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ON THE FUNCTIONAL EXPANSION OF THE SCATTERING MATRIX IN NORMAL PRODUCTS OF ASYMPTOTIC FIELDS

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ON THE FUNCTIONAL EXPANSION OF THE SCATTERING MATRIX IN NORMAL PRODUCTS OF ASYMPTOTIC FIELDS

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Abstract. *The properties of the scattering matrix are investigated by the axiomatic approach, without the perturbation theory. S-matrix is presented as a series of the normal products of asymptotic field operators. Formulae are established formally expressing the coefficient functions of such an expansion in terms of chronological products of the current operator and some sequence of operators* Λ_ν. *Some infinite sets of coupled equations are also derived for these coefficient functions.*

1. Introduction. Notations.

The usual approach to the quantum field theory based on the Hamiltonian formalism encounters a number of well-known difficulties. The most important one is the impossibility to go beyond the limits of the perturbation theory: one of the basic stones of the theory - the removing-of-divergences prescriptions -cannot be even formulated otherwise than in terms of successive powers of small coupling parameter.

This circumstance has stimulated the attempts to attack the problem from the very opposite end. Namely, instead of writing down the Lagrangian or the equations of motion and trying to solve them, one attempts to formulate those physically evident basic requirements which the solutions must satisfy, and to find all the totality of solutions satisfying these requirements. Assuming furthermore the perturbation theory and the hypothesis of the adiabatic switching on and off the interaction to be valid such a method was systematically employed by Bogolubov[1] and by Bogolubov and Shirkov [2]. It may be seen, that this method leads then to the conclusions essentially coinciding with those obtained with the Hamiltonian one.

Without these simplifying assumptions such a method - it is often called "axiomatic" one, though this name doesn't seem to us the best - has been intensively developed recently in connection with the dispersion relations - the only exact result in the quantum field theory.

The basic physical principles of the axiomatic method may be formulated in different ways. So, for instance. we could require from the very beginning the Heisenberg fields commuting on any space-like hypersurface to exist at each point. The attempts in this direction have been made in the papers by Lehmann, Symanzik, Zimmermann and other authors (see[3,4] and numerous further investigations). On the other hand, we can follow the programme suggested by Heisenberg[5] and restrict ourselves to treating the scattering matrix. The latter way was chosen by Bogolubov, Polivanov and the author[6]* in connection with the theory of dispersion relations.

It is to be emphasized at once that we are not quite exact when saying we follow the original Heisenberg programme. In point of fact, the manifold of objects under study and the system of the basic physical principles will now be wider, but the class of the admissible theories becomes more narrow. Namely, the Heisenberg programme is dealing with the "on - the-energy-shell" S-matrix elements only, which correspond to transitions between the asymptotic stable states, i.e. transitions in which the total energy and momentum are conserved and the squares of all initial and final 4-momenta are equal to the corresponding masses. The manifold of such matrix elements can be expressed as a functional series of the form PTDR, eq. /2.14/ in the creation and annihilation operators or, instead, as a functional expansion of the form PTDR, eq. /2.15/ (or eq. /10/ below) i.e. expansion in the normal products of the asymptotic fields satisfying the equation

$$(\Box - m^2)\ \phi(x) = 0. \qquad (1)$$

However, we cannot formulate the condition for exact causality making use of the scattering matrix on the energy shell only (one may see this already from the fact that we can't construct a four-dimensional δ-function from the solutions of equation (1)). Hence, it would be impossible to distinguish between the theories with the exact and with the macroscopic causality. To be able to formulate the exact causality condition, one has unavoidably to extend the functional expansion (10) and to treat it as defined on the class of arbitrary functions $\phi(x)$ not necessarily satisfying the equation (1).

This brings the method of PTDR closer to those adopting the existence of Heisenberg fields. In the latter case, however, the class of admissible theories becomes still more narrow, there being no need of it physically.

We shall start from the system of basic principles as formulated in PTDR, Sec.2. For the sake of simplicity let us consider one self-interacting scalar (or pseudo-scalar) field

$$\phi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d\underline{k}}{\sqrt{2k^0}} \{ e^{ikx} a^{(+)}(\underline{k}) + e^{-ikx} a^{(-)}(\underline{k}) \} ; \qquad (2)$$

$$k^0 = + \sqrt{\underline{k}^2 + m^2}.$$

....

* Hereafter referred to as PTDR.

Here $a^{(\pm)}(\underline{k})$ are the creation / annihilation operators for the asymptotic particles (strictly speaking out-particles), with the usual commutation relations

$$[a^{(-)}(\underline{k}), a^{(+)}(\underline{k}')] = \delta(\underline{k} - \underline{k}') . \qquad (3)$$

Let us assume that there are no bound states. Then one may take the manifold of states such as

$$|\underline{k}_1, \ldots, \underline{k}_n> = a^{(+)}(\underline{k}_1) \ldots a^{(+)}(\underline{k}_n) | 0> \qquad (4)$$

as a complete system provided by the requirement PTDR, I, (4) . Then the equation

$$<\alpha | AB | \beta> = \sum_{s=0}^{\infty} \frac{1}{s!} \int d\underline{s}_1 \ldots d\underline{s}_s <\alpha| A | \underline{s}_1, \ldots, \underline{s}_s><\underline{s}_1, \ldots, \underline{s}_s | B | \beta> . \qquad (5)$$

holds for any two operators A and B.

We shall write the four-dimensional Fourier-transformation in the form

$$F(x) = \frac{1}{(2\pi)^4} \int e^{-ikx} \tilde{F}(k)\, dk ; \; \tilde{F}(k) = \int e^{ikx} F(x)\, dx . \qquad (6)$$

and exploit similar formulae for many arguments.

