1}{m^2}\frac{\partial}{\partial x_\mu}\frac{\partial A_\nu}{\partial x_\nu} \tag{14}$$

where the first part

$$A_\mu - \frac{1}{m^2}\frac{\partial}{\partial x_\mu}\frac{\partial A_\nu}{\partial x_\nu} \tag{15}$$

describes the quanta with spin 1, while the second one

$$\frac{1}{m^2}\frac{\partial}{\partial x_\mu}\frac{\partial A_\nu}{\partial x_\nu} \tag{16}$$

- the zero spin quanta. This can be shown by means of the invariant operator of the square of the spin momentum for the field $A_\mu(x)$ (see Appendix II):

$$\left(\Gamma^2\right)_{\mu\nu} = 2\left(\delta_{\mu\nu}\Box - \frac{\partial^2}{\partial x_\mu \partial x_\nu}\right) \tag{17}$$

with the eigenvalues $s(s+1)\Box$ for spin s . Indeed, by using Eq. (13), it is easy to verify that

$$\left(\Gamma^2\right)_{\mu\nu}\left(A_\nu - \frac{1}{m^2}\frac{\partial}{\partial x_\nu}\frac{\partial A_\lambda}{\partial x_\lambda}\right) = 2\Box\left(A_\mu - \frac{1}{m^2}\frac{\partial}{\partial x_\mu}\frac{\partial A_\lambda}{\partial x_\lambda}\right) \tag{18}$$

$$\left(\Gamma^2\right)_{\mu\nu}\frac{1}{m^2}\frac{\partial}{\partial x_\nu}\frac{\partial A_\lambda}{\partial x_\lambda} = 0\,. \tag{19}$$

Naturally, that the same quanta of zero spin are described by the scalar $\frac{\partial A_\nu}{\partial x_\nu}$ itself.

It should be emphasized that under condition (10) the gauge invariance (9) changes only the part of A_μ with spin 0

$$\frac{1}{m^2}\frac{\partial}{\partial x_\mu}\frac{\partial A'_\nu}{\partial x_\nu} = \frac{1}{m^2}\frac{\partial}{\partial x_\mu}\frac{\partial A_\nu}{\partial x_\nu} + \frac{\partial \Lambda}{\partial x_\mu} \tag{20}$$

while the part with spin 1 remains unaltered

$$A'_\mu - \frac{1}{m^2}\frac{\partial}{\partial x_\mu}\frac{\partial A'_\nu}{\partial x_\nu} = A_\mu - \frac{1}{m^2}\frac{\partial}{\partial x_\mu}\frac{\partial A_\nu}{\partial x_\nu}. \tag{21}$$

If we wish to consider the quanta with spin 1 only, it would seem necessary to eliminate the quanta with zero spin.

Our statement is that no special measures (for instance, the imposition of the supplementary condition) should be taken in the case of non-zero rest mass of the vector field. It follows from the gauge-invariant field equations (5) and (6) that the quanta with spin 0 do not interact with other fields and with each other: the part of A_μ describing them obeys the free equation

$$\left(\Box - m^2\right)\frac{1}{m^2}\frac{\partial}{\partial x_\mu}\frac{\partial A_\nu}{\partial x_\nu} = 0 \tag{22}$$

(a trivial consequence of Eq. (10)).

Hence, the part of A_μ with spin 1 obeys the equation with interaction (5)

$$\left(\Box - m^2\right)\left(A_\mu - \frac{1}{m^2}\frac{\partial}{\partial x_\mu}\frac{\partial A_\nu}{\partial x_\nu}\right) = -j_\mu . \tag{23}$$

Thus, the quanta with spin 0 do not affect the physical aspect of the problem: if a certain assembly of these quanta is present in the initial state, then the same one will be in the finite state . Otherwise, the matrix element of the S-matrix is zero.
This statement may be formulated in terms of the conservation laws. The conservation laws of the total four-momentum $P^{(0)}_\mu$ and of the angular momentum $M^{(0)}_{\mu\nu}$ of the scalar field $\frac{1}{m}\frac{\partial A_\nu}{\partial x_\nu}$ follow from free equation (13). Moreover, from this equation follows the conservation law of the number of quanta with every value of the momenta. Thereby, the S-matrix is diagonal with respect to all the quantum numbers of the spin 0 quanta.

