ЛАБОРАТОРИЯ ТЕОРЕТИЧЕСКОЙ ФИЗИКИ

ОБЪЕДИНЕННЫЙ ИНСТИТУТ ЯДЕРНЫХ ИССЛЕДОВАНИЙ

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Abstract

The elastic nucleon-nucleon scattering is studied by using the two dimensional spectral representation of Mandelstam. In this first paper, the convergence problem arising from the combination of the forward dispersion relation with the unitarity condition on the physical cut, is solved by use of a conformal transformation.

1. A system of integral equations for the elastic scattering of protons on protons in the singlet state was obtained by using the two-dimensional spectral representation of Mandelstam. The method is based on the approach recently introduced by Efremov, Meshcheryakov, Shirkov and Tzu [1]. We briefly recall their idea. The dispersion relations are written only for such angles (forward and backward scattering) for which they are simple and at which no spectral functions are met in the unphysical region. Then, the dispersion relations are differentiated with respect to $\cos\theta$ at these points. In the whole region $-1 \leq \cos\theta \leq 1$ the amplitude $M(q^2, \cos\theta)$ is represented by a Taylor expansion around these points and this series is inserted into the unitarity condition. The partial waves can also be defined in terms of these Taylor coefficients.

However, as it will be shown, this approach leads, on the physical cut, to a problem of convergence, due to the fact that the cosine singularities of $M(q^2, \cos\theta)$ located at the non-physical region $\cos\theta > 1$ restrict the convergence of the Taylor expansion also in the physical region. Indeed, the cosine of the nearest singularity is $1 + \mu^2/2q^2$ and approaches asymptotically the forward scattering point, when q^2 increases* (see Fig.1). In the physical region, the expansion around $\cos\theta = 1$ converges only for $\cos\theta > 1 - \mu^2/2q^2$ and, consequently, a certain q^2_{max} exists

$$1 - \frac{\mu^2}{2q^2_{max}} = -1 \qquad i.\,e. \qquad q^2_{max} = \frac{\mu^2}{4}$$

above which the convergence circle of the Taylor expansion around $\cos\theta = 1$ does not cover the whole physical region $-1 \leq \cos\theta \leq 1$. Thus, for $q^2 > \mu^2/4$ the Taylor expansion in $\cos\theta$ cannot be used anymore in the unitarity condition.

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* q is the length of the center of mass momentum in the reaction $N+N \to N+N$, θ is the c.m. scattering angle. M is the so-called causal amplitude defined, for instance, in [2] or [3].

In the $N+N \to N+N$ process, the corresponding $E_{max} = 2\sqrt{q_{max}^2 + m^2}$ is extremely small, having the value of 10 MeV in the laboratory system. (If symmetry is taken into account and an expansion in $\cos^2\theta$ is used, E_{max} can be shifted up to 49.4 MeV). But it is desirable to ensure the convergence of the Taylor series at least up to the threshold of the first inelastic process $N+N \to N+N+\pi$ (290Me V), for which the simple unitarity condition still holds.

2. In this purpose, we made a conformal transformation of the complex cosine plane

$$w = \frac{1-z^2}{\alpha^2 - z^2} \quad , \quad z \equiv \cos\theta \, , \quad \alpha^2 = 1 + 2\frac{\mu^2}{q^2} + \frac{\mu^4}{2q^4} \qquad (1)$$

(see Fig. 2) and we showed that the series

$$\sum \frac{\partial^n M}{\partial w^n}\bigg|_{z^2=1} \frac{w^n}{n!}$$

converges in the whole interval $-1 \leq z \leq +1$ for any observable energy. Below 49.4 MeV, where the power series in $\cos^2\theta$ also converges, the series in w^n yields a better approximation if the same number of terms is taken. In this way, a quickly convergent set of integral equations is obtained.

The properties of this conformal transformation : as well as the reasons which led to this form are collected in Appendix I. Some quantitative estimations are given in Appendix II.