Define

$$<0|\phi(x)\phi(y)|0> = -iD^{(-)}(x-y) ;$$

$$[\phi(x), \phi(y)]_{-} = -iD(x-y), \qquad (7)$$

$$-<0|\phi(y)\phi(x)|0> = -iD^{(+)}(x-y);$$

Then by eq. (2), (3) and (6),

$$\tilde{D}^{(-)}(k) = 2\pi i\,\theta(k^0)\,\delta(k^2 - m^2);$$

$$\tilde{D}(k) = 2\pi i\,\mathcal{E}(+k^0)\,\delta(k^2 - m^2); \qquad (8)$$

$$\tilde{D}^{(+)}(k) = -2\pi i\,\theta(-k^0)\,\delta(k^2 - m^2);$$

Note, that in our notations

$$[\tilde{D}^{(-)}(-k)]^{*} = \tilde{D}^{(+)}(k); \qquad \tilde{D}^{(-)}(-k) = -\tilde{D}^{(+)}(k);$$
$$[D^{(-)}(x)]^{*} = D^{(+)}(x); \qquad D^{(-)}(-x) = -D^{(+)}(x). \tag{9}$$

2. Some Properties of Scattering Matrix.

The main quantity we are going to deal with is the scattering matrix out of the energy shell. We shall write it down as a functional expansion in the normal products of the asymptotic fields

$$S = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int dx_1 \dots dx_n \Phi^n(x_1,\dots,x_n) : \phi(x_1) \dots \phi(x_n) : \tag{10}$$

Here the coefficient functions $\Phi^n(x_1,\dots,x_n)$ are c-functions symmetrical in all their arguments. Let us emphasize that we don't intend the functions $\phi(x)$ here to be restricted by the condition eq. (1).Performing variational differentiations one may obtain from the scattering matrix a sequence of operators depending upon the space - like points. However, it is more useful to consider not the variational derivatives themselves, but the radiative operators:

$$S^{(n)}(x_1,\dots,x_n) = \frac{\delta^n S}{\delta\phi(x_1)\dots\delta\phi(x_n)} S^{+}. \tag{11}$$

Now,we are going to prove some Lemmas which establish the relations between the vacuum expectation values (VEV) of these radiative operators, S-matrix's coefficient functions and their matrix elements.

Lemma 1. The coefficient functions of the scattering matrix coincide (up to a factor) with the VEV of the radiative operators (11)

$$\Phi^n(x_1,\dots,x_n) = i^n \langle 0| \frac{\delta^n S}{\delta\phi(x_1)\dots\delta\phi(x_n)} S^{+}|0\rangle = i^n \langle 0|S^{(n)}(x_1,\dots,x_n)|0\rangle. \tag{12}$$

Proof: Taking the n-th variation derivative of the expansion (10), we get

$$\frac{\delta^n S}{\delta\phi(x_1)\dots\delta\phi(x_n)} = \sum_{\nu=n}^{\infty} \frac{(-i)^\nu}{\nu!}\int dz_1\dots dz_\nu \Phi^\nu(z_1,\dots,z_\nu)\frac{\nu!}{(\nu-n)!}\,\delta(z_1-x_1)\dots\delta(z_n-x_n):\phi(z_{n+1})\dots\phi(z_\nu):$$

or, after the renumeration of the variables

$$\frac{\delta^n S}{\delta\phi(x_1)\dots\delta\phi(x_n)} = (-i)^n\sum_{\nu=0}^{\infty}\frac{(-i)^\nu}{\nu!}\int dz_1\dots dz_\nu \Phi^{n+\nu}(x_1,\dots,x_n,z_1,\dots,z_\nu):\phi(z_1)\dots\phi(z_\nu):$$

After taking the VEV only the term without normal products survices in the right-hand side

$$i^n\langle 0|\frac{\delta^n S}{\delta\phi(x_1)\dots\delta\phi(x_n)}|0\rangle = i^n(-i)^n\,\Phi^n(x_1,\dots,x_n).$$

Thus, to prove the eq.(12), it remains but to introduce the operator S^+ under the VEV sign in the left-hand side. This can be done in virtue of the vacuum stability condition PTDR I, (6).

Corollary : The scattering matrix can be written down in the form

$$S=\sum_{n=0}^{\infty}\frac{1}{n!}\int dx_1\dots dx_n\,\langle 0|\frac{\delta^n S}{\delta\phi(x_1)\dots\delta\phi(x_n)}S^+|0\rangle:\phi(x_1)\dots\phi(x_n): \qquad (13)$$

Lemma I expresses the coefficient functions of the scattering matrix in terms of the VEV of the operators (11). Let us now express (on the energy shell) the matrix elements of the S-matrix in terms of the same expectation values as well. To this end we will establish the

Lemma 2:

$$\langle\vec\ell_1,\dots,\vec\ell_n|S|\vec k_1,\dots,\vec k_m\rangle = \sum_{s=0}^{\min(n,m)} P\left(\frac{\ell_1,\dots,\ell_{n-s}}{\ell_{n-s+1},\dots,\ell_n}\right)P\left(\frac{k_1,\dots,k_{m-s}}{k_{m-s+1},\dots,k_m}\right)P(k_{m-s+1},\dots,k_m)\cdot$$

$$\cdot\,\delta(\vec\ell_{n-s+1}-\vec k_{m-s+1})\dots\delta(\vec\ell_n-\vec k_m)\,\cdot \qquad (14)$$

$$\cdot\int\frac{dx_1\dots dx_{n-s}dy_1\dots dy_{m-s}\,e^{i(\sum_1^{n-s}\ell x-\sum_1^{m-s}ky)}}{(2\pi)^{\frac{3(n+m)}{2}-3s}\sqrt{2\ell_1^0\dots 2\ell_{n-s}^0\,2k_1^0\dots 2k_{m-s}^0}}\,\langle 0|\frac{\delta^{n+m-2s}S}{\delta\phi(x_1)\dots\delta\phi(x_{n-s})\,\delta\phi(y_1)\dots\delta\phi(y_{m-s})}S^+|0\rangle\,;$$

$$k_j^{\circ} = +\sqrt{\vec{k}_j^{\,2} + m^2}\,; \quad \ell_j^{\circ} = +\sqrt{\vec{\ell}_j^{\,2} + m^2}$$

Proof : Consider now the matrix element $\langle \vec{\ell}_1, \ldots \vec{\ell}_n \mid S \mid \vec{k}_1, \ldots, \vec{k}_m \rangle$ which, according to (4), may be written as

$$\langle 0 |\, a(\vec{\ell}_1) \ldots a(\vec{\ell}_n)\, S\, a^+(\vec{k}_1) \ldots a^+(\vec{k}_m)\, | 0 \rangle .$$

If all the momenta $\vec{k}_j$ and $\vec{\ell}_j$ of various groups were different we could push them immediately through the S-matrix by means of the formulae PTDR (2.20). Then one would obtain the expression PTDR (2.21).

