ЛАБОРАТОРИЯ ТЕОРЕТИЧЕСКОЙ ФИЗИКИ

ОБЪЕДИНЕННЫЙ ИНСТИТУТ ЯДЕРНЫХ ИССЛЕДОВАНИЙ

B.M.Barbashov, G.V.Efimov

D 612

A MODEL OF THE LOCAL FIELD THEORY WITH THE FINITE CHARGE RENORMALIZATION

ЖЭТФ, 1961, т40, в. 3, с 848.

Дубна 1960

B.M.Barbashov, G.V.Efimov

D 612

A MODEL OF THE LOCAL FIELD THEORY WITH THE FINITE CHARGE RENORMALIZATION

Submitted to JETP

Abstract

A model of the local field theory suggested by Bialynicki-Birula is treated using the method developed in the previous papers. The S-matrix and the renormalization constants have been obtained. It is proved that the charge renormalization is finite for all the orders and does not contain the logarithmic divergencies. It is shown that the contribution to the series from the ultra-violet region is summed towards the finite limit. It is also proved that a series for the Green function of the nucleon is absolutely convergent at small times and has a branch point at $t = 0$, the singularity at zero being integrable.

Introduction

In investigating the consistency of the quantum field theory, the study of different models became rather popular as there are still unavoidable difficulties on the way of obtaining the solutions for exact field equations. The investigation of the Lee model /1/ has led, as it then seemed, to the conclusion about the internal contradictions in the theory. However, it was shown later on /2/, that the nonconsistency of the model is connected with the simplifications which are made for obtaining the exactly soluble Hamiltonian. In particular, these simplications violate an important requirement of the crossing symmetry.

Using the method developed by the authors in the previous papers /3,4/ a modified Lee model suggested by Bialynicki-Birula /5/ is investigated. In this model the condition of the crossing symmetrv is fulfilled. The solutions are obtained as series in a renormalized constant Δm (Δm is the physical parameter upon which the mass difference between two fermion states in the model depend). The convergence of these series in the ultra-violet region $E \gg \Delta m$ is proved. An important property of the model is the finite charge renormalization in all the orders of Δm for the point interaction in contrast to the Lee model where exists a well-known problem of zero-charge.

The Green function of the model under consideration possesses all the properties of the Green function of the renormalization theory /6,7/. 1. The renormalized propagator is analytical in the plane t and at $t = 0$ it is a branch point, 2. For $g^2/\pi^2 < 1$, there exists a Fouriertransform which allows the expansion in a series near the point $g^2 = 0$.

1. The S-matrix of the Model

Bialynicki-Birula's paper treated a model of the local field theory with the fixed nucleon. According to this model the nucleon may be in two states differing in mass (we agree to call these states - proton and neutron).

The Hamiltonian of the system is as follows

$$H = m_0(\psi^+\psi) + \frac{1}{2}\int d\vec{x} : \left[\pi^2(\vec{x}) + (\nabla\varphi(\vec{x}))^2 + \mu^2\varphi^2(\vec{x})\right] : + g(\psi^+\tau_1\psi)\int d\vec{x}\,\varphi(\vec{x})\delta(\vec{x}) + \Delta m_0(\psi^+\tau_3\psi) \tag{1}$$

where $\psi = v_p c_p + v_n c_n$ is the operator of the nucleon field, c_N $(N = p, n)$ is the annihilation operator of the nucleon, v_N $\left[v_p = \binom{1}{0},\ v_n = \binom{0}{1} \right]$ is the spinor describing the nucleon, $\pi(\vec{x})$ and $\varphi(\vec{x})$ are the operators of the meson field, τ_1 and τ_3 are the matrices of the isotopic spin $\frac{1}{2}$. Noting that by $\Delta m_0 = 0$, an exactly soluble case of the scalar mesons with the fixed source is obtained, it is possible to apply the perturbation theory in the constant Δm_0, without restricting the interaction force between the nucleon and mesons. In this way an interisting result has been obtained in /5/. It consists in the fact that the charge renormalization turns out to be finite, i.e., it does not contain the logarithmic divergencies.

According to the method developed in papers[3,4/*] the Hamiltonian (1) is remarkable because a series of Lappo-Danilevsky coincides here with that of the perturbation theory in the constant Δm_o. However, a new method makes it possible to get the n-th order term of the series what the perturbation theory fails to yield.

So, let us consider the equation for the S-matrix in the interaction representation. We shall look for the "adiabatic" S^α-matrix in order to make use of formulas (I.4.2) and (I.4.3).

In the interaction representation we have

$$i\frac{\partial}{\partial t} S^\alpha(t,t_o) = H_I(t) e^{-\alpha|t|} S^\alpha(t,t_o)$$

$$S^\alpha(t,t_o)\Big|_{t=t_o} = 1$$

where

$$H_I(t) = g(\psi^+\tau_1\psi)\hat{\varphi}(t) + \Delta m_o(\psi^+\tau_3\psi) \tag{2}$$

$$\hat{\varphi}(t) = \sum_{\vec{k}} \frac{1}{\sqrt{2\omega}}\left(a_{\vec{k}} e^{-i\omega t} + a^+_{\vec{k}} e^{i\omega t}\right)$$

Following the procedure set forth in Sections (I. 1 - 3) we get the following expression for the $S^\alpha(t,t_o)$ - matrix:

$$S^\alpha(t,t_o) = 1 - [2(\psi^+\psi) - (\psi^+\psi)^2] +$$

$$+\sum_{q=0}^{\infty} \frac{[-i(\psi^+\tau_3\psi)\Delta m_o]^q}{q!} \int_{t_o}^{t} d\xi_1 \ldots \int_{t_o}^{t} d\xi_q\, e^{-\alpha(|\xi_1|+\ldots+|\xi_q|)} : \exp\Big\{-ig(\psi^+\tau_1\psi)\int_{t_o}^{t} ds\, e^{-\alpha|s|} \prod_{j=1}^{q} \varepsilon(\xi_j - s)\hat{\varphi}(s)\Big\} : \times \tag{3}$$

$$\times \exp\Big\{-\frac{ig^2}{2}\int_{t_o}^{t}\int_{t_o}^{t} ds_1\, ds_2\, e^{-\alpha(|s_1|+|s_2|)} \prod_{j=1}^{q} \varepsilon(\xi_j - s_1)\Delta(s_1 - s_2)\varepsilon(\xi_j - s_2)\Big\}$$