3. Let us now study the unitarity condition. On the physical cut, it has the following form (in the c.m. system)

$$\mathrm{Im}\, M(q^2, z) = \frac{1}{2}\frac{m^2}{(4\pi)^2}\frac{q}{\sqrt{q^2+m^2}}\int M^+(q^2, z_1)\, M(q^2, z_2)\, d^{(2)} z_1\, d\varphi$$

where

$$z_2 = z z_1 - \sqrt{1-z^2}\,\sqrt{1-z_1^2}\,\cos\varphi$$

At this stage, an expansion of M into partial waves is frequently performed which allows to write the unitarity condition for the partial waves. However, in the forward scattering method only the properties of M near $\cos\theta = 1$ are used; thus the partial wave amplitudes can be defined only by integrating the Taylor series. Since each power of w contains an infinite number of partial waves, the simultaneous use of the w power series and the partial wave expansion would lead to a reordering problem. Each coefficient of one expansion would be written as a combination of an infinite number of coefficients of the other expansion. In practical calculations, however, one must restrict oneself to a finite number of terms and thus the transition from one expansion to the other leads to supplementary errors. For this reason, we avoided the partial wave expansion and used only the power series in w. All integrals in (2) are to be performed by direct calculation.

Writing now the singlet amplitude $M_{ss}(q^2, z)$ as

$$\mathcal{M}_{ss}(q^2, z)\,\frac{1-(\vec{\sigma}_1\cdot\vec{\sigma}_2)}{4}$$

we expand $\mathcal{M}_{ss}(q^2, z)$ in w^n and take only three terms

$$\mathcal{M}_{ss}(q^2, z) = a_0(q^2) + a_1(q^2)\,w + a_2(q^2)\,w^2. \qquad (3)$$

Then, we obtain from (2) and its derivatives with respect to z

$$\mathrm{Im}\,\mathcal{M}_{ss}(q^2, 1) = \frac{1}{2}\,\frac{m^2}{(4\pi)^2}\,\frac{q}{\sqrt{q^2+m^2}}\sum_{i,j=0}^{2} a_i^* a_j\, K_{ij}$$

$$\left.\frac{\partial\,\mathrm{Im}\,\mathcal{M}_{ss}}{\partial z}\right|_{z=1} = \frac{1}{2}\,\frac{m^2}{(4\pi)^2}\,\frac{q}{\sqrt{q^2+m^2}}\sum_{i,j=0}^{2} a_i^* a_j\, K'_{ij}$$

$$\left.\frac{\partial^2\,\mathrm{Im}\,\mathcal{M}_{ss}}{\partial z^2}\right|_{z=1} = \frac{1}{2}\,\frac{m^2}{(4\pi)^2}\,\frac{q}{\sqrt{q^2+m^2}}\sum_{i,j=0}^{2} a_i^* a_j\, K''_{ij}$$

where

$$K_{ij}^{(n)} = \left.\frac{\partial^n}{\partial z^n}\int_0^{2\pi}\int_{-1}^{1} w^i(z_1)\, w^j(z_2)\, dz_1\, d\varphi\right|_{z=1}.$$

The $K_{ij}^{(n)}$-s depend only on the energy (via α^2, see (1)) and are easily obtained. Their explicit form is

$$K_{00} = 4\pi$$

$$K_{10} = 4\pi - 2\pi\,\frac{\alpha^2-1}{\alpha}\,\ln\frac{\alpha+1}{\alpha-1}$$

$$K_{11} = K_{20} = 2\pi\,\frac{3\alpha^2-1}{\alpha^2} - \pi\,\frac{(3\alpha^2+1)(\alpha^2-1)}{\alpha^3}\,\ln\frac{\alpha+1}{\alpha-1}$$

$$K_{21} = \frac{\pi}{2\alpha^4}\left(15\alpha^4 - 4\alpha^2 - 1\right) - \frac{3\pi}{4\alpha^5}\left(5\alpha^4 + 2\alpha^2 + 1\right)\left(\alpha^2 - 1\right) \ln\frac{\alpha+1}{\alpha-1}$$

$$K'_{00} = K'_{10} = K'_{20} = 0$$

$$K'_{11} = \frac{\pi}{6\alpha^4}\left(3\alpha^4 - 2\alpha^2 + 3\right) - \frac{\pi}{4\alpha^5}\left(\alpha^2+1\right)\left(\alpha^2-1\right)^2 \ln\frac{\alpha+1}{\alpha-1}$$

$$K'_{21} = \frac{\pi}{24\alpha^6}\left(15\alpha^6 - 7\alpha^4 - 7\alpha^2 + 15\right) - \frac{\pi}{16\alpha^7}\left(5\alpha^4 + 6\alpha^2 + 5\right)\left(\alpha^2-1\right)^2 \ln\frac{\alpha+1}{\alpha-1}$$