However, we can not restrict ourselves to this case only since the commutations (3) being singular, the cases of coincidence of some $\vec{\ell}$ with some $\vec{k}$ will bring nonvanishing contributions to occasional further integrations.

Now the coincidences of $\vec{\ell}$ and $\vec{k}$ give rise to some additional terms. Namely, each of the creation operators $a^+(\vec{k})$ may contract not with the S-matrix, but with anyone of the annihilation operators $a(\vec{\ell})$. Hence, it is evident that the whole expression for the matrix element considered must be composed of a sum of terms of the form PTDR eq. (2.21), the order of differentiation being successively decreased by 2, and the factors $\delta(\vec{\ell} - \vec{k})$ appearing accordingly.

For writing down such expressions explicitly it is convenient to use the symmetrizing operators introduced in [2]. The first of them $P\left(\frac{x_1, \ldots x_s}{x_{s+1}, \ldots, x_n}\right)$ prescribes to sum over all the possible $\frac{n!}{s!(n-s)!}$ divisions of the set $\{x_1, \ldots, x_n\}$ into two subsets of s and n-s elements. The divisions differing only by permutations inside one of the subsets, are to be counted only once. The second operator $P(x_1, \ldots, x_n)$ denotes the sum over all $n!$ permutations inside $\{x_1, \ldots, x_n\}$. It is easily seen that in terms of such operators the sum mentioned above takes just the form of (14) * (the factor S^+ may be put under the sign of the vacuum expectation value due to the vacuum stability). The upper limit in the sum over S in the right-hand side of (14) is determined by the number of the creation and annihilation operators;

The first summation over symmetrizations P corresponds to dividing the whole set of annihilation operators into two subsets: those contracting with creation operators and those contracting with the S-matrix; the second summation $P\,(k_{m-s+1}, \ldots, k_m)$ corresponds to similar division of the creation operators. Finally, the last sum comes from all possible different combinations of the operators $a(\vec{\ell})$ and $a^+(\vec{k})$ to be contracted.

It is clear that the formula analogous to Lemma 2 will hold also for the matrix elements of the S-matrix variational derivatives of any order - the additional differentiations will come unchanged under the VEV sign. The situation will be quite different in case of the radiative operators (11). Due to the factor S^+ an additional expansion in the complete system of functions will be necessary, and the formulae become much more troublesome. Exceptional are the cases when the ket state is the vacuum or one-particle state. Then, in virtue of the stability condition PTDR, I, (6) the operator S^+ can be absorbed in the ket, and the proof will be the same as for Lemma 2. Thus, one can formulate two more lemmas

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* Compare the technique developed in [8]. Its application would make it possible to replace the reasoning by calculation (to say the truth, a complicated one).

Lemma 3 :

$$<\vec{\ell}_1,\dots,\vec{\ell}_n|S^{(\kappa)}(z_1,\dots,z_\kappa)|0> = \int \frac{dx_1\dots dx_n e^{i\sum_1^n \ell x}}{(2\pi)^{\frac{3n}{2}}\sqrt{2\ell_1^\circ}\dots\sqrt{2\ell_n^\circ}} <0|S^{(\kappa+n)}(z_1,\dots,z_\kappa,x_1,\dots x_n)|0> \qquad (15)$$

By complex conjugation we obtain

Corollary :

$$<0|S^{+(\kappa)}(z_1,\dots,z_\kappa)|\vec{k}_1,\dots,\vec{k}_m> = \int \frac{dy_1\dots dy_m e^{-i\sum_1^m ky}}{(2\pi)^{\frac{3m}{2}}\sqrt{2k_1^\circ}\dots\sqrt{2k_m^\circ}} <0|S^{+(\kappa+m)}(z_1,\dots,z_\kappa,y_1,\dots y_m)|0> \qquad (16)$$

Lemma 4 :

$$<\vec{\ell}_1,\dots,\vec{\ell}_n|S^{(\kappa)}(z_1,\dots,z_\kappa)|\vec{k}> = (-i)^{\kappa+n+1}\int \frac{dx_1\dots dx_n\, dy\, e^{i\sum_1^n(\ell x - ky)}}{(2\pi)^{\frac{3(n+1)}{2}}\sqrt{2\ell_1^\circ\dots 2\ell_n^\circ . 2k^\circ}}\Phi^{\kappa+n+1}(z_1,\dots,z_2,x_1,\dots,x_n,y) +$$

$$(17)$$

$$+ P\left(\frac{\ell_1\dots\ell_{n-1}}{\ell_n}\right)(-i)^{\kappa+n-1}\int \frac{dx_1\dots dx_{n-1}\ell^{i\sum_1^{n-1}\ell x}}{(2\pi)^{\frac{3(n-1)}{2}}\sqrt{2\ell_1^\circ\dots 2\ell_{n-1}^\circ}}\Phi^{\kappa+n-1}(z_1,\dots,z_\kappa,x_1,\dots,x_{n-1})\delta(\vec{\ell}_n-\vec{k}) .$$

Taking the complex conjugate expression one can obtain from here the eq. of the type of eq. (16).