The imposition of the supplementary condition of the form

$$\frac{\partial A_\mu}{\partial x_\mu} = 0 \tag{24}$$

or

$$\left(\frac{\partial A_\mu}{\partial x_\mu}\right)_- \Phi = 0^{*} \tag{25}$$

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* This condition fixes the gauge in this theory and signifies the absence of the spin 0 quanta in the physical states Φ

is superfluous for the elimination of spin 0, since it concerns only that part of $A_\mu(x)$ which describes the free zero spin quanta interacting with nothing.

One more reason, sometimes the main one,/1,2/ for imposing condition (24) is that otherwise the operator of the total energy P_0 is not positive-definte. However, the operator of the energy $P_0^{(0)}$ of always free quanta with spin 0 is conserved. Therefore, the operator*

$$P_0^{Phys} = P_0 - P_0^{(0)} \tag{26}$$

may be taken as the physical operator of the energy. This operator is conserved, it is gauge-invariant and its spectrum is positive-definite. Such a subtraction might not essentially have been done. Then the energy would have been counted off not from zero but from a certain negative level dependent on the gauge. Thus, from this point of view there is no necessity in supplementary condition (24) either.

4. The Equivalence with the Conventional Theory with the Supplementary Condition

Although the scalar field $\frac{1}{m}\frac{\partial A_\nu}{\partial x_\nu}$ turned out to be free, it is not yet eliminated from Dirac equation (6). Let us now decompose the operator ψ multiplicatively into the gauge dependent and gauge independent parts

$$\psi(x) = \exp\left[ig\frac{1}{m^2}\frac{\partial A_\nu}{\partial x_\nu}\right]\varphi(x) \tag{27}$$

where

$$\varphi(x) = \exp\left[-ig\frac{1}{m^2}\frac{\partial A_\nu}{\partial x_\nu}\right]\psi(x) \tag{28}$$

is the gauge independent** part of the ψ. Then in terms of the variables φ the Lagrangian (4) and Dirac equation (6) are written as

$$\mathcal{L}(x) = -\frac{1}{2}\frac{\partial A_\nu}{\partial x_\mu}\frac{\partial A_\nu}{\partial x_\mu} - \frac{m^2}{2}A_\mu A_\mu + \frac{ig}{2}\bar{\varphi}\gamma_\mu(\varphi A_\mu^{(1)} + A_\mu^{(1)}\varphi) - \bar{\varphi}\left(\gamma_\mu\frac{\partial}{\partial x_\mu} + M\right)\varphi \tag{29}$$

$$\left(\gamma_\mu\frac{\partial}{\partial x_\mu} + M\right)\varphi = \frac{ig}{2}\gamma_\mu\left(\varphi A_\mu^{(1)} + A_\mu^{(1)}\varphi\right) \tag{30}$$

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* An explicit form of the operator $P_0^{Phys.}$ for the free field A_μ see in Appendix III.

** Up to the constant phase factor: after the transformation (8), (9), (10).

$$\varphi' = \exp\left[-\frac{ig}{m^2}(\Box - m^2)\Lambda\right]\varphi = \exp\left[-\frac{ig}{m^2}\,\mathrm{const}\right]\varphi$$

where

$$A_\mu^{(1)} \equiv A_\mu - \frac{1}{m^2}\frac{\partial}{\partial x_\mu}\frac{\partial A_\nu}{\partial x_\nu} . \qquad (31)$$