The S^α - matrix is determined with an accuracy up to a unitary phase factor and satisfies the equality

$$S^\alpha(\infty,-\infty)|0\rangle = |0\rangle$$

$$S^\alpha(\infty,-\infty)|N\rangle = e^{i\frac{Const}{\alpha}}|N\rangle \tag{4}$$

---

* Paper[4] will be further reffered to as I.

where $|0\rangle$ and $|N\rangle$ denote the vacuum and one-nucleon state of zero Hamiltonian. However, usual conditions for the vacuum and one-particle state to be stable require that

$$S(\infty,-\infty)|0\rangle = |0\rangle$$
$$S(\infty,-\infty)|N\rangle = |N\rangle \tag{5}$$

In all further calculations we shall make use of the adiabatic hypothesis which will allow, in particular, to remove correctly the phase factor /10/ which is indefinite for 0, and, hence, to satisfy (5) (see Appendix A).

Consider now the most important elements of the S-matrix.

- The eigenvalue of the energy of the one-fermion state is /10/ (see Appendix B).

$$E_N = \lim_{\alpha\to 0} \frac{\langle N|HS^{\alpha}(0,-\infty)|N\rangle}{\langle N|S^{\alpha}(0,-\infty)|N\rangle} =$$

$$= m + \delta_N \Delta m \sum_{q=0}^{\infty} (-\delta_N \Delta m)^q \int_0^{\infty} dx_1 \ldots \int_0^{\infty} dx_q \cdot x_1 \cdot \ldots \cdot x_q \times \tag{6}$$

$$\times \frac{\partial^q}{\partial x_1 \cdots \partial x_q} \exp\left\{2g^2 \sum_{\vec{k}} \frac{1}{\omega^3} \sum_{\ell=1}^{q} \sum_{m=1}^{\ell} (-)^{\ell+m} e^{-\omega \sum_{j=m}^{\ell} x_j}\right\}$$

where $\delta_N = \begin{cases} +1 \text{ for the proton } (N=p) \\ -1 \text{ for the neutron } (N=n) \end{cases}$ (6′)

$$m = m_0 - \frac{1}{2} g^2 \sum_{\vec{k}} \frac{1}{\omega^2}$$

$$\Delta m = \Delta m_0 \exp\left\{-g^2 \sum_{\vec{k}} \frac{1}{\omega^3}\right\} \tag{6''}$$

The requirement that the observed m and Δm should be finite leads to the necessity of considering the values m_o and Δm_o as infinite. Note, that the renormalization m_o just coincides with the case of the scalar mesons in the field of the fixed source. Upon making the renormalizations (6′) and (6″), each term of series (6) is finite if $g^2/\pi^2 < 1$ *)(see in more detail Appendix B).

The renormalization constant of the fermion field Z_2^N is determined, according to its probability meaning, by the square of the matrix element

$$Z_2^N = |\langle N|\mathbf{N}\rangle|^2 = |\langle N|\, S^\alpha(0,-\infty)|N\rangle|^2 =$$

$$= Z_2^{CK}\Big[\sum_{q=0}^{\infty}(\delta_N\Delta m)^q\int_0^\infty dx_1\ldots\int_0^\infty dx_q\, x_1\cdot\ldots\cdot x_q\ \times \tag{7}$$

$$\times\frac{\partial^q}{\partial x_1\cdots\partial x_q}\exp\Big\{g^2\sum_{\vec{k}}\frac{1}{\omega^3}\Big[-\sum_{\ell=1}^{q}(-)^\ell e^{-\omega\sum_{j=1}^{\ell}x_j}+2\sum_{\ell=2}^{q}\sum_{m=2}^{\ell}(-)^{\ell+m}e^{-\omega\sum_{j=m}^{\ell}x_j}\Big]\Big\}\Big]^2$$

where

$$Z_2^{CK} = \exp\Big\{-\frac{1}{2}g^2\sum_{\vec{k}}\frac{1}{\omega^3}\Big\}$$

$|\mathbf{N}\rangle$ denotes the one-nucleon state of the total Hamiltonian.

Let us emphasize that the constant Z_2^N in the model under consideration is equal to the product of the constant Z_2^{CK} of the scalar neutral theory with the fixed source by a series of finite terms since for $g^2/\pi^2 < 1$ all the terms of the series are finite. Further this circumstance will allow to make the conclusion about the constant Z_1. The renormalized coupling constant is determined as usual

$$\frac{g_r}{g} = \langle \mathbf{p}|(\psi^+\tau_1\psi)|\mathbf{n}\rangle = \lim_{\alpha\to 0}\frac{\langle p|\,S^\alpha(\infty,0)(\psi^+\tau_1\psi)\,S^\alpha(0,-\infty)\,|n\rangle}{\sqrt{\langle p|\,S^\alpha(\infty,-\infty)|p\rangle\langle n|\,S^\alpha(\infty,-\infty)|\,n\rangle}} =$$

$$= 1+\sum_{q=1}^{\infty}(\Delta m)^{2q}\int_0^\infty dx_1\ldots\int_0^\infty dx_{2q-1}\, x_1\cdot\ldots\cdot x_{2q-1}\sum_{j=1}^{q}x_{2j-1}\ \times \tag{8}$$

$$\times\frac{\partial^{2q-1}}{\partial x_1\cdots\partial x_{2q-1}}\exp\Big\{2g^2\sum_{\vec{k}}\frac{1}{\omega^3}\sum_{\ell=1}^{2q-1}\sum_{m=1}^{\ell}(-)^{\ell+m}e^{-\omega\sum_{j=m}^{\ell}x_j}\Big\}$$

---

*) In paper [5] is given an incorrect condition for the terms of the series to be finite $g^2/\pi^2 < 4e^{-1}$.