Up till now we were not concerned with the causality condition. Under the main assumptions of PTDR it was expressed in the form

$$\frac{\delta}{\delta\phi(x)}\left(\frac{\delta S}{\delta\phi(y)}S^+\right) = \frac{\delta S^{(1)}(y)}{\delta\phi\ (x)} = 0 \quad \text{for} \quad x \lesssim y . \qquad (18)$$

(The inequality sign comes from the fact that we consider the asymptotic fields $\phi\ (x)$ to be out-fields . For the in-fields it would be replaced by the opposite one). Let us prove some new relations following from it.

Lemma 5 : It follows from the causality condition Eq. (18) that

$$\frac{\delta}{\delta\phi(x)} S^{(n)}(y_1, \ldots, y_n) = 0, \tag{19}$$

if

$$x \lesssim \{ y_1, \ldots, y_n \} \tag{19a}$$

(The symbol (19a) means that the point x is earlier (or space-like) with respect to all the points $y_1, \ldots, y_n$)

Proof : Suppose the Lemma is valid up to a certain N. Then

$$\frac{\delta}{\delta\phi(x)} \left(\frac{\delta^N S}{\delta\phi(y_1)\ldots\delta\phi(y_N)} S^+ \right) = 0, \qquad \text{if} \qquad x \lesssim \{ y_1, \ldots, y_N \} \tag{*}$$

Differentiating this equality over $\phi\ (y_{N+1})$, one obtains

$$\frac{\delta}{\delta\phi(x)} \left(\frac{\delta^{N+1} S}{\delta\phi(y_1)\ldots\delta\phi(y_{N+1})} S^+ \right) + \frac{\delta}{\delta\phi(x)} \left(\frac{\delta^N S}{\delta\phi(y_1)\ldots\delta\phi(y_N)} S^+ \cdot S \frac{\delta S^+}{\delta\phi(y_{N+1})} \right) = 0,$$

Hence, making use of the unitarity, PTDR I, (5) and performing the differentiation over $\phi\ (x)$, one finds that

$$\frac{\delta}{\delta\phi(x)} \left(\frac{\delta^{N+1} S}{\delta\phi(y_1)\ldots\delta\phi(y_{N+1})} S^+ \right) = \frac{\delta}{\delta\phi(x)} \left(\frac{\delta^N S}{\delta\phi(y_1)\ldots\delta\phi(y_N)} S^+ \right) \cdot \frac{\delta S}{\delta\phi(y_{N+1})} S^+ +$$

$$+ \frac{\delta^N S}{\delta\phi(y_1)\ldots\delta\phi(y_N)} S^+ \cdot \frac{\delta}{\delta\phi(x)} \left(\frac{\delta S}{\delta\phi(y_{N+1})} S^+ \right).$$

The first term vanishes here by($*$), the second one - in virtue of eq. (18), if we require that $x \lesssim y_{N+1}$. So, if ($*$) holds for $n = N$, ($*$) holds for $n = N+1$, too. Since ($*$) holds for $n = 1$, Lemma 5 is proved by induction.

Lemma 5 allows one to increase arbitrarily the number of internal arguments in the causality condition eq. (18). It is also possible to increase the number of external arguments :

Lemma 6: It follows from the causality condition that

$$\frac{\delta^m}{\delta\phi(x_1)\dots\delta\phi(x_m)} \bar{S}^{(n)}(y_1,\dots y_n) = 0\,, \tag{20}$$

if, at least for one $1 \leq j \leq m$,

$$x_j \lesssim \{ y_1,\dots, y_n \} \tag{20a}$$

The proof is evident.

Both the original form of the causality condition (18) and the corollaries thereof, eqs. (19), (20) contain not only radiative operators but their variational derivatives as well. In some respects that form of the causality condition is more convenient which includes only the operators (11) themselves. This form is analogous to the "integral" causality condition of the perturbation theory [7]. Such a form is established by

Lemma 7 : It follows from the causality condition that for any $0 \lesssim s \lesssim n$ the radiative operators $S^n(x_1,\dots, x_n)$ are factorized in the form

$$S^{(n)}(x_1,\dots,x_n) = S^{(s)}(x_{j_1},\dots, x_{j_s})\, S^{(n-s)}(x_{j_{s+1}},\dots, x_{j_n}) \tag{21}$$

if

$$\{ x_{j_1},\dots, x_{j_s} \} \gtrsim \{ x_{j_{s+1}},\dots,x_{j_n} \} \tag{21a}$$

holds

Proof : Suppose the Lemma is valid for $n \leq N$ and let the arguments $\{x_1, \dots, x_{N+1}\}$ be such that

$$\{x_{j_1}, \dots, x_{j_s}\} \geq \{x_{j_{s+1}}, \dots, x_{j_{N+1}}\}. \qquad (*)$$

Expressing $S^{(k)}$ as usual in terms of the variational derivatives, write down the identity

$$\frac{\delta^{N+1} S}{\delta\phi(x_1)\dots\delta\phi(x_{N+1})} S^+ = \frac{\delta}{\delta\phi(x_{j_{N+1}})} \left(\frac{\delta^N S}{\delta\phi(x_{j_1})\dots\delta\phi(x_{j_s})\,\delta\phi(x_{j_{s+1}})\dots\delta\phi(x_{j_N})} S^+ \right) -$$

$$- \frac{\delta^N S}{\delta\phi(x_{j_1})\dots\delta\phi(x_{j_s})\,\delta\phi(x_{j_{s+1}})\dots\delta\phi(x_{j_N})} S^+ . \; S \frac{\delta S^+}{\delta\phi(x_{j_{N+1}})} .$$