Note further, that the Lagrangian density (29) may be put as

$$\mathcal{L}(x) = -\frac{1}{4}F_{\mu\nu}^{(1)}F_{\mu\nu}^{(1)} - \frac{m^2}{2}A_\mu^{(1)}A_\mu^{(1)} + \frac{1}{2}\left(\frac{\partial}{\partial x_\mu}\frac{1}{m}\frac{\partial A_\nu}{\partial x_\nu}\right)^2 + \frac{m^2}{2}\left(\frac{1}{m}\frac{\partial A_\nu}{\partial x_\nu}\right)^2 + \qquad (32)$$
$$+ \frac{ig}{2}\bar{\varphi}\gamma_\mu\left(\varphi A_\mu^{(1)} + A_\mu^{(1)}\varphi\right) - \bar{\varphi}\left(\gamma_\mu\frac{\partial}{\partial x_\mu} + M\right)\varphi$$

up to the four-dimensional divergence. Here

$$F_{\mu\nu}^{(1)} \equiv \frac{\partial A_\nu^{(1)}}{\partial x_\mu} - \frac{\partial A_\mu^{(1)}}{\partial x_\nu} . \qquad (33)$$

This Lagrangian corresponds to the conventional theory/17/ of the interaction of the spinor φ and the vector $A_\mu^{(1)}$ fields, i.e., to the theory with the supplementary condition

$$\frac{\partial A_\mu^{(1)}}{\partial x_\mu} = 0. \qquad (34)$$

Besides, Lagrangian (32) describes the free scalar field $\frac{1}{m}\frac{\partial A_\nu}{\partial x_\nu}$.

The commutation relations for the fields φ and $A_\mu^{(1)}$ are the same as in the conventional vector theory with the supplementary condition* . At the same time the interacting fields φ and $A_\mu^{(1)}$ commute with the scalar field $\frac{1}{m}\frac{\partial A_\nu}{\partial x_\nu}$ and with all its derivatives. Therefore, the scalar field $\frac{1}{m}\frac{\partial A_\nu}{\partial x_\nu}$ is dynamically quite independent. Thus, the theory under consideration is completely equivalent to the conventional theory of the neutral vector field with the supplementary condition. The equivalence of both theories may be established in another manner, by means of the unitary Dyson transformation/18,11/, which also "switch off" the vector interaction of the scalar field.

Let us emphasize that the imposition of the supplementary condition does not only give the non-gauge invariant form to the theory, but also causes some other troubles. For instance, the use of the propagators corresponding to such a theory in the perturbation theory makes the renormalizability not quite evident (see/12/, page 343).

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* They may be obtained from the commutation relations (7) and the equations of motion (5), (6).

5. Conclusion

Let us make a remark on the mass renormalization. Only that part of A_μ is subject to the gauge transformations which describes the quanta with spin 0. Therefore, only for these quanta one can expect that their mass must not be renormalized. Indeed, this is so, since they interact with nothing. As for the quanta with spin 1, their mass is, naturally, renormalized owing to the interaction. With account of the mass remormalization of the quanta with spin 1, Eq. (5) may be, e.g., put as

$$(\Box - m^2)A_\mu = -j_\mu - \delta m^2\left(A_\mu - \frac{1}{m^2}\frac{\partial}{\partial x_\mu}\frac{\partial A_\nu}{\partial x_\nu}\right) . \qquad (35)$$

Finally, note, that in the considered theory of the massive vector field the Green functions obey the same gauge transformation laws as in the electrodynamics /19-23/. These laws connect the expectation values of the products of operators written in two gauges. It is implied the expectation values are taken for the same state which is the vacuum in both gauges for the spin 1 quanta, but only from the point of view of one of them this state does not contain the spin 0 quanta. (Appendix 1). From the above-mentioned laws just as in electrodynamics follow the Ward's identities.

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Appendix I

The Conservation Law Following from the Gauge Invariance

The Lagrangian density (4) after the gauge transformation (8), (9) with account of condition (10) differs from the original one by the divergence

$$\mathcal{L}'(x) = \mathcal{L}(x) - \frac{\partial}{\partial x_\mu}\left(A_\nu \frac{\partial^2 \Lambda}{\partial x_\mu \partial x_\nu}\right) - \frac{1}{2}\frac{\partial}{\partial x_\mu}\left(\frac{\partial \Lambda}{\partial x_\nu}\frac{\partial^2 \Lambda}{\partial x_\mu \partial x_\nu}\right) \qquad (1.1)$$

or infinitesimally

$$\delta \mathcal{L} = -\frac{\partial}{\partial x_\mu}\left(A_\nu \frac{\partial^2 \Lambda}{\partial x_\mu \partial x_\nu}\right) \qquad (1.2)$$

