ОБЪЕДИНЕННЫЙ ИНСТИТУТ ЯДЕРНЫХ ИССЛЕДОВАНИЙ

Лаборатория теоретической физики

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PION-KAON SCATTERING IN THE LOW-ENERGY REGION

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Abstract

In the approximation of the effective-range theory the explicit expressions for s and p phase shifts of the $\pi - K$ scattering are derived.

1

In the paper/1/ the approximate equations for partial waves of the $\pi - K$ scattering have been obtained. There are great difficulties at present in finding exact solutions for these equations: first it is necessary to know s and p phase shifts of the pion-pion scattering and secondly up to the present time there is no reliable selection rule of a physical solution among solutions obtained by the interaction way on computers.

However in rough approximation in the low-energy region one can obtain one of possible solutions of the system of Eqs. (25) — (26) and (30) — (33) from the paper/1/ namely, the solution $T_\ell^{1/2} = T_\ell^{3/2}$ where $T_\ell^{\mathcal{J}}(q^2)$ is the partial wave of the $\pi - K$ scattering depending on the square of the π-meson momentum q^2 in the c.m.s. in the state where isotopic spin is equal to $\mathcal{J}$ and orbital momentum $\ell = 0$ or I. It is advisable to use it as a first approximation when solving more exact equations by the interaction method on computers since it is known, that the form of solution obtained with this method depends strongly on the choice of the first approximation.

So, we assume that $T_\ell^{1/2} = T_\ell^{3/2}$. Then the system of Eqs. (25) — (26) from/1/ written for partial waves of the $\pi - K$ scattering averaged over all angles takes the form:

$$Re\, T_\ell(q^2) = Re\, T_\ell(0) + \frac{1}{16\pi^2}\mathcal{P}\int_0^\infty dx \frac{\sqrt{x}}{W(x)} \Big\{ \big[n_\ell(x,q^2) - n_\ell(x,0) \big] |T_0(x)|^2 + 3\big[\varphi_\ell(x,q^2) - \varphi_\ell(x,0) \big] |T_1(x)|^2 \Big\} \qquad (1)$$

and the system of Eqs. (30) — (33) from/1/ for the $\pi - K$ scattering amplitudes in the case of the backwards scattering will be of the form:

$$Re\, T_0(q^2) = Re\, T_0(0) + \frac{1}{8\pi^2}\mathcal{P}\int_0^\infty dx \frac{\sqrt{x}}{W(x)} \Big\{ \frac{|T_0(x)|^2}{x(x-q^2)} \Big[x\big(V(x,q^2) - V(x,0) \big) + q^2 V(x,0) \Big] + 3\frac{|T_1(x)|^2}{x(x-q^2)} \Big[x\big(L(x,q^2) - L(x,0) \big) + q^2 L(x,0) \Big] \Big\} \qquad /2/$$

$$
\mathrm{Re}\, T_1(q^2) = \frac{1}{24\pi^2}\, \mathcal{P}\int_0^{\infty} dx \frac{\sqrt{x}}{W(x)} \left\{ \frac{|T_0(x)|^2}{x(x-q^2)} \Big[x\Big(C(x,q^2) - C(x,0)\Big) + q^2 C(x,0)\Big] + \right.
$$

$$
\left. + 3\frac{|T_1(x)|^2}{x(x-q^2)} \Big[x\Big(Q(x,q^2) - Q(x,0)\Big) + q^2\, Q(x,0)\Big]\right\} \qquad /3/
$$

In Eqs. (1)–(3) $h_\ell, \varphi_\ell, V, L, C$ and Q - are complex functions of the variables q^2 and x whose complete notations are given in /1/. $W(x)$ is the total energy of the $\pi - K$ system.

It is interesting to note that the p -phase shift of the $\pi - \pi$ scattering does not enter Eqs. (1) –(3) which gives greater freedom in choosing an explicit expression for the S-phase shift of the $\pi - \pi$ scattering since it is not necessary now to take care of the self-consistency of the s and p phase shifts.

In the present paper the expression for the S phase shift of the $\pi - \pi$ scattering is taken from the paper /2/.

The S phase shift of the $\pi - \pi$ scattering enters Eqs. (1) – (3) by means of the exponent $\exp\,[U(q^2, z_1) - U(x, z_1)]$. The calculation shows that the account of the S phase shift leads for the exponent to values close to the unit: deflections do not exceed 5 percent up to the values of $x \sim 10\mu^2$ (μ is the mass of the π -meson at rest) in Eqs. (1) and do not exceed 15 percent up to the same values of x in Eqs. (2)–(3). If $x \geq 10\mu^2$ then the subtraction procedure must count greatly and the integral contribution in Eqs. (1)–(3) in the region $x > 10\mu^2$ must be negligibly small.