By supposition , Lemma 7 is valid for the radiative operators of the N-th order. Therefore, one can proceed by applying the Lemma and performing differentiation over $\phi(x_{j_{N+1}})$ in the first term :

$$\frac{\delta}{\delta\phi(x_{j_{N+1}})} \left(\frac{\delta^s S}{\delta\phi(x_{j_1})\dots\delta\phi(x_{j_s})} S^+ \right) . \frac{\delta^{N-s} S}{\delta\phi(x_{j_{s+1}})\dots\delta\phi(x_{j_N})} S^+ +$$

$$+ \frac{\delta^s S}{\delta\phi(x_{j_1})\dots\delta\phi(x_{j_s})} S^+ . \frac{\delta^{N-s+1} S}{\delta\phi(x_{j_{s+1}})\dots\delta\phi(x_{j_{N+1}})} S^+ + \frac{\delta^s S}{\delta\phi(x_{j_1})\dots\delta\phi(x_{j_s})} S^+ .$$

$$. \frac{\delta^{N-s} S}{\delta\phi(x_{j_{s+1}})\dots\delta\phi(x_{j_N})} . \frac{\delta S^+}{\delta\phi(x_{j_{N+1}})} - \frac{\delta^s S}{\delta\phi(x_{j_1})\dots\delta\phi(x_{j_s})} S^+ . \frac{\delta^{N-s} S}{\delta\phi(x_{j_{s+1}})\dots\delta\phi(x_{j_N})} S^+ S \frac{\delta S^+}{\delta\phi(x_{j_{N+1}})} .$$

Here the first term vanishes by Lemma 5 as the point $x_{j_{N+1}}$ belongs to the second group in $(*)$; the last two terms cancel. Returning to the operators $S^{(k)}$, we see that the Lemma holds also for $n = N+1$. Since for $n = 1$ it is trivial, Lemma 7 is proved by induction.

The integral causality condition eq. (21) is not only necessary, but also sufficient for the fulfilment of the differential condition, eq. (18). Moreover, to obtain the latter eq. (21) must hold only for n = 2. Indeed, let us perform the variational differentiation over $\phi(x)$ in the left-hand side of eq. (18):

$$\frac{\delta}{\delta\phi(x)}\left(\frac{\delta S}{\delta\phi(y)} S^{+}\right) = \frac{\delta^2 S}{\delta\phi(x)\,\delta\phi(y)} S^{+} + \frac{\delta S}{\delta\phi(y)} S^{+} \cdot S \frac{\delta S^{+}}{\delta\phi(x)} .$$

The first term, by eq. (21) is equal to

$$\frac{\delta S}{\delta\phi(y)} S^{+} \cdot \frac{\delta S}{\delta\phi(x)} S^{+} \text{ if } x \lesssim y \text{ and } \frac{\delta S}{\delta\phi(x)} S^{+} \cdot \frac{\delta S}{\delta\phi(y)} S^{+} \text{ if } y \lesssim x ,$$

whereas the second one - in virtue of the unitarity - will always give $-\frac{\delta S}{\delta\phi(y)} S^{+} \cdot \frac{\delta S}{\delta\phi(x)} S^{+}$. Thus, we see that once the eq. (21) holds, then

$$\frac{\delta}{\delta\phi(x)}\left(\frac{\delta S}{\delta\phi(y)} S^{+}\right) = \begin{cases} = \left[\frac{\delta S}{\delta\phi(x)} S^{+}, \frac{\delta S}{\delta\phi(y)} S^{+}\right]_{-} ; & x \gtrsim y \\ = 0 ; & x \lesssim y \end{cases} \quad (22)$$

Thereby we proved

<u>Lemma 8</u> :For the differential causality condition to hold it is sufficient that the integral causality condition would be fulfilled in the form

$$S^{(2)}(x_1, x_2) = S^{(1)}(x_{j_1})\, S^{(2)}(x_{j_2}) \quad \text{if} \quad x_{j_1} \gtrsim x_{j_2} \quad (23)$$

Why is condition eq. (23) sufficient for the derivation of eq. (18), while in perturbation theory it had been necessary to require [7] a fulfilment of the integral condition for all n ? The reason is that now our radiation operators $S^{(k)}$ depend not only upon the arguments $x_1,\ldots, x_k$ written in explicit form, but are the functionals $\phi(x)$ as well. This latter functuonal dependence connects the operators $S^{(k)}$ of different orders with each other.

The results of the investigation we have performed may be formulated in the form of

Theorem 1: The causality conditions in the forms eq. (18) and eq. (23) are equivalent. The relations between the operators $S^{(k)}$ follow from either of them. These relations are expressed by eqs. (19), (20) and (21).

3. Coupled Equations for the Coefficient Functions.

Various forms of the causality condition obtained in the previous section were the operator equations. In many respects it is better to deal not with the operators but with the c-functions - the coefficient functions or matrix elements. The relations proved above allow us to derive a number of equations for these functions.

Note that the causality condition in the form eq. (21) may be written (by neglecting possible coincidences of the arguments, cf. below) as the equality

$$\frac{\delta^n S}{\delta\phi(z_1)\ldots\delta\phi(z_n)} = P\left(\frac{z_1,\ldots,z_s}{z_{s+1},\ldots,z_n}\right)\Theta(z_1,\ldots,z_s;\, z_{s+1},\ldots,z_n)\cdot \frac{\delta^s S}{\delta\phi(z_1)\ldots\delta\phi(z_s)}\, S^{+}\cdot \frac{\delta^{n-s} S}{\delta\phi(z_{s+1})\ldots\delta\phi(z_n)}\, S^{+}, \tag{24}$$

where $\Theta(z_1,\ldots, z_s;\, z_{s+1},\ldots, z_n)$ is equal to 1, if all $z_1^0,\ldots, z_s^0$ are greater than all $z_{s+1}^0,\ldots, z_n^0$ and is equal to zero in the opposite case. If one takes the VEV of this equality, then in the left-hand side will appear the function $(-i)^n\,\Phi^n(z_1,\ldots, z_n)$ by Lemma 1. On the other hand, the right-hand side is to be expanded in the complete system of states making use of eq. (4). In doing this, we meet two possibilities.