$$K''_{00} = K''_{10} = K''_{20} = 0$$

$$K''_{11} = -\frac{\pi}{120\alpha^6(\alpha^2-1)}\left(45\alpha^8 - 90\alpha^6 - 64\alpha^4 + 90\alpha^2 - 45\right) + \frac{\pi}{16\alpha^7}\left(3\alpha^4 + 2\alpha^2 + 3\right)\left(\alpha^2-1\right)^2 \ln\frac{\alpha+1}{\alpha-1}$$

$$K''_{21} = -\frac{\pi}{480\alpha^8(\alpha^2-1)}\left(225\alpha^{10} - 375\alpha^8 + 90\alpha^6 - 398\alpha^4 + 645\alpha^2 - 315\right) + \frac{\pi}{64\alpha^9}\left(15\alpha^6 + 15\alpha^4 + 13\alpha^2 + 21\right)\left(\alpha^2-1\right)^2 \ln\frac{\alpha+1}{\alpha-1}$$

$$K^{(n)}_{ij} = K^{(n)}_{ji} \quad ; \quad K^{(n)}_{22} \text{ are neglected}$$

The two-pion contribution to the unphysical cut yields an inhomogeneous term in the integral equations, which will be denoted by χ. This term, as well as other contributions to the unphysical cut, have not been studied by us in detail. They can be taken from other works [3, 4] which give a complete analysis of them. Nevertheless, needing only the forward scattering, we hope to obtain a simpler form of them.

4. Writing the integral equations only for the singlet scattering, we take advantage of the fact that in the expansion of M into the "physical" amplitudes

$$M = \alpha + \beta\,(\vec{\sigma}_1\cdot\vec{n})(\vec{\sigma}_2\cdot\vec{n}) + \gamma\left((\vec{\sigma}_1\cdot\vec{n}) + (\vec{\sigma}_2\cdot\vec{n})\right) + \delta\,(\vec{\sigma}_1\cdot\vec{m})(\vec{\sigma}_2\cdot\vec{m}) + \varepsilon(\vec{\sigma}_1\cdot\vec{l})(\vec{\sigma}_2\cdot\vec{l})$$

the dispersion relations for α, β, δ and ε have a simple form in the nonrelativistic approximation (see [2]). The singlet amplitude $\mathcal{M}_{ss}$ is given in terms of them as follows

$$\mathcal{M}_{ss} = \alpha - \beta - \delta - \varepsilon$$

and does not contain the γ -term, which fulfills a more involved dispersion relation.

The integral equations have the following form

$$a_0 = -\frac{f^2}{\mu^2}\frac{4q^2}{4q^2+\mu^2} + \chi_{ss}(t=0) + \frac{1}{2\pi}\frac{m^2}{(4\pi)^2}\int_0^\infty \frac{q'}{\sqrt{q'^2+m^2}} \frac{\sum_{i,j=0}^{2} K_{ij}\, a_i^* a_j}{q'^2-q^2}\, dq'^2$$

$$a_1 = (\alpha^2-1)q^2 f^2\left(\frac{1}{(4q^2+\mu^2)^2} - \frac{1}{\mu^4}\right) - (\alpha^2-1)q^2 \frac{d\chi_{ss}}{dt}\Big|_{t=0} -$$

$$- \frac{1}{2\pi}\frac{m^2}{(4\pi)^2}(\alpha^2-1)q^2 \int_0^\infty \frac{1}{2q'\sqrt{q'^2+m^2}} \frac{\sum_{i,j=1}^{2} K'_{ij}\, a_i^* a_j}{q'^2-q^2}\, dq'^2$$

$$2a_2 = \frac{\alpha^2+3}{2}a_1 + (\alpha^2-1)^2 q^4\, 2f^2\left(\frac{1}{(4q^2+\mu^2)^3} - \frac{1}{\mu^6}\right) + (\alpha^2-1)^2 q^4 \frac{d^2\chi_{ss}}{dt^2}\Big|_{t=0} +$$

$$+ \frac{1}{2\pi}\frac{m^2}{(4\pi)^2}(\alpha^2-1)^2 q^4 \int_0^\infty \frac{1}{4q'^3\sqrt{q'^2+m^2}} \frac{\sum_{i,j=1}^{2} K''_{ij}\, a_i^* a_j}{q'^2-q^2}\, dq'^2$$

$$(\text{here } f^2 = \frac{\mu^2}{4m^2} g^2.) \qquad (4)$$

As usual in the forward scattering case, one subtraction must be performed. However, this subtraction is to be applied only to the first equation, because the second and the third one have a correct behaviour at infinity. This was achieved by using constant momentum transfer dispersion relations instead of the constant angle ones.