The situation in this model is essentially unlike to that for the charge scalar theory (see(14,10) and further on), since at the point $g^2 = 0$ all the integrals are limited in contrast to expression (1.4.12). Therefore, here in applying the perturbation theory, i.e., in representing the solution as a power series in g^2, the logarithmic divergencies do not arise which are so characteristic for the local field theory. In this connection this model, in our opinion, does not reflect certain fundamental difficulties of the mesodynamics equations.

The renormalization constant of the vertex Z_1 may be found from the well-known equation

$$g_r = Z_1^{-1} \left(Z_2^p Z_2^n \right)^{1/2} Z_3^{1/2} g$$

In the given model the renormalization constant of the meson field Z_3 is equal to 1. By using (7) and (8), one can make a conclusion that the constant Z_1 has the following structure

$$Z_1 = Z_2^{ck} \; \sigma(g^2, \Delta m) \tag{9}$$

where $\sigma(g^2, \Delta m)$ is a series in Δm consisting of the finite quantities if $\frac{g^2}{\pi} < 1$.

We write down the matrix element of meson-nucleon elastic scattering (see Appendix A)

$$S_{f\leftarrow i} = \lim_{\alpha \to 0} \frac{\langle N | a_{\vec{p}_f} S^{\alpha}(\infty, -\infty) a^{+}_{\vec{p}_i} | N \rangle}{\langle N | S^{\alpha}(\infty, -\infty) | N \rangle} =$$

$$= \delta(\vec{p}_i - \vec{p}_f) - 2\pi i \, \delta(\omega_i - \omega_f) \, M_{f\leftarrow i}(\omega_f) \tag{10}$$

where

$$M_{f\leftarrow i}(\omega_f) = -\frac{2\delta_N g^2}{\omega_f^2} \cdot \frac{\Delta m}{\omega_f} \cdot \sum_{q=0}^{\infty} (-i\delta_N \Delta m)^q \int_0^{\infty} dx_1 \ldots \int_0^{\infty} dx_q \left[q + 1 - 2 \sum_{\ell=1}^{q} \sum_{m=1}^{\ell} (-1)^{\ell+m} \cos\left(\omega_f \sum_{j=m}^{\ell} x_j\right) \right] \times$$

$$\times(-)^q \int_{x_1}^{\infty} dy_1 \cdots \int_{x_q}^{\infty} dy_q \frac{\partial^q}{\partial y_1 \cdots \partial y_q} \exp\left\{2g^2 \sum_{\vec{k}} \frac{1}{\omega^3} \sum_{\ell=1}^{q} \sum_{m=1}^{\ell} (-)^{\ell+m} e^{-i\omega \sum_{j=m}^{\ell} y_j}\right\}$$

2. On Series Convergence for E_p, Z_2^n, g_r.

A proof of series convergence (6) - (10) is extremely complicated, since the estimation of the n-th order term requires very refined methods of approximations.

Only this estimation allows one to judge about the series behaviour on the whole. However, the series for E_p , Z_2^n , and for g_r can be summed in a certain generalized sense /9/ (we shall say in the " $\mathcal{E}$ " sense) which is illustrated by the series for E_p .

Consider the series

$$\delta m = \sum_{q=0}^{\infty} (-\Delta m)^q \int_0^{\infty} dx_1 \cdots \int_0^{\infty} dx_q \cdot x_1 \cdot \ldots \cdot x_q \frac{\partial^q}{\partial x_1 \cdots \partial x_q} F_q(x_1, \ldots, x_q) \tag{11}$$

where
$$F_q(x_1, \ldots, x_q) = \exp\left\{2g^2 \sum_{\vec{k}} \frac{1}{\omega^3} \sum_{\ell=1}^{q} \sum_{m=1}^{\ell} (-)^{\ell+m} e^{-\omega \sum_{j=m}^{\ell} x_j}\right\}$$

After the partial integration in each term, we get (see formula (B.8) in Appendix B)

$$\delta m = \sum_{q=0}^{\infty} (\Delta m)^q \int_0^{\infty} dx_1 \cdots \int_0^{\infty} dx_q \prod_{j=1}^{q} (1 - \hat{Q}_j) F_q(x_1, \ldots, x_q) \tag{12}$$

where the operator $\hat{Q}_j$ is determined by the equality

$$\hat{Q}_j F_q(\ldots, x_j, \ldots) = F_q(\ldots, \infty, \ldots)$$

Introduce another series

$$\delta m^{\varepsilon} = \sum_{q=0}^{\infty} (\Delta m)^q \int_0^{\infty} dx_1 \ldots \int_0^{\infty} dx_q \, e^{-\varepsilon \sum_{j=1}^{q} x_j} \prod_{i=1}^{q} (1 - \hat{Q}_i) F_q(x_1, \ldots, x_q) \tag{13}$$

which at $\varepsilon = 0$ passes into (12).

It can be shown that for ε satisfying the inequality

$$\frac{\Delta m}{\varepsilon} \leq 1 - \Delta m . \int_0^{\infty} dx \, e^{-\varepsilon x} F_1(x) \tag{14}$$

series (13) is limited by the quantity (see Appendix C) .

$$\delta m^{\varepsilon} < \left\{ 1 - \Delta m \int_0^{\infty} dx \, e^{-\varepsilon x} \left[F_1(x) - 1 \right] \right\}^{-1} \tag{15}$$

With $\varepsilon \to 0$, the left-hand side of inequality (15) goes over into initial series (12), whereas the right-hand side has a finite positive limit when $\Delta m \int_0^{\infty} dx \left[F(x) - 1 \right] < 1$.