If the first subset in eq. (24) contains only one argument $s = 1$ then the unitarity property of the S-matrix makes possible to put the first factor of the right-hand side in the form

$$\frac{\delta S}{\delta\phi(z_1)} S^+ = - S \frac{\delta S^+}{\delta\phi(z_1)} .$$

After this by the Corollary of Lemma 3 it may be expressed in terms of the functions $\overset{*}{\Phi}{}^{\nu}(z_1,\dots z_\nu)$ The second factor will just have the form reducing to the coefficient functions Φ^ν by Lemma 3. Thus we obtain

$$\Phi^n(z_1,\dots, z_n) = P\left(\frac{z_1}{z_2,\dots, z_n}\right) \Theta(z_1; z_2,\dots, z_n) \sum_{m=0}^{\infty} \frac{1}{m!} \int dx_1\dots dx_m dy_1\dots dy_m \cdot$$

$$\cdot \int \frac{d\vec{k}_1\dots d\vec{k}_m}{(2\pi)^{3m}\, 2k_1^o\dots 2k_m^o} e^{-i\sum_1^m k(y-x)} \Phi^{*m+1}(z_1, y_1,\dots, y_m)\, \Phi^{m+n-1}(x_1,\dots,x_m, z_2,\dots, z_n)$$

Noting now that according to eqs. (6), (8), the integrals over each of $\vec{k}$

$$\frac{1}{(2\pi)^3} \int \frac{d\vec{k}_j}{2k_j^o} e^{-ik_j(y_j - x_j)} = -i\, \mathfrak{D}^{(-)}(y_j - x_j) .$$

reduce to the functions $\mathfrak{D}^{(-)}(y-x)$ we are led to an infinite set of coupled equations for the functions Φ^ν

$$\Phi^n(z_1,\dots, z_n) = P\left(\frac{z_1}{z_2,\dots, z_n}\right) \Theta(z_1; z_2,\dots, z_n) \sum_{m=0}^{\infty} \frac{(-i)^m}{m!} \int dx_1\dots dx_m dy_1\dots dy_m \cdot$$

$$\cdot \Phi^{*1+m}(z_1, y_1,\dots, y_m)\, \mathfrak{D}^{(-)}(y_1 - x_1)\dots \mathfrak{D}^{(-)}(y_m - x_m)\, \Phi^{n-1+m}(x_1,\dots, x_m, z_2,\dots, z_n) . \tag{25}$$

This set is analogous to the " system A " derived by Lehman, Symanzik and Zimmerman/4/ starting from other basic assumptions.

Another possibility, or, to be more exact, another set of possibilities is realized if we choose $1 < s < n-1$ in (24) . Then the unitarity condition may not be used to bring the first factor of the right-hand side of eq.(24) to the form allowing the application of the Corollary of Lemma 3. In order to reduce the right-hand side to the functions Φ^ν we have but to resort to the division into three factors, i.e., to use the expansion in the complete system of states two times. Then we shall come to the set

$$\Phi^n(x_1,\dots,x_n) = P\left(\frac{x_1,\dots,x_s}{x_{s+1},\dots,x_n}\right)\Theta(x_1,\dots,x_s;x_{s+1},\dots,x_n)\sum_{\varkappa=0}^{\infty}\sum_{\nu=0}^{\infty}\sum_{\mu=0}^{\infty}\frac{i^{\varkappa+\nu+\mu}}{\varkappa!\,\nu!\,\mu!}\cdot$$

$$\cdot\int dz_1\dots dz_\varkappa du_1\dots du_\nu \int du'_1\dots du'_\nu\, dv_1\dots dv_\mu \int dz'_1\dots dz'_\varkappa\, dv'_1\dots dv'_\mu \cdot$$

$$\cdot\,\Phi^{s+\varkappa+\nu}(x_1,\dots,x_s,z_1,\dots,z_\varkappa,u_1,\dots,u_\nu)\,\mathfrak{D}^{(-)}(z_1-z'_1)\dots\mathfrak{D}^{(-)}(z_\varkappa-z'_\varkappa)\,\mathfrak{D}^{(-)}(u_1-u'_1)\dots\mathfrak{D}^{(-)}(u_\nu-u'_\nu)\,\cdot$$

$$\cdot\,\overset{*}{\Phi}{}^{\nu+\mu}(u'_1,\dots,u'_\nu,v_1,\dots,v_\mu)\,\mathfrak{D}^{(-)}(v_1-v'_1)\dots\mathfrak{D}^{(-)}(v_\mu-v'_\mu)\,\cdot \qquad (26)$$

$$\cdot\,\Phi^{n-s+\varkappa+\mu}(z'_1,\dots,z'_\varkappa,v'_1,\dots,v'_\mu,x_{s+1},\dots,x_n)$$

(strictly speaking, this is a set of sets correspondingly to the choice of the value of s). The structure of the set obtained may be visualized by the graphic scheme