5. The integral equations are written for the first three coefficients of (3) and not for the partial waves. As it has been already said, we were led to this unusual expansion (3) in order to avoid the supplementary errors arising from the simultaneous use of two expansions. But there is also another circumstance which confirms this point of view.

As it is known, the $N+N \to N+N$ scattering amplitude contains many waves so that the partial wave expansion converges very slowly. To obtain a true picture of the reality without using many approximation terms, we must expand $M(q^2, z)$ into a more convenient set of functions. Due to the method used we shall consider only the power expansions around $z^2 = 1$.

Emphasizing this idea we can say that the most rapidly convergent expansion of $M(q^2,z)$ is that in the powers of M itself, where it is reduced to the trivial identity $M=M$. But M is not known before solving the equations; the only information known about M concerns the location of the poles and cuts. In the limits of this information, the most rapidly convergent expansion will be that in powers of a function having the same location of singularities as M . An example of such a function w_M is given in Appendix 1, but it is complicated for practical calculation. Our w -function, which is much simpler, approximates this ideal case somewhat worse, but, as the numerical estimations show (see Appendix 11), three coefficients in (2) are quite sufficient.

It is pleasure for us to acknowledge the frequent stimulating discussions we had with Professor H.Y.Tzu and Dr. D.V.Shirkov. We wish also to thank to Dr. V.S.Vladimirov and E.V.Maikov for their helpful discussions concerning the mathematical aspects of this work.

Appendix 1

1. In order to secure the convergence of the power expansion around $\cos^2\theta = 1$ in the whole physical region $0 \leq \cos^2\theta \leq 1$ we transformed the left half - plane defined by the straight line C into the inside of the unit circle C in the complex plane $w = \frac{1-\cos^2\theta}{\alpha^2-\cos^2\theta}$ (see Fig.2). The point $\cos^2\theta = 1$ transforms evidently into $w = 0$.

The power expansion around $w=0$ is convergent in each point inside the unit circle, because the amplitude has no singularities in this region. All physical points are contained in this circle, thus the w -power expansion can be used in the integrals of the unitarity condition without supplementary precautions.

2. There is a great variety of conformal transformations which ensure the convergence of the corresponding Taylor expansions in the whole physical region. As it is known, the Taylor expansion behaves for $n\to\infty$ as the geometrical series $\left(\frac{r}{R}\right)^n$ where R is the convergence radius, and $r=|w|$. It would be natural to find such a transformation for which in all physical points $\frac{r}{R}$ would have a minimal value. Such an optimal transformation $w_M(z)$ actually exists and we shall show that it is that one which transforms the cut complex plane $D_{max.}$ (See Fig.3) into the unit circle. This is equivalent with the statement that for any other conformal transformation w_1 transforming an arbitrary domain $D_1 \subset D_M$ (not containing points of the z-cut !) into the unit circle, the inequality

$$r \equiv |w_1(P)| \geqslant r_M \equiv |w_M(P)| \qquad (\text{here } R_1 = R_M = 1)$$

takes place for each point $P \in D_1$.

To demonstrate this, we observe that $w_M(z)$ transforms D_1 into a domain $\tilde{D}_1$ contained in the unit circle $\tilde{D}_M$. The function $w_M = f(w_1)$ which transforms the w_1-unit circle into $\tilde{D}_1$ is analytical and has the following properties: ①. $|f(P)| \leq 1$ for each P contained in the w_1-unit circle, and
②. $f(0) = 0$, because both $w_1(z^2)$ and $w_M(z^2)$ transform the point $\cos^2\theta = 1$ into the origin.

Now, the well known Schwartz Lemma* requires

$$|w_M| \leq |w_1|$$

for all points w_1 contained in the unit circle.

3. The optimal transformation w_M can readily be constructed with the help of the Joukowsky transformation. By means of the transformation $u = \frac{2\tau^2 - z^2 - 1}{z^2 - 1}$ the cut of the z^2 plane goes into a cut between $u = -1$ and $u = +1$; and $u(z^2 = 1) = \infty$. This cut can be afterwards transformed into the unit circle (and $u = \infty$, into the origin) by means of the transformation $\frac{1}{2}(w_M + \frac{1}{w_M}) = u$. Thus,

$$w_M = 2(\tau^2 - 1)^{1/2}\,\frac{(\tau^2 - 1)^{1/2} - (\tau^2 - z^2)^{1/2}}{z^2 - 1} - 1$$

