ОБЪЕДИНЕННЫЙ ИНСТИТУТ ЯДЕРНЫХ ИССЛЕДОВАНИЙ

L.D. Solovyov

D - 580

PHOTOPRODUCTION OF PIONS ON PIONS

ЖЭТФ, 1961, т.40, в2, с597.
Межд. конф. по физике высоких энергий в Рочестере, 1960.
p. 366.

Дубна 1960

L.D. Solovyov

D - 580

PHOTOPRODUCTION OF PIONS ON PIONS

Abstract

An exact solution has been found for the equation describing the photoproduction of pions on pions at low energies. A requirement choosing a unique solution has been formulated. The solution is determined by the high energy singularities of the amplitude. It has a resonance character if there is a resonance in pion-pion scattering in the state with $J = I = 1$.

1. Introduction

The photoproduction of pions on pions

$$\gamma + \pi \rightarrow \pi + \pi \tag{1}$$

should be considered in studying the photoproduction of pions on nucleons with the aid of the Mandelstam representation/1/ as well as it is necessary to consider the pion-pion scattering for studying the pion-nucleon scattering.

In treating the pion-pion scattering a new constant of pion-pion interaction is introduced into the theory/2/. Must one more similar constant be taken into account in considering the photoproduction of pions on pions? The perturbation theory answers this question in the negative. Indeed, if the four-pion vertex with four internal nucleon lines is divergent, what makes us introduce into the Lagrangian the corresponding term and the pion-pion interaction constant (see, e.g./3/), then an analogous vertex with one photon and three pion external lines is convergent, so that there is no need in new terms and in a new constant. Moreover, such photon-three-pion terms cannot be simply introduced due to the covariance and renormalizability considerations. Thus, from the point of view of the perturbation theory the amplitude for the photoproduction must be expressed in terms of the "old" constants (say, the constants of electromagnetic, pion-pion and pion-nucleon interaction). Further, if after the electromagnetic interaction is taken into account once in switching on the photon, one considers only strong interactions and does not take into account pion-two- kaon (πKK) interaction (which cannot be stronger than the electromagnetic one/4/), then any graph of process (1) must contain the nucleon or nucleon-hyperon loop (Fig.1). Thus, process (1) is essentially associated with