For the sake of simplicity of calculations it is natural to assume that $\exp\,[U(q^2, z_1) - U(x, z_1)] = 1$ in the whole region of change of the variable x . Under the same assumption, namely, that the integrals over x in Eqs. (1)–(3) contribute mainly in the region of small x , we can use the following approximate expressions for the functions $h_\ell, \varphi_\ell, V, C, L, Q$ and W:

$$
h_\ell(x,q^2) = \frac{1}{x-q^2}\int_{-1}^{+1} P_\ell(z_1)\, dz_1 \; ; \qquad \varphi_\ell(x,q^2) = \frac{1}{x-q^2}\int_{-1}^{+1} z_1 \cdot P_\ell(z_1)\, dz_1 \; ;
$$

$$
V(x,q^2) = 1 \; ; \qquad L(x,q^2) = 0 \; ; \qquad C(x,q^2) = 0 \; ;
$$

$$
Q(x,q^2) = 1 \; ; \qquad W(x) = \sqrt{(M+\mu)^2 + x\left(2 + \frac{M}{\mu}\right)}
$$

M bieng the mass of the K -meson.

By inserting these approximate expressions into Eqs. (1)–(3) we obtain in both cases the same system of splitting up equations:

$$Re\, T_0(q^2) = Re\, T_0(0) + \frac{q^2}{8\pi^2} \mathcal{P} \int_0^\infty dx \frac{\sqrt{x}}{W(x)} \frac{|T_0(x)|^2}{x(x-q^2)} \tag{4}$$

$$Re\, T_1(q^2) = \frac{q^2}{8\pi^2} \mathcal{P} \int_0^\infty dx \frac{\sqrt{x}}{W(x)} \cdot \frac{|T_1(x)|^2}{x(x-q^2)} . \tag{5}$$

2

The connection of isotopic spin states T^J with scattering phase shifts is of the form:

$$T^J = \frac{8\pi W(x)}{\sqrt{x}} \sum_{\ell=0}^{\infty} (2\ell+1) \cdot exp\{i\delta_\ell^J\} \sin\delta_\ell^J \cdot P_\ell(z_1) . \tag{6}$$

From which it is easy to obtain the following relation :

$$\frac{Im\, T_\ell(x)}{|T_\ell(x)|^2} = \frac{\sqrt{x}}{8\pi W(x)} \tag{7}$$

3

We shall seek the solution of the equation (4) in the form

$$T_0(q^2) = \frac{b}{V(q^2)} \tag{8}$$

where $Re\, V(0)$ is the quantity related to the scattering wave length, and b is a some constant. From (8) it follows that the analytical properties of the function $V(q^2)$ are defined by those of the function $T_0(q^2)$. By using the condition (7) we obtain an explicit expression for $V(q^2)$:

$$V(q^2) = Re\, V(0) - \frac{bq^2}{8\pi^2} \int_0^\infty dx \frac{\sqrt{x}}{W(x)} \cdot \frac{1}{x(x-q^2)} \tag{9}$$

or

$$V(q^2) = Re\, V(0) - \frac{b}{8\pi^2\sqrt{\lambda+\frac{\nu}{q^2}}}\, \ln \frac{\sqrt{\lambda+\frac{\nu}{q^2}}-\sqrt{\lambda}}{\sqrt{\lambda+\frac{\nu}{q^2}}+\sqrt{\lambda}} - \frac{i\,b}{8\pi\sqrt{\lambda+\frac{\nu}{q^2}}} \tag{10}$$

where $\nu = (M+\mu)^2$, $\lambda = 2+\frac{M}{\mu}$ *.

From (8) and (10) it is seen that the only constant, namely, $\frac{b}{Re\, V(0)}$ enters practically T_0 . In the following we assume therefore the value b to be equal to the unit.

We must demand further that in the whole plane of the complex variable z the function $V(z)$ be different from zero. It follows from here that the subtraction constant can have only positive values, i.e. $Re\, V(0) > 0$ **.

By using the expression (10) we get the following expression for the S -phase shift:

$$\frac{q}{\sqrt{\lambda q^2+\nu}}\, ctg\, \delta_0 = 8\pi\, Re\, V(0) - \frac{1}{\pi\sqrt{\lambda+\frac{\nu}{q^2}}}\, \ln \frac{\sqrt{\lambda+\frac{\nu}{q^2}}-\sqrt{\lambda}}{\sqrt{\lambda+\frac{\nu}{q^2}}+\sqrt{\lambda}} \tag{11}$$

To compare the S phase shift obtained in the present paper with that of the paper /3/ it should be assumed in (10a) that the quantity $q_0^2 = \mu^2$ (this corresponds very close to the subtraction point from the paper/3/). The results of the comparison are given on the diagram I (a small difference is connected to the choice of the approximate expression for $W(x)$).