Appendix II

In order to compare the convergence rates of the different expansions, we shall treat, as an example, the case of an amplitude of the form

$$A = \frac{1}{\tau + \cos\theta} + \frac{1}{\tau - \cos\theta} \qquad \left(\tau^2 \equiv \frac{1+\alpha^2}{2}\right)$$

To check the accuracies of the first, second, etc., approximations of the $\cos^2\theta$ and $w = \frac{1 - \cos^2\theta}{\alpha^2 - \cos^2\theta}$ expansions, we shall calculate some partial coefficients of $A = \frac{1}{\tau + \cos\theta} + \frac{1}{\tau - \cos\theta}$ first exactly, by integrations, and then with the help of the approximative expressions provided for A by the $\cos^2\theta$ and expansions. These approximations are listed below, in the first and second columns respectively, the errors being expressed in percents. Table 1 is calculated for $\alpha^2 = 5/3$, for which both expansions are convergent. Table II is calculated for $\alpha^2 = 1.28$ for which the expansion in $\cos^2\theta$ is divergent, the w-expansion remaining, naturally, convergent. This corresponds to the laboratory nucleon energy of 72 MeV, but if the one-pion pole is subtracted and τ represents the location of the beginning of the two-pion singularity, this value corresponds to the inelastic threshold 290 MeV.

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* If $f(z)$ is analytical for all $|z| < R$ and if $|f(z)| \leq M$ and $f(0) = 0$, then it follows $|f(z)| \leq \frac{M|z|}{R}$.

TABLE I ($\alpha^2 = \frac{5}{3}$).

	S-wave		D-wave		G-wave	
EXACT VALUES	= 1,386		= 0,0831		= 0,0071	
EXPANSION:	in $\cos^2\theta$.	in w.	in $\cos^2\theta$.	in w.	in $\cos^2\theta$.	in w.
ONE TERM APPROX.	1,9 35%	1,9 35%	0	0	0	0
TWO TERMS APPROX.	1,17 -15%	1,30 -6%	0,14 69%	0,105 26%	0	0,005 -30%
THREE TERMS APPROX.	1,49 7%	1,403 1%	0,05 -39%	0,078 -5%	0,015 113%	0,0082 17%
FOUR TERMS APPROX.	1,33 -4%	1,383 -0,2%	0,10 22%	0,0836 0,6%	0,001 -83%	0,0064 -9%

TABLE II ($\alpha^2 = 1,28$)

	S-wave		D-wave		G-wave	
EXACT VALUES	= 2,10		= 0,265		= 0,047	
EXPANSION:	in $\cos^2\theta$.	in w.	in $\cos^2\theta$.	in w.	in $\cos^2\theta$.	in w.
ONE TERM APPROX.	4 91%	4 91%	0	0	0	0
TWO TERMS APPROX.	-0,2 -108%	1,4 -33%	0,8 216%	0,4 58%	0	0,045 -4%
THREE TERMS APPROX.	5 142%	2,4 12%	-0,7 -350%	0,19 -27%	0,2 361%	0,056 16%
FOUR TERMS APPROX.	-2 -195%	1,99 -5%	1,7 528%	0,298 12%	-0,4 -1034%!	0,040 -14%