Calculating the variation $\mathcal{L}(x)$ in the usual manner by using the Euler equations we are lead to the conservation law (11)

$$\frac{\partial}{\partial x_\mu}\left[-j_\mu \Lambda - \frac{\partial A_\nu}{\partial x_\mu}\frac{\partial \Lambda}{\partial x_\nu} + A_\nu \frac{\partial^2 \Lambda}{\partial x_\mu \partial x_\nu}\right] = 0 \qquad (1.3)$$

Any functions $\Lambda(x)$ restricted only by Eq .. (10) may be substituted into this conservation law. Therefore, (1.3) is the continuum of the conservation laws in consistence with the continium of the parameters of the gauge group.

One can easily verify that Eq. (1.3) is equivalent to Eq. (5).

Note, that the conservation law (1.3) may be also written as

$$\frac{\partial}{\partial x_\mu}\left[\left(\frac{\partial}{\partial x_\mu}\frac{\partial A_\nu}{\partial x_\nu}\right)\Lambda - \frac{\partial A_\nu}{\partial x_\nu}\frac{\partial \Lambda}{\partial x_\mu} + A_\mu(\Box - m^2)\Lambda\right] = 0 . \qquad (1.4)$$

The conserving operator in the quantum field theory is the infinitesimal operator of the state vectors transformation which connects the eigen vectors with the identical eigenvalues in two different frames of reference, gauges etc. In passing from one gauge to another one such a transformation is

$$\Psi' = U \Psi \qquad (1.5)$$

where

$$U = \exp\left\{\int d\vec{x}\left[-j_4 \Lambda - \frac{\partial A_\nu}{\partial x_4}\frac{\partial \Lambda}{\partial x_\nu} + A_\nu \frac{\partial^2 \Lambda}{\partial x_4 \partial x_\nu}\right]\right\} = \qquad (1.6)$$

$$= \exp\left\{\int d\vec{x}\left[\left(\frac{\partial}{\partial x_4}\frac{\partial A_\nu}{\partial x_\nu}\right)\Lambda - \frac{\partial A_\nu}{\partial x_\nu}\frac{\partial \Lambda}{\partial x_4} + A_4(\Box - m^2)\Lambda\right]\right\} . \qquad (1.7)$$

Now the law of the gauge transformations of the field operators (8) and (9) may be written also

$$\psi'(x) = U \psi(x) U^{-1} = \exp\left[ig \Lambda(x)\right]\psi(x) \qquad (1.8)$$

$$A'_\mu(x) = U A_\mu(x) U^{-1} = A_\mu(x) + \frac{\partial \Lambda(x)}{\partial x_\mu} \qquad (1.9)$$

what can be easily verified by means of the commutation relations (7).

From the point of view of different gauges the same state posses different assemblies of non-interacting spin 0 quanta.

We demonstrate this by the transformation of the vaccum. Here we restrict ourselves to $\Lambda(x)$ satisfying to

$$(\Box - m^2)\Lambda(x) = 0 \tag{1.10}$$

what allows to represent $\Lambda(x)$ in the form

$$\Lambda(x) = \int \frac{d\vec{p}}{\sqrt{(2\pi)^3 2p_0}} \left[\lambda(\vec{p}) e^{ipx} + \lambda^*(\vec{p}) e^{-ipx}\right] \quad / p_0 \equiv \sqrt{\vec{p}^2 + m^2} / . \tag{1.11}$$

Besides, owing to free equation (13) it is possible to write down $\frac{\partial A_\mu}{\partial x_\mu}$ as

$$\frac{\partial A_\mu}{\partial x_\mu} = i \int \frac{d\vec{p}}{\sqrt{(2\pi)^3 2p_0}} \left[a_\mu p_\mu e^{ipx} - a^+_\mu p_\mu e^{-ipx}\right] \quad / p_0 \equiv \sqrt{\vec{p}^2 + m^2} / \tag{1.12}$$

where $a_\mu p_\mu$ and $a^+_\mu p_\mu$ are the <u>creation</u> and <u>annihilation</u> operators* of the spin 0 quanta with the commutation relations **

$$\left[a_\mu(\vec{p}_1) p_{1\mu}, a^+_\nu(\vec{p}_2) p_{2\nu}\right] = -m^2 \delta(\vec{p}_1 - \vec{p}_2) \tag{1.13}$$

consistent with (7).