It is important to note that the region of change of the constant $Re\, V(0)$ depends explicitly on the choice of the subtraction point. By choosing suitably constants the expression (11) coincides practically with the expression for the S- phase shift from the paper /5/.

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* In the case when in the expression (9) the subtraction procedure is carried out not at zero but at a certain point $0 > q_0^2 > -\frac{\nu}{\lambda}$ and the integral is cut off for a certain value of the parametre L , we shall have for the function $V(q^2)$ the following expression:

$$V(q^2) = Re\, V(q_0^2) - \frac{1}{8\pi\sqrt{\lambda+\frac{\nu}{q^2}}}\, \ln \frac{\sqrt{\lambda+\frac{\nu}{q^2}}-\sqrt{\lambda+\frac{\nu}{L}}}{\sqrt{\lambda+\frac{\nu}{q^2}}+\sqrt{\lambda+\frac{\nu}{L}}} + \tag{10a}$$

$$+ \frac{1}{4\pi^2\sqrt{\lambda+\frac{\nu}{q_0^2}}} \left(\frac{\pi}{2} - \mathrm{arctg}\, \frac{\sqrt{\lambda+\frac{\nu}{L}}}{\sqrt{\lambda+\frac{\nu}{q_0^2}}}\right) - \frac{i}{8\pi\sqrt{\lambda+\frac{\nu}{q^2}}}$$

(In the formula (10a) instead of q_0^2 is should insert positive values of q_0^2 , namely $\frac{\nu}{\lambda} > q_0^2 \geq 0$). It is advisable to use the expression (10a) in comparing the results of the present paper with those of the paper/3/.

** Similar restrictions imposed on the value of the coupling constant have been treated in papers/3/ and /4/. These restrictions can be obtained in our work too, if we introduce the subtraction procedure at a suitable point and demand that the function $V(q^2)$ have no zero in the region $-\mu^2 \leq q^2 \leq 0$

4

The expression (5) contains no inhomogeneous term and therefore, in finding solution by the reverse function method it turns out to be necessary to take into account the negative cut. Indeed, if we shall seek solution of the equation (5) in the form:

$$T_1(q^2) = \frac{q^2}{V(q^2)} \tag{12}$$

use the expression for $\mathrm{Im}\, V(q^2)$:

$$\mathrm{Im}\, V(q^2) = -\frac{q^3}{V(q^2)}$$

and assume as in the case of the S phase shift, the analytical properties of the function $V(q^2)$ to be defined by those of the function $T_1(q^2)$ then writing the Cauchy theorem for the function $V(q^2)$ and performing two subtractions we get:

$$\mathrm{Re}\, V(q^2) = A + Bq^2 - \mathcal{J}(q^2) \tag{13}$$

where

$$A = \mathrm{Re}\, V(0)$$

$$B = \frac{\partial\, \mathrm{Re}\, V(q^2)}{\partial q^2}\Big/_{q^2=0}$$

$$\mathcal{J}(q^2) = \frac{q^4}{8\pi^2}\, \mathcal{P}\int_0^\infty \frac{\sqrt{x}\, dx}{x(x-q^2)W(x)}$$

The elementary check shows however that the function $V(q^2)$ obtained in this way have zero in the region $q^2 < 0$. To avoid zero in this region it is necessary

a) or take into account the influence of the negative cut*

b) or introduce the cut-off that leads to the appearence of the third constant.

For our purpose - obtaining approximate solution of the equation (5) in the region of small q^2 - it is sufficient to introduce the cut off L provided $q^2 \ll L$

In this case the solution takes the form:

$$\frac{q^3}{\sqrt{\nu + \lambda q^2}}\, \mathrm{ctg}\, \delta_1 = 8\pi\left(A + Bq^2\right) - \frac{q^2}{\pi\sqrt{\lambda + \frac{\nu}{q^2}}}\, \ln \frac{\sqrt{\lambda + \frac{\nu}{q^2}} - \sqrt{\lambda + \frac{\nu}{L}}}{\sqrt{\lambda + \frac{\nu}{q^2}} + \sqrt{\lambda + \frac{\nu}{L}}} \tag{14}$$

In the region of $q^2 \approx 0$ the last term in the relation (14) tends to zero faster than the second one, and the expression can be written in the form:

$$\frac{q^3}{\sqrt{\nu + \lambda q^2}}\, \mathrm{ctg}\, \delta_1 \approx 8\pi A\left(1 + \frac{B}{A} q^2\right). \tag{15}$$

which is in good agreement with the results of the paper[5].