This procedure means that series (11) is summed up in a generalized sense*) (ε- sense) by

$$\Delta m \int_0^{\infty} dx \left[F_1(x) - 1 \right] < 1$$

---

*) This method is regular, i.e. convergent series of form (11) are summed up to their usual sum /9/.

and its sum is restricyed by the quantity

$$\delta m < \left\{ 1 - \Delta m \int_0^\infty dx \left[F_1(x) - 1 \right] \right\}^{-1} \quad (\mathcal{E}) \qquad (16)$$

The series for Z_2^n and g_r are summed up in the $\mathcal{E}$- sense. Let us give here the result

$$Z_2^n < Z_2^{cK} \left\{ 1 - \Delta m \int_0^\infty dx \left[F_1(x) - 1 \right] \right\}^{-2} \quad (\mathcal{E}) \qquad (17)$$

$$\left| \frac{g_r}{g} - 1 \right| < \frac{2(\Delta m)^2 \int_0^\infty dx \cdot x \left[F_1(x) - 1 \right]}{\left\{ 1 - \left(\Delta m \int_0^\infty dx \left[F_1(x) - 1 \right] \right)^2 \right\}^2} \quad (\mathcal{E}) \qquad (17')$$

Consider the physical meaning of the summation suggested. We have seen that series (13) is convergent in a usual manner for sufficiently large $\mathcal{E}$ (condition (14)), , but series (13) differs from (12) by a reduced contribution from large times (large x_j), i.e., small energies. Therefore, the contribution from the ultra-violet region turns out to be summed up towards the finite quantity, what seems unexpected from the point of view of the local field theory.

In this connection it is interesting to investigate the behaviour of the Green function at small times i.e., in the ultra-violet region. The Green function is determined by

$$\begin{aligned} G(t) &= \langle 0 | T \{ \psi(t) \psi^+(0) S(\infty, -\infty) \} | 0 \rangle = \\ &= Z_2^{cK} G_{cK}(t) \sum_{q=0}^{\infty} (-i\tau_3 \Delta m)^q \int_0^t d\xi_1 \int_0^{\xi_1} d\xi_2 \ldots \int_0^{\xi_{q-1}} d\xi_q \, \times \\ &\times \exp \Big\{ -g^2 \sum_{\vec{k}} \frac{1}{\omega^3} \Big[\sum_{l=1}^{q} (-)^l \left(e^{-i\omega(t-\xi_l)} - e^{-i\omega \xi_l} \right) + \\ &\qquad + 2 \sum_{l=2}^{q} \sum_{m=1}^{l-1} (-)^{l+m} e^{-i\omega(\xi_m - \xi_l)} \Big] \Big\} \end{aligned} \qquad (18)$$

where

$$G_{cK}(t) = \theta(t) \exp \left\{ -imt + \frac{1}{2} g^2 \sum_{\vec{k}} \frac{1}{\omega^3} e^{-i\omega t} \right\}$$

The function $G_{cx}(t)$ is the Green function of the scalar neutral theory and at small times it behaves as t^{-g^2/π^2} /11/.

It can be shown (see Appendix D) that the series in Δ m small t ($\mu t << 1$) is convergent absolutely if $g^2/\pi^2 < 1$ and has a branch point when t $\Rightarrow$ 0. Thus, the function $G(t)$ has at t = 0 a singular point, provided this singularity is integrable due to the condition $g^2/\pi^2 < 1$, what permits to make a Fourier transform.

Conclusion

The described example of the model with the finite charge renormalization shows another possibility (compared with the Lee model) which may be realized in the rigorous theory. However, in our opinion, this model does not also reflect a real situation in the field theory as the investigation of more complex models (see /4/) leads, very likely, to the conclusion that in the exact solution exists a singular point $g^2=0$, at which the expansion of the solution into a Taylor series leads to additional divergences. As it was pointed out above the model we are considering has no such a property.

In conclusion the authors express their deep gratitude to Prof. D.I.Blokhintsev for his permanent interest in the research and stimulating discussions, as well as to L.D.Zastavenko for the discussion of the mathematical problems.

Appendix A

Since the S-matrix is set as a series, then the matrix elements are represented as a limit by $\alpha \to 0$ of the ratio of two series. It turns out that if we divide one series by another and collect the terms with the identical powers Δm, then in the terms thus obtained the phase reduces, and, hence, it is possible to go over to the limit with $\alpha \to 0$ in each term separatelv.

Let us illustrate the procedure of removing the phase on the example of the matrix element of meson-nucleon scattering (see formula (10))

$$M_{f\leftarrow i}(\omega_f) = \frac{g^2}{2i\omega_f}\int_{-\infty}^{\infty} d\tau\, e^{-i\omega_f\tau} \lim_{\alpha\to 0} \frac{\sum_{q=1}^{\infty}(-i\delta_N \Delta m_0)^q B_q^{\alpha}(\tau)}{\sum_{q=0}^{\infty}(-i\delta_N \Delta m_0)^q b_q^{\alpha}} \tag{A.1}$$

where

$$B_q^{\alpha}(\tau) = \frac{1}{q!}\int_{-\infty}^{\infty} d\xi_1 \dots \int_{-\infty}^{\infty} d\xi_q \, \exp\Big\{-\alpha\sum_{j=1}^{q}|\xi_j|\Big\} \prod_{\ell=1}^{q} \varepsilon(\xi_\ell)\varepsilon(\xi_\ell - \tau) \times \tag{A.2'}$$

$$\times \exp\Big\{-\frac{ig^2}{2}\iint_{-\infty}^{\infty} ds_1 ds_2\, e^{-\alpha(|s_1|+|s_2|)} \prod_{j=1}^{q}\varepsilon(\xi_j - s_1)\Delta(s_1 - s_2)\varepsilon(\xi_j - s_2)\Big\}$$

$$b_q^{\alpha} = \frac{1}{q!}\int_{-\infty}^{\infty} d\xi_1 \dots \int_{-\infty}^{\infty} d\xi_q \, \exp\Big\{-\alpha\sum_{j=1}^{q}|\xi_j| - \frac{ig^2}{2}\iint_{-\infty}^{\infty} ds_1 ds_2\, e^{-\alpha(|s_1|+|s_2|)} \times \tag{A.2''}$$

$$\times \prod_{j=1}^{q}\varepsilon(\xi_j - s_1)\,\Delta(s_1 - s_2)\,\varepsilon(\xi_j - s_2)\Big\}$$