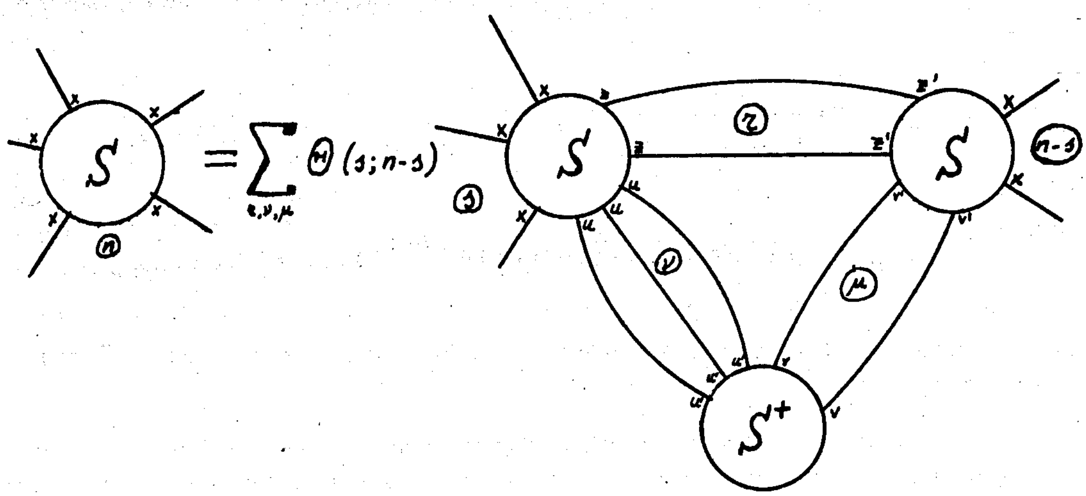

Fig. 1

An infinite set of eq. (25) is much simpler than the set, eq. (26). However, to find the functions Φ^{ν} one cannot restrict oneself only to this set. The reason for this lies in the following, very essential circumstance. When in the previous section we wrote down different forms of the causality conditions eqs. (18)-(23), we have always used strict inequalities between the coordinates - e.g., the equality (23) is valid only if $x_1 \lesssim x_2$ or if $x_2 \lesssim x_1$, but not for $x_1 = x_2$. Therefore, expressions (21) or (23) determine the functions $\Phi^{\nu}(x_1,\dots, x_{\nu})$ not for all the values of the arguments: if any two points entering different subsets in the condition 21a (or two points in condition (23)) coincide, then the corresponding value of Φ^{ν} remains undertermined. Therefore, it is necessary to introduce additional terms ("counterterms") in the right-hand sides of eqs. (25) and (26) corresponding to such coincidences of the arguments*.

A consistent treatment of these counterterms seems to lead to great combinatorial difficulties. If, however, we shall allow ourselves to use simultaneously the sets, eq. (26) for all the values s, then it is easy to see that a set of points $x_1,\dots x_n$ will permit no divisions eq. (21a), only if all of them coincide. In other words, for each function Φ^{ν} it will be necessary to introduce only one counterterm - the remaining one being taken account of automatically by one of the eqs. (26). At the same time, however, one has to resort to a peculiar "equilibristics" - to obtain one and the same function it is necessary to use different sets of equations depending upon the value of the arguments.

Still another set of equations may be obtained from condition eq. (23) which, according to proof, is sufficient for the fulfilment of the causality. A great advantage of this set is that it contains only one θ- function. Therefore, one may hope that the combinatorial difficulties will disappear. However, in this case one cannot restrict oneself to the VEV of the operator condition to obtain a closed set, it will be necessary to consider the matrix elements between all the states. Such a set was studied by M.K. Polivanov and the author elsewhere. Therefore we will not write it down here.

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* The formal use of the sets, eqs. (25), (26) without taking account of this circumstance, may lead to well-known divergences. They arise due to the multiplication of θ-functions by the functions insufficiently regular near zero. As shown in PTDR, sec. 4,5 this is equivalent to an incorrect application of the integral Cauchy formula to the Fourier-transforms, not tending towards zero at infinity.

4. Functional Expansion of the Scattering Matrix

In the previous section we established some infinite sets of coupled equations for the coefficient functions Φ^{ν}, entering the functional expansion, eq. (10), of the scattering matrix in the normal products of the asymptotic fields. An attempt to solve these equations even approximately, leads to enourmous difficulties. Yet we shall see now that some evidence about the functional expansion may be obtained in another way. Indeed, we will show that all the coefficient functions may be expressed in terms of the VEV of the T- products of a certain set of operators. At the same time it will turn out that the structure of the functional expansion (10) closely resembles that of the functional expansion [1-2] in powers of the "switching on and off" function, or roughly speaking, of the usual perturbation theory expansion.

Let us turn again to the radiative operators eq. (11). The causality condition is expressed in terms of of them by eq. (21). On the other hand, if n variational differentiations of the unitarity condition PTDR, I, (5) are performed, then the requirements

$$\sum_{m=0}^{\infty} P\left(\frac{x_1,\dots, x_{n-m}}{x_{n-m+1},\dots, x_n}\right) S^{(n-m)}(x_1,\dots, x_{n-m})\, \overset{+}{S}{}^{(m)}(x_{n-m+1},\dots, x_n) = \delta_{n0} \qquad (27)$$

will be imposed upon the operators $S^{(n)}$

Now we may forget about the way (11) of forming the radiative operators $S^{(n)}$ and set a purely algebraic problem of finding a general form of the operators satisfying the condition (21) and (27). Now, the conditions (21) and (27) are just those imposed on the operator coefficient functions entering the S-matrix expansion in powers of the "switching on and off function" g(x) (see [2], (18.1) [7], eq. (1). The condition (21) coincides algebraically with the "integral causality condition" /7/ from [7], whereas condition (27) is the unitarity condition /18.9/ from [2].

Thus, it turns out that the problem of finding a general form of the radiative operators $S^{(n)}$ is identical with that of finding a general form of the coefficient functions of the scattering matrix in the perturbation theory. The latter problem has been solved in [2], and, just for case of the causality condition analogous to eq. (21), in [7]. It has been shown that all the conditions (21) and (27) are compatible and each of the operator functions $S^{(n)}(x_1,\dots x_n)$ is expressed in terms of the lower order operator functions up to its anti-hermitian part for all arguments coinciding. This latter may be given arbitrary value.

Therefore, in our case the general expression for $S^{(n)}$ will be of the form

$$S^{(n)}(x_1,\dots,x_n) = (-i)^n\, T[\Lambda_1(x_1)\dots\Lambda_1(x_n).] +$$

$$+ \sum_{\substack{2\leq m\leq n-1\\ \nu_1+\dots+\nu_m=n\\ \nu_j\geq 1}} \frac{(-i)^m}{m!}\, P(x_1,\dots,x_{\nu_1}\,|\,x_{\nu_1+1},\dots,x_{\nu_1+\nu_2}\,|\dots|\,x_{\nu_1+\dots+\nu_{m-1}+1},\dots,x_n)\cdot \tag{28}$$

$$\cdot T[\Lambda_{\nu_1}(x_1,\dots,x_{\nu_1})\dots\Lambda_{\nu_m}(x_{\nu_1+\dots+\nu_{m-1}+1},\dots,x_n)$$

where $P(x_1,\dots,x_{\nu_1}\,|\,\dots|\dots x_n)$ –

is[2] the operator of summation over all possible $\dfrac{n!}{\nu_1!\dots\nu_m!}$ divisions of the set of n points into m subsets with $\nu_1, \nu_2,\dots,\nu_m$ points in each.