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* One can show[6] that the account of the influence of the negative cut under some assumptions is reduced to the appearence of the additional multiplier x^{α} $(0 < \alpha < \frac{1}{2})$ in the denominator of the integrand $\mathcal{J}(q^2)$. In this case the integral turns out to be convergent and the constants A and B can be choosen so that $V(q^2)$ should not vanish for all $q^2 < 0$.

Conclusion

Lately the problem of the $\pi - K$ scattering is being investigated very intensively. Two approaches are used:

a) perturbation method /5/ and close to it Chew-Low /7/ method and

b) Mandelstam double representation method /1,3,4/

In the latter the attempts of investigating the $\pi - K$ scattering are somewhat different one from another: in the paper /3/ the investigation is carried out by means of the 'classical' Mandelstam representations, in the paper /4/ the Cini-Fubini method is used, and in the paper /1/ one uses a way of symmetrization and antisymmetrization of amplitudes, in order to avoid difficulties connected with the kinematic cut /2/. However, in all three cases the obtaining of exact solutions of equations for the $\pi - K$ scattering is confronted by great difficulties and solutions can be obtained only after per forming some simplifications. In particular, in papers 3,4 solutions have been sought in the effective- range approximation. The comparison of solutions for the s and p phase shifts obtained in the present paper and papers/3,4,5/ shows that the results coincide.

Thus, solutions (11) and (15) get in a rather rough approximation can nevertheless turn out to be useful in the two cases: a) they can be considered with great reliability to be true for the $\pi - K$ scattering in the π meson low-energy region $< \mu c^2$;

b) the solutions (11) and (15) can be used as a first approximation in solving more exact equations from /1/ by the iteration method on computers.

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References

1. P.S. Isaev and M.V. Sewerynski. Preprint JINR P-550; Nucl.Phys. - in print.
2. A.V. Efremov, V.A. Meshcheryakov, D.V. Shirkov. JETP , 39, 438 (1960).
3. B.M. Lee. Phys.Rev. 120, 325 (1960).
4. M. Gourdin, Y. Noirot and Ph. Salin. Nuovo Cim. XYIII, 651 (1960).
5. S. Okubo. Phys.Rev. 118, 357 (1960).
6. A.V. Efremov - private communication.
7. G. Costa, L. Tenaglia. Nuovo Cim., XYIII, 368 (1960).
8. M. Cini and S. Fubini. Ann. Phys. 3, 352 (1960).

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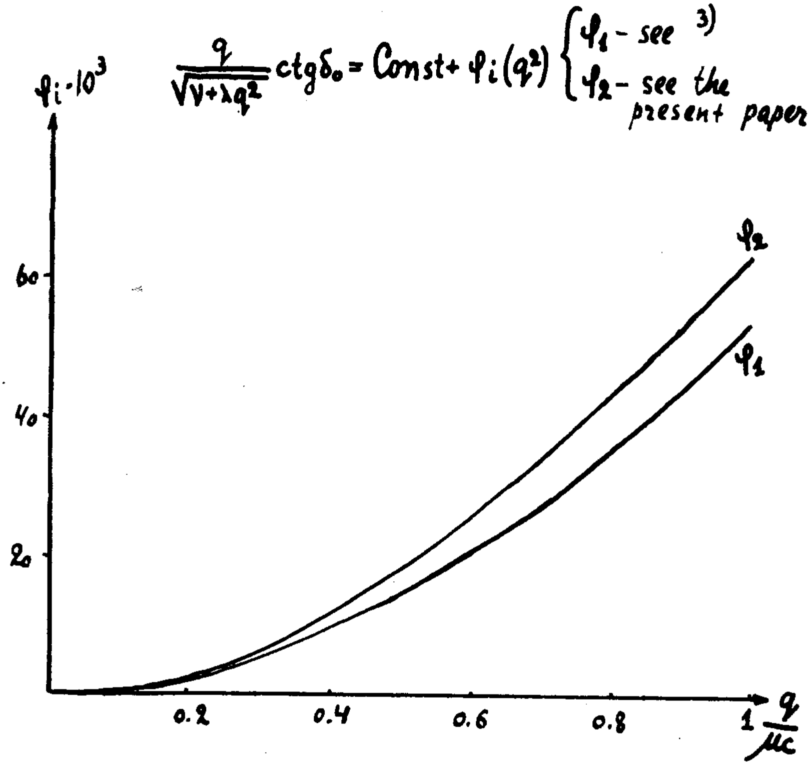

$\varphi_i \cdot 10^3$
$\frac{q}{\sqrt{\nu+\lambda q^2}} \operatorname{ctg}\delta_0 = Const + \varphi_i(q^2)$
φ_1 - see 3)
φ_2 - see the present paper
60
40
20
0.2
0.4
0.6
0.8
1
$\frac{q}{\mu c}$
φ_2
φ_1

Fig. 1.