The integral of the exponent is equal to

$$-\frac{ig^2}{2}\iint_{-\infty}^{\infty} ds_1 ds_2\, e^{-\alpha(|s_1|+|s_2|)} \prod_{j=1}^{q}\varepsilon(\xi_j - s_1)\,\Delta(s_1 - s_2)\,\varepsilon(\xi_j - s_2) = -\frac{g^2}{2i\alpha}\sum_{\vec{k}}\frac{1}{\omega^2} -$$

$$-g^2\sum_{\vec{k}}\frac{1}{\omega^3}\Big\{q + 2\sum_{\nu_1=2}^{q}\sum_{\nu_2=1}^{\nu_1-1}\prod_{j_1\neq\nu_1}\varepsilon(\xi_{\nu_1}-\xi_{j_1})\prod_{j_2\neq\nu_2}\varepsilon(\xi_{\nu_2}-\xi_{j_2})\, e^{-i\omega|\xi_{\nu_1}-\xi_{\nu_2}|}\Big\} \tag{A.3}$$

The first component is identical for all the terms of a series and therefore, it is cancelled.

The presence of the phase in the expression for the matrix element is demonstrated in the fact that the integrals in time (by ξ_j) are divergent linearly at the infinity by $\alpha = 0$. However, as was mentioned above, a series which is the ratio of the series divergent when $\alpha=0$, contains the integrals finite at $\alpha = 0$. This due to the fact that near each degree Δm the divergent integrals are grouping so that the infinities reduce. The procedure of regularization is the following. In the integrals (A.2′) and (A.2″) we go over to the integration over simplex and make a substitution of the variables

$$\xi_1 = \xi$$

$$\xi_\ell = \xi - \sum_{j=1}^{\ell-1} x_j \qquad (\ell \geq 2) \tag{A.4}$$

then

$$B_q^\alpha(\tau) = \int_{-\infty}^{\infty} d\xi \int_0^{\infty} dx_1 \ldots \int_0^{\infty} dx_{q-1}\, e^{-\alpha\left(|\xi| + \sum_{\ell=1}^{q-1} |\xi - \sum_{j=1}^{\ell} x_j|\right)} \mathcal{E}(\xi)\mathcal{E}(\xi-\tau) \prod_{\ell=1}^{q-1} \mathcal{E}(\xi - \sum_{j=1}^{\ell} x_j)\mathcal{E}(\xi-\tau-\sum_{j=1}^{\ell} x_j) \times$$

$$\times \left(\frac{\Delta m}{\Delta m_0}\right)^q F_{q-1}(x_1, \ldots, x_{q-1}) \tag{A.5}$$

$$b_q^\alpha = \int_{-\infty}^{\infty} d\xi \int_0^{\infty} dx_1 \ldots \int_0^{\infty} dx_{q-1}\, e^{-\alpha\left(|\xi| + \sum_{\ell=1}^{q-1} |\xi - \sum_{j=1}^{\ell} x_j|\right)} \left(\frac{\Delta m}{\Delta m_0}\right)^q F_{q-1}(x_1, \ldots, x_{q-1})$$

where

$$F_{q-1}(x_1, \ldots, x_{q-1}) = \exp\left\{ 2g^2 \sum_{\vec{k}} \frac{1}{\omega^3} \sum_{\ell=1}^{q-1} \sum_{m=1}^{\ell} (-)^{\ell+m} e^{-i\omega \sum_{j=m}^{\ell} x_j} \right\}$$

The integral of the function F_{q-1} is divergent linearly at the infinity by any argument x_j. However, the function $F_{q-1}(\ldots, x_j, \ldots) - F_{q-1}(\ldots, \infty, \ldots)$ can be already integrated over x_j at the infinity. Note further, that

$$F_{q-1}(x_1, \ldots, x_{j-1}, \infty, x_{j+1}, \ldots, x_{q-1}) = F_{j-1}(x_1, \ldots, x_{j-1}) F_{q-j-1}(x_{j+1}, \ldots, x_{q-1}) \tag{A.6}$$

It follows from what has been said that in order to regularize the function $B_q^\alpha(\tau)$, it is necessary from the function F_{q-1} subtract its values at the infinity by each its arguments i.e., to replace F_{q-1} by the function

$$\prod_{\ell=1}^{q-1} (1 - \hat{Q}_\ell) F_{q-1}(x_1, \ldots, x_{q-1}) \tag{A.7}$$

where the operator $\hat{Q}_j$ is determined by the equality $\hat{Q}_j F_{q-1}(\ldots, x_j, \ldots) = F_{q-1}(\ldots, \infty, \ldots)$. Such a substitution of the function F_{q-1} for the function (A.7) is just accomplished by dividing a series by a series and by grouping the terms near the identical powers Δm. The same holds for the regularization of the integral over ξ (A.5). Finally, the matrix element of scattering with the removed phase is put as

$$M_{f\leftarrow i}(\omega_f) = \frac{g^2}{2i\omega_f} \int_{-\infty}^{\infty} d\tau\, e^{-i\omega_f \tau} \sum_{q=1}^{\infty} (-i\delta_\nu \Delta m)^q \int_0^{\infty} dx_1 \ldots \int_0^{\infty} dx_q \prod_{j=1}^{q} (1 - \hat{Q}_j) F_q(x_1, \ldots, x_q) \times \tag{A.8}$$

$$\times \int_{-\infty}^{\infty} d\xi \left[\mathcal{E}(\xi)\mathcal{E}(\xi-\tau)\prod_{\ell=1}^{q}\mathcal{E}(\xi-\sum_{j=1}^{\ell}x_j)\mathcal{E}(\xi-\tau-\sum_{j=1}^{\ell}x_j)-1\right]$$

From (A.8) immediately follows (10), if we take into account that the operator ($1-\hat{Q}_j$) may be represented as

$$1-\hat{Q}_j = -\int_{x_j}^{\infty} dy_j \frac{\partial}{\partial y_j} \tag{A.9}$$

In an analogous manner the phase for any other matrix element is removed.