Substituting (1.11) and (1.12) into (1.7), we get

$$U = \exp\left\{-i \int d\vec{p} \left[a_\mu p_\mu \lambda^*(\vec{p}) + a^+_\mu p_\mu \lambda(\vec{p})\right]\right\}. \tag{1.14}$$

If we determine the vacua in two different gauges according to ***

$$a^+_\mu p_\mu \Psi_0 = 0 \qquad a'^+_\mu p_\mu \Psi'_0 = 0 \tag{1.15}$$

then the transformation U (1.14) connecting Ψ'_0 and Ψ_0 yields

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* See Appendix III.

** We may consider them as a consequence of more general commutation relations

$$\left[a_\mu(\vec{p}_1), a^+_\nu(\vec{p}_2)\right] = \delta_{\mu\nu} \delta(\vec{p}_1 - \vec{p}_2)$$

which are meaningful only in the free case.

*** We do not write the conditions that the spin 1 quanta are absent in Ψ_0 and Ψ'_0 . These conditions are gauge invariant.

$$\Psi_0' = U\Psi_0 = \exp\left\{-\frac{m^2}{2}\int d\vec{p}\,\lambda(\vec{p})\lambda^*(\vec{p}) - i\int d\vec{p}\,\lambda^*(\vec{p})a_\mu p_\mu\right\}\Psi_0 . \qquad (1.16)$$

The latter expression shows the expansion of the vacuum of the new gauge Ψ_0' over the old states which are created of the old creation operators $a_\mu p_\mu$ from the old vacuum Ψ_0 .

Thus, the vacuum from the point of view of one gauge is not a vacuum from the point of view of another.

This is understandable since condition (1.15) as well as a more general condition (25) are not gauge-invariant for they are the conditions which fix the gauge.

Appendix II.

The Operator of the Square of the Spin Momentum for the Field A_μ .

A general definition has been given for the operator of the square of the spin momentum (one of the invariants of an inhomogeneous Lorentz group) for arbitrary many-component functions transforming according to the representations of the inhomogeneous of the Lorentz group/13-16/. This definition is/16/

$$\Gamma^2 = -p_\lambda^2 m_{\rho\sigma} m_{\rho\sigma} + m_{\lambda\rho} m_{\lambda\sigma} p_\rho p_\sigma \qquad (11.1)$$

where

$$p_\lambda = \frac{1}{i}\frac{\partial}{\partial x_\lambda} \qquad (11.2)$$

$$m_{\rho\sigma} = \frac{1}{i}\left(x_\rho \frac{\partial}{\partial x_\sigma} - x_\sigma \frac{\partial}{\partial x_\rho}\right) + s_{\rho\sigma} \qquad (11.3)$$

are the infinitesimal operators of the translation and 4-rotation for the given function. The matrices $s_{\rho\sigma}$ are the infinitesimal operators of a rotation of its components.

Since for the vector function

$$(s_{\rho\sigma})_{\mu\nu} = i\left(\delta_{\rho\mu}\delta_{\sigma\nu} - \delta_{\rho\nu}\delta_{\sigma\mu}\right) \qquad (11.4)$$

then the substitution (11.2), (11.3) and (11.4) in (11.1) gives just expression (17)

$$(\Gamma^2)_{\mu\nu} = 2\left(\delta_{\mu\nu}\Box - \frac{\partial^2}{\partial x_\mu \partial x_\nu}\right). \qquad (11.5)$$

The law of the vector function $A_\mu(x)$ transformation, is independent if $A_\mu(x)$ obeys some equation or not. Therefore, the same is valid for the infinitesimal operators p_λ and $m_{\rho\sigma}$ and the operator of the square of the spin momentum (11.1) or (11. 5) constructed of them. $A_\mu(x)$ may be decomposed into independent parts with spins 0 and 1 when there is no interaction. However, this might be not so if there is any interaction.