The operators Λ_ν are arbitrary operators with the properties :

(1) locality :

$$\Lambda_\nu(x_1,\dots,x_\nu) = 0 \quad \text{except the case} \quad x_1 = x_2 = \dots = x_\nu , \tag{29}$$

(2) hermiticity

$$\Lambda^+_\nu(x_1,\dots,x_\nu) = \Lambda_\nu(x_1,\dots,x_\nu) \tag{30}$$

(3) symmetry

$$\Lambda_\nu(x_{a_1},\dots,x_{a_\nu}) = \Lambda_\nu(x_1,\dots,x_\nu) \tag{31}$$

and (4) commutativity in spacelike points:

$$\begin{cases} [\Lambda_\nu(x_1,\dots x_\nu)\ ;\ \Lambda_\mu(y_1,\dots,y_\mu)]_- = 0 \\ \text{if}\quad (x_1=\dots=x_\nu) \sim (y_1=\dots=y_\mu) \end{cases} \tag{32}$$

They reflect the just mentioned arbitrariness of choosing the values of the radiative operators $S^{(\nu)}$ for the coinciding arguments. The first of the operators Λ coincides with the current operator

$$\Lambda_1(x) = j(x) = iS^{(1)}(x) \tag{33}$$

There is yet one point to be emphasized. When using eq.(28), it is to be understood that we fix somehow for each ν the arbitrariness contained in the definition of the T-product for the coinciding arguments (the integration rules near zero). In doing this we should keep the unitary condition fulfilled. Only after this do we add the operator Λ_ν. Therefore, one can say that the operators Λ_ν contain all the arbitrariness of the T-products. They play the part of the counterterms which were told about above in connection with infinite sets of equations. Should we not like to separate the counter terms, then it would be sufficient to restrict ourselves in eq. (28) to the T-product of the currents, but in every case of the coinciding arguments the arbitrariness would arise *.

Since (Lemma 1) the coefficient functions Φ^n of expansion (10) are obtained by taking the VEV of the radiative operators $S^{(n)}$ then, with the aid of (28), they may be expressed also in terms of the successive operators $\Lambda_1, \ldots, \Lambda_\nu, \ldots$:

$$\begin{aligned} \Phi^n(x_1, \ldots, x_n) &= \langle 0 | T[j(x_1) \ldots j(x_n)] | 0 \rangle + \\ &+ \sum_{\substack{2 \leq m \leq n-1 \\ \Sigma\nu = n}} \frac{i^{n-m}}{m!} P(x_1, \ldots, x_{\nu_1} | \ldots | \ldots, x_n) \langle 0 | T[\Lambda_{\nu_1}(x_1, \ldots, x_{\nu_1}) \ldots \Lambda_{\nu_m}(\ldots, x_n)] | 0 \rangle + \\ &+ i^{n-1} \langle 0 | \Lambda_+(x_1, \ldots, x_n) | 0 \rangle. \end{aligned} \tag{34}$$

....

* Note, that according to our approach the counter terms play a more noticeable role than in the perturbation theory. If in the perturbation theory they were essential practically only for the problems related to the infinities and renormalizations, then, here, - as one can be easily make sure by using the perturbation theory expression for the current, the operators Λ_ν with $\nu > 1$ will contribute to the processes which do not involve divergencies, e.g., in the Compton effect in the lower order. This is caused by the fact that the main term here ((28) or (34)) is T-product of currents, and not of Lagrangians.

5. Discussion.

Expressions (28) and (34) obtained for the radiative operators and coefficient functions of the S-matrix considerably clear up the structure of the functional expansion (10) of the scattering matrix. In particular, they may prove to be a valuable tool for the attempts to find an approximate solution of infinite equation systems for the functions Φ^{ν}. In no way, however, they can claim to resolve the fundamental problem to find a scattering matrix without perturbation theory.

Indeed, first of all in the perturbation theory the expression externally identical to eq. (28) solves the problem of only the formal construction of the scattering matrix. We have to speak only about the formal construction since its immediate application would lead to the divergent expressions when the arguments of the T-products coincide. As is well-known, a further step is to impose the condition of the absence of divergencies upon the whole sum analogous to the sum eq. (28). At the same time one is able to demonstrate (see [2], Sec. 26), that such a condition may be be always satisfied by appropriately choosing the arbitrary operators Λ_{ν}. In our case this would be still done.

However, the main difference of our formulae from the perturbation theory expansion consists in another point. In the perturbation theory the quasilocal operators Λ_{ν} are dependent only upon the coordinates $x_1, \dots, x_{\nu}$ (strictly speaking upon the free field operators at these, coinciding points). In our case the operators Λ_{ν} depend not only upon the coordinates $x_1, \dots, x_{\nu}$ but also functionally upon $\phi(y)$:

$$\Lambda_{\nu}(x_1, \dots, x_{\nu}) = \Lambda_{\nu}(x_1, \dots, x_{\nu} | \phi(y)) \qquad (35)$$

The same holds, evidently, for the operators $S^{(n)}$. But we do not know yet anything about this dependence in detail; it is only clear that it will connect the operators of different orders.

By taking into account this circumstance formula (28) expresses the operators $S^{(n)}$ for a certain "fixed" $\phi(y)$ in terms of products of the operators Λ_{ν} related to the same "value" $\phi(y)$ therefore, in terms of $\Lambda_{\nu}(\dots | \phi(y))$ with any $\phi(y)$ (the sum over the complete system!). Therefore, in order eqs. (28), (34) could serve as a basis for solving the problem of finding the scattering matrix without the perturbation theory, it is still necessary to clear up the character of the functional dependence of the operators Λ_{ν} upon $\phi(y)$. This problem may be the subject of another investigation which will be performed elsewhere.

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