Appendix B

According to formula (6) the eigenvalue of the one-fermion state is determined by

$$E_N = \lim_{\alpha\to 0} \frac{\langle N|HS^{\alpha}(0,-\infty)|N\rangle}{\langle N|S^{\alpha}(0,-\infty)|N\rangle} = m_0 + \delta_N \Delta m_0 + \delta E_N \tag{B.1}$$

where

$$\delta E_N = \lim_{\alpha\to 0} \frac{\langle N|g(\psi^{+}\tau_1\psi)\hat{\varphi}(0)S^{\alpha}(0,-\infty)|N\rangle}{\langle N|S^{\alpha}(0,-\infty)|N\rangle}\,. \tag{B.1'}$$

The matrix element in the numerator is

$$\begin{aligned} M_1^{\alpha} &= \langle N|g(\psi^{+}\tau_1\psi)\hat{\varphi}(0)S^{\alpha}(0,-\infty)|N\rangle = \\ &= -\sum_{q=1}^{\infty}(-i\delta_N\Delta m_0)^q \int_{-\infty}^{0}d\xi_1\ldots\int_{-\infty}^{\xi_{q-1}}d\xi_q\, e^{\alpha\sum_{j=1}^{q}\xi_j}\, ig^2\int_{-\infty}^{0}ds\, e^{\alpha s}\, i\Delta(s)\prod_{j=1}^{q}\mathcal{E}(s-\xi_j)\times \\ &\quad\times \exp\left\{-\frac{ig^2}{2}\iint_{-\infty}^{0}ds_1 ds_2\, e^{\alpha(s_1+s_2)}\prod_{j=1}^{q}\mathcal{E}(s_1-\xi_j)\Delta(s_1-s_2)\mathcal{E}(s_2-\xi_j)\right\}\,. \end{aligned} \tag{B.2}$$

The matrix element in the denominator (B.1') is obtained in an analogous manner

$$M_2^{\alpha} = \langle N|S^{\alpha}(0,-\infty)|N\rangle = \sum_{q=0}^{\infty}(-i\delta_N\Delta m_0)^q\tau_q \tag{B.3}$$

where

$$\tau_q = \int_{-\infty}^{0}d\xi_1\ldots\int_{-\infty}^{\xi_{q-1}}d\xi_q\, e^{\alpha\sum_{j=1}^{q}\xi_j}\, \exp\left\{-\frac{ig^2}{2}\iint_{-\infty}^{0}ds_1 ds_2\, e^{\alpha(s_1+s_2)}\prod_{j=1}^{q}\mathcal{E}(s_1-\xi_j)\Delta(s_1-s_2)\mathcal{E}(s_2-\xi_j)\right\}$$

The integral of the exponent, is equal to (by $\xi_1>\xi_2>\ldots>\xi_q$)

$$\begin{aligned} I_q(\xi_1,\ldots,\xi_q) &= -\frac{ig^2}{2}\iint_{-\infty}^{0}ds_1 ds_2\, e^{\alpha(s_1+s_2)}\prod_{j=1}^{q}\mathcal{E}(s_1-\xi_j)\Delta(s_1-s_2)\mathcal{E}(s_2-\xi_j) = \\ &= -\frac{g^2}{4}\sum_{\vec{k}}\frac{1}{\omega^2}\left(\frac{1}{i\alpha}+\frac{1}{\omega}\right) - g^2\sum_{\vec{k}}\frac{1}{\omega^3}\left[q+\sum_{\ell=1}^{q}(-)^{\ell}e^{i\omega\xi_\ell}+2\sum_{\ell=2}^{q}\sum_{m=1}^{\ell-1}(-)^{\ell+m}e^{-i\omega(\xi_m-\xi_\ell)}\right] \end{aligned} \tag{B.4}$$

The first component in (B.4) is identical for all the terms of the series both of the numerator and the denominator. Therefore, it reduces. Calculating in (B.2) the integral over S_1 we get

$$M_1^{\alpha} = -\frac{g^2}{2}\sum_{\vec{K}}\frac{1}{\omega^2}\,M_2^{\alpha} - \sum_{q=1}^{\infty}(-i\delta_\nu \Delta m_0)^q \int_{-\infty}^{0} d\xi_1 \dots \int_{-\infty}^{\xi_{q-1}} d\xi_q\, e^{\alpha\sum_{j=1}^{q}\xi_j} \times \tag{B.5}$$

$$\times\, i\left(\frac{\partial}{\partial\xi_1}+\dots+\frac{\partial}{\partial\xi_q}\right)\exp\{I_q(\xi_1,\dots,\xi_q)\}$$

Consider now the q-th term of the series (B.5)

$$\int_{-\infty}^{0} d\xi_1 \dots \int_{-\infty}^{\xi_{q-1}} d\xi_q\, e^{\alpha\sum_{j=1}^{q}\xi_j}\, i\left(\frac{\partial}{\partial\xi_1}+\dots+\frac{\partial}{\partial\xi_q}\right)\exp\{I_q(\xi_1,\dots,\xi_q)\} = i\tau_{q-1} - i\alpha q \tau_q \tag{B.6}$$

Further, by substituting (B.6) into (B.5) for E_N, we get according to (A.1) the expression

$$E_N = m_0 - \frac{g^2}{2}\sum_{\vec{K}}\frac{1}{\omega^2} + \lim_{\alpha\to 0} i\alpha\,\frac{\sum_{q=1}^{\infty}(-i\delta_\nu\Delta m_0)^q q\tau_q}{\sum_{q=0}^{\infty}(-i\delta_\nu\Delta m_0)^q \tau_q} \tag{B.7}$$

Dividing in (B.7) a series by a series by terms, we exclude thereby the phase factor in each order of Δm. This allows a limiting transition by α in each order in Δm (see Appendix A).