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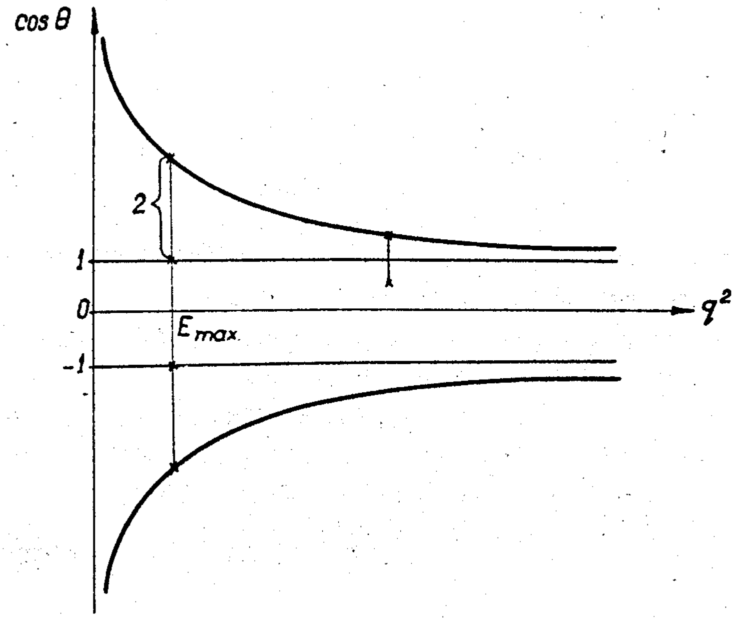
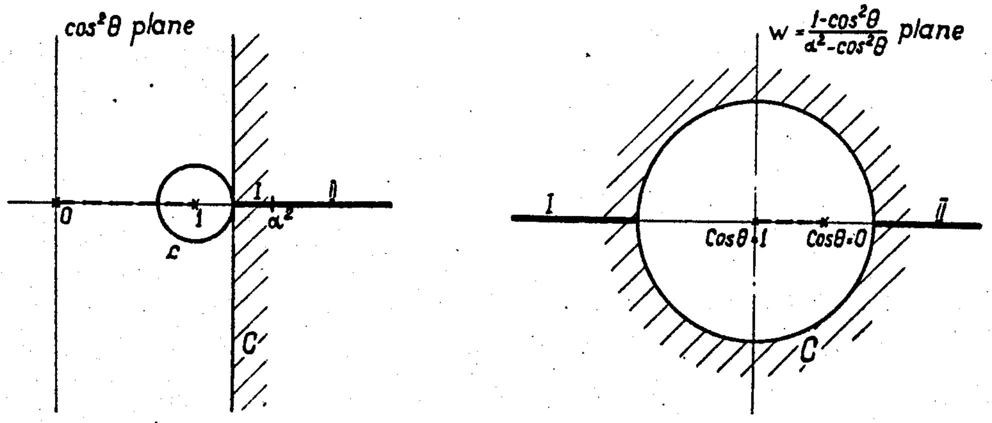


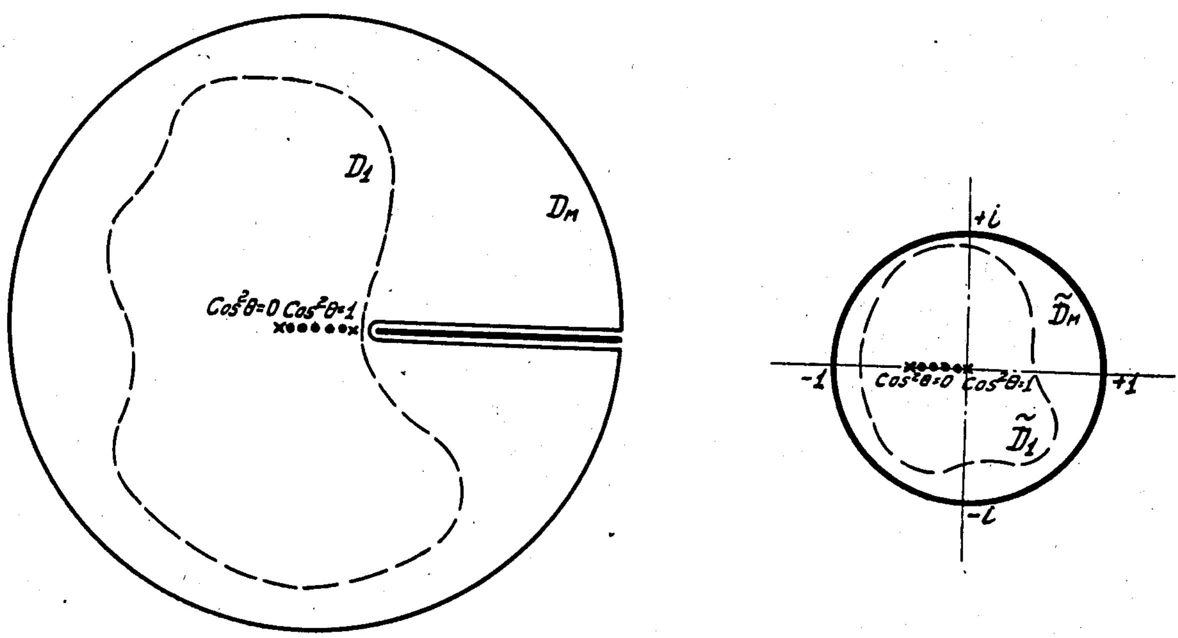
Fig. 1



c - convergence limit of the $\cos^2\theta$ expansion

C - convergence limit of the W expansion.

Fig. 2.



$z^2 \equiv \cos^2\theta$ plane

W_M - plane

Fig.3