In our case, as we saw (§ 3), $A_\mu(x)$ decomposes into dynamically independent parts with spins 0 and 1 even in the presence of the interaction.

Appendix III

Normal Product of the Vector Field Operators and the Definition of Physical Operator of Energy in the Free Case

In the free case it is convenient to determine all the operators as normal products, i.e., in such a manner, that the creation operators would be in the left-hand side from the annihilation operators.

For instance, in such a sense we should like to understand the 4-momentum operator

$$P_\mu = \int d\vec{p}\; p_\mu \; :a_\nu^+ a_\nu: \qquad \begin{array}{l} / \; p_0 \equiv \sqrt{\vec{p}^2 + m^2} \; / \\ / \; p^2 = -m^2 \; / \end{array} \tag{111.1}$$

what is denoted by colons.

The operators a_ν and a_ν^+ obey the commutation relations

$$[a_\mu(\vec{p}_1), a_\nu^+(\vec{p}_2)] = \delta_{\mu\nu}\, \delta(\vec{p}_1 - \vec{p}_2) . \tag{111.2}$$

It is clear from these relations that a_m and a_0^+ are the annihilation operators, while a_m^+ and a_0 are the creation operators. However, the writing of the normal product in (111.1) in the form

$$:a_\nu^+ a_\nu: \; = a_m^+ a_m - a_0 a_0^+ \tag{111.3}$$

is not acceptable, at least, because of the non-covariance. The covariant definition can be given if we decompose the operators a_μ and a_μ^+ into the parts with spin 0 and 1.

	Annihilation Operators	Creation Operators
$s=1$	$a_\mu + \frac{a_\nu p_\nu}{m^2} p_\mu$	$a_\mu^+ + \frac{a_\nu^+ p_\nu}{m^2} p_\mu$
$s=0$	$-\frac{a_\nu^+ p_\nu}{m^2} p_\mu$	$-\frac{a_\nu p_\nu}{m^2} p_\mu$

The spin of these parts may be determined by means of the invariant operator of the square of the spin momentum (17)*, written in the momentum representation. The meaning of them as the annihilation and creation operators follows from (111.2).

Now the normal products are written as, e.g. ;

$$:\left(a_\mu^+ + \frac{a^+p}{m^2}p_\mu\right)\left(a_\nu + \frac{ap}{m^2}p_\nu\right): = :\left(a_\nu + \frac{ap}{m^2}p_\nu\right)\left(a_\mu^+ + \frac{a^+p}{m^2}p_\mu\right): = \left(a_\mu^+ + \frac{a^+p}{m^2}p_\mu\right)\left(a_\nu + \frac{ap}{m^2}p_\nu\right) \qquad (111.5)$$

$$:\frac{a^+p}{m^2}p_\mu \frac{ap}{m^2}p_\nu : = :\frac{ap}{m^2}p_\nu \frac{a^+p}{m^2}p_\mu : = \frac{ap}{m^2}p_\nu \frac{a^+p}{m^2}p_\mu \,. \qquad (111.6)$$

In the mixing products with $s=1$ and $s=0$ the order is not essential due to the commutation.

The normal product in (111.1) may be now expanded as

$$P_\mu = \int d\vec{p}\; p_\mu \left\{ \left[a_\nu^+ a_\nu + \frac{(a^+p)(ap)}{m^2} \right] - \frac{(ap)(a^+p)}{m^2} \right\} \qquad (111.7)$$

The first term in the curved brackets is a positive-definite operator of the number of quanta with spin 1, whereas the second one is a negative-definite operator of the number of quanta with spin 0. Accordingly, the energy of quanta with spin 1 is positive-definite, and the quanta with spin 0 is negative-definite.

If we subtract from P_μ a four-vector of the energy-momentum of the spin 0 quanta, we get the operator of the energy-momentum of the spin 1 (physical) quanta

$$P_\mu^{Phys} = \int d\vec{p}\; p_\mu \left[a_\nu^+ a_\nu + \frac{(a^+p)(ap)}{m^2} \right] \qquad (111.8)$$

with the positive-definite energy.

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* See Appendix II.

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