Finally we obtain

$$E_N = m_0 - \frac{g^2}{2}\sum_{\vec{K}}\frac{1}{\omega^2} + \sum_{q=1}^{\infty}(-i\delta_\nu\Delta m)^q \int_0^{\infty} dx_1 \dots \int_0^{\infty} dx_{q-1} \prod_{\ell=1}^{q-1}(1-\hat{Q}_j)\,F_{q-1}(x_1,\dots,x_{q-1}) \tag{B.8}$$

where F_{q-1} is given by (A.5). From (B.8) immediately follows (6), if we take into account (A.9).

Appendix C

Consider formula (13). We make use of the relations:

$$\prod_{j=1}^{q}(1-\hat{Q}_j) = \sum_{\ell=0}^{q}(-)^{\ell}\sum_{s_1=\ell}^{q}\dots\sum_{s_i=\ell-i+1}^{s_{i-1}-1}\dots\sum_{s_\ell=1}^{s_{\ell-1}-1}\hat{Q}_{s_1}\dots\hat{Q}_{s_i}\dots\hat{Q}_{s_\ell} \tag{C.1}$$

$$\int_0^{\infty} dx_1 \dots \int_0^{\infty} dx_q\, e^{-\varepsilon(x_1+\dots+x_q)}\hat{Q}_{s_1}\dots\hat{Q}_{s_i}\dots\hat{Q}_{s_\ell}\,F_q(x_1,\dots,x_q) = \tag{C.2}$$

$$= \left(\frac{1}{2}\right)^{\ell} I^{\varepsilon}_{s_\ell-1}\, I^{\varepsilon}_{s_{\ell-1}-s_\ell-1}\dots I^{\varepsilon}_{s_i-s_{i+1}-1}\dots I^{\varepsilon}_{s_1-s_2-1}\, I^{\varepsilon}_{q-s_1}$$

$$I_s^{\varepsilon} = \int_0^{\infty} dx_1 \ldots \int_0^{\infty} dx_s \, e^{-\varepsilon(x_1+\ldots+x_s)} F_s(x_1,\ldots,x_s) \qquad \text{(C.3)}$$

Substituting these relations into (13), changing the order of summation in the sums by q, ℓ and S_i and replacing the summation indices (the calculations are simple, but somewhat cumbersome, therefore, we will not give them here) we get

$$\delta m_*^{\varepsilon} = \sum_{q=0}^{\infty} (\Delta m)^q \bar{I}_q^{\varepsilon} = \sum_{\ell=0}^{\infty} \frac{(-\Delta m)^{\ell}}{\varepsilon^{\ell}} \left\{ \sum_{q=0}^{\infty} (\Delta m)^q I_q^{\varepsilon} \right\}^{\ell+1} \qquad \text{(C.4)}$$

where

$$\bar{I}_q^{\varepsilon} = \int_0^{\infty} dx_1 \ldots \int_0^{\infty} dx_q \, e^{-\varepsilon(x_1+\ldots+x_q)} \prod_{j=1}^{q} (1-\hat{Q}_j) F_q(x_1,\ldots,x_q)$$

Perform formally the summation in (C.4)

$$\delta m^{\varepsilon} = \left[\frac{\Delta m}{\varepsilon} + \left\{ \sum_{q=0}^{\infty} (\Delta m)^q I_q^{\varepsilon} \right\}^{-1} \right]^{-1} \qquad \text{(C.5)}$$

Since $I_q^{\varepsilon} \leq (I_1^{\varepsilon})^q$, then

$$\delta m^{\varepsilon} \leq \left[\frac{\Delta m}{\varepsilon} + 1 - \Delta m I_1^{\varepsilon} \right]^{-1} = \left[1 - \Delta m \bar{I}_1^{\varepsilon} \right] \qquad \text{(C.6)}$$

The summation in (C.4) is valid if ε satisfies the conditions

$$\Delta m I_1^{\varepsilon} < 1 \qquad \text{(C.7a)}$$

$$\frac{\Delta m}{\varepsilon} < 1 - \Delta m I_1^{\varepsilon} \qquad \text{(C.7b)}$$

from which the second one is stronger.

Appendix D

Consider the q-th term of series (18) for the Green function

$$J_q(t)=(-i\Delta m)^q\int_0^t d\xi_1\ldots\int_0^{\xi_{q-1}} d\xi_q\, \exp\Big\{-g^2\sum_{\vec{k}}\frac{1}{\omega^3}\Big[\sum_{\ell=1}^{q}(-)^\ell\big(e^{-i\omega(t-\xi_\ell)}-e^{-i\omega\xi_\ell}\big)+2\sum_{\ell=2}^{q}\sum_{m=1}^{\ell-1}(-)^{\ell+m}e^{-i\omega(\xi_m-\xi_\ell)}\Big]\Big\} \tag{D.1}$$

As we are interested in the behaviour of the Green function at small times ($\mu t \ll 1$) then the functions in the exponent may be replaced for their asymptotic expansion for small arguments.

$$g^2\sum_{\vec{k}}\frac{1}{\omega^3}e^{-i\omega\xi}=-G^2\ln(i\mu\xi)+G^2(\ln 2-C-1)+O(\mu\xi) \tag{D.2}$$

where C is the Euler constant, $G^2=\frac{g^2}{2\pi^2}$.

Replacing further the integration variables in (D.1) $\xi_j=\mu t\cdot x_j$ and substituting there the asymptotic expansion (D.2), we get

$$J_q(t)=\Big(-\frac{\Delta m}{\mu}\Big)^q(i\mu t)^{q-2G^2[\frac{q}{2}]}\,e^{2G^2(\ln 2-C-1)[\frac{q}{2}]}\,I_q \tag{D.3}$$

$$I_q=\int_0^1 dx_1\ldots\int_0^{x_{q-1}}dx_q\prod_{z=1}^{q}\Big(\frac{1-x_z}{x_z}\Big)^{(-)^z G^2}\prod_{\ell=2}^{q}\prod_{m=1}^{\ell-1}(x_m-x_\ell)^{(-)^{\ell+m}2G^2}$$

$$\Big[\frac{q}{2}\Big]=\begin{cases}\frac{q}{2}, & \text{if } q \text{ is even}\\ \frac{q-1}{2}, & \text{if } q \text{ is odd}\end{cases}$$

Consider first the numbers I_q for the even values $q=2n$

$$I_{2n}=\int_0^1 dx_1\ldots\int_0^{x_{2n-1}}dx_{2n}\prod_{z=1}^{n}\Big[\frac{(1-x_{2z})x_{2z-1}}{x_{2z}(1-x_{2z-1})}\Big]^{G^2}\prod_{\ell=1}^{n}\frac{1}{(x_{2\ell-1}-x_{2\ell})^{2G^2}}\prod_{m=1}^{\ell-1}\Big[\frac{(x_{2m}-x_{2\ell})(x_{2m-1}-x_{2\ell-1})}{(x_{2m}-x_{2\ell-1})(x_{2m-1}-x_{2\ell})}\Big]^{2G^2} \tag{D.4}$$

Make the following substitution of the variables in the integrals

$$x_\ell=\prod_{j=1}^{\ell}z_j \tag{D.5}$$

then we obtain

$$I_{2n}=\int_0^1 dz_1\ldots\int_0^1 dz_{2n}\prod_{s=1}^{2n}z_s^{(2n+1-s)(1-G^2)-1}\times$$

$$\times\prod_{\ell=1}^{n}\Bigg(\frac{1-\prod_{j=1}^{2\ell}z_j}{1-\prod_{j=1}^{2\ell-1}z_j}\Bigg)^{G^2}\frac{1}{(1-z_{2\ell})^{2G^2}}\prod_{m=1}^{\ell-1}\Bigg[\frac{(1-\prod_{j=2m+1}^{2\ell}z_j)(1-\prod_{j=2m}^{2\ell-1}z_j)}{(1-\prod_{j=2m+1}^{2\ell-1}z_j)(1-\prod_{j=2m}^{2\ell}z_j)}\Bigg]^{2G^2} \tag{D.6}$$

It can be easily seen that the integrals are convergent at the upper limit, provided

$$2G^2 < 1 \qquad \text{or} \qquad \frac{g^2}{\pi^2} < 1 \tag{D.7}$$

To estimate the upper value of I_{2n} we use the inequality

$$\frac{1-x}{1-x\beta} \leqslant 1 \qquad \text{by} \quad 0 \leqslant x \leqslant 1, \; 0 \leqslant \beta \leqslant 1 \tag{D.8}$$

The estimation yields

$$I_{2n} \leqslant \int_0^1 \frac{dz_1 \cdot z_1^{2n(1-G^2)-1}}{(1-z_1)^{G^2}} \prod_{\ell=2}^{2n} \int_0^1 dz_\ell \, z_\ell^{(2n+1-\ell)(1-G^2)-1} (1-z_\ell)^{-2G^2} \tag{D.9}$$

The calculation of the integrals leads to

$$I_{2n} \leqslant \frac{\left[(1-G^2)\Gamma(1-2G^2)\right]^{2n}}{\Gamma\left(\frac{1-2G^2}{1-G^2}\right)} \Gamma\left(\frac{(2n+1)(1-G^2)-1}{1-G^2}\right) \left[\frac{\Gamma(1-G^2)}{\Gamma((2n+1)(1-G^2))}\right]^2 \tag{D.10}$$

When n is tending to infinity the inequality has the form

$$I_{2n} \leqslant \frac{\left[\Gamma(1-2G^2)\right]^{2n}}{\Gamma((2n+1)(1-2G^2))} \cdot \frac{(1-G^2)^{2n}\left[\Gamma(1-G^2)\right]^2}{\Gamma\left(\frac{1-2G^2}{1-G^2}\right)} \left[\frac{(1-2G^2)^{(1-2G^2)}}{(1-G^2)^{2(1-2G^2)}}\right]^{2n+1} \tag{D.11}$$

by $$(2n+1)(1-2G^2) \gg 1$$

The presence of the factor $\Gamma((2n+1)(1-2G^2))$ in the denominator in the right-hand side of the inequality provides the absolute convergence of the series (D.1) at small t ($\mu t \ll 1$) and Δm is arbitrary and $g^2/\pi^2 < 1$.

The estimation of the odd numbers I_{2n+1} is made in an analogous manner and yields the same result.

References

1. T.D.Lee, Phys.Rev. 95, 1329 (1954).
2. Th.W.Ruijgrok, L.Van Hove, Physica, 22, 880 (1956).
3. Б.М.Барбашов, Г.В.Ефимов. ЖЭТФ, 38,198 /1960/.
4. Б.М.Барбашов, Г.В.Ефимов. ЖЭТФ, 39,450 /1960/.Препринт, ОИЯИ Д-498/1960/.
5. I.Bialynicki-Birula, Nucl.Phys. 12, 309 (1959).
6. R.Arnowitt and S.Deser, Phys.Rev. 100, 349 (1955).
7. L.N.Cooper. Phys.Rev. 100, 362 (1955).
8. Н.Н.Боголюбов, Д.В.Ширков. Введение в теорию квантованных полей. Гостехиздат, 1957г.
9. Г.Харди. Расходящиеся ряды. ИЛ, 1951г.
10. M.Gell-Mann, F.Low, Phys.Rev. 84, 350 (1951). ПСФ n.10 (1955).
11. S.F.Edwards, R.E.Peierls. Proc.Roy.Soc. 224, 24 (1954).

Received by Publishing Department
on September 27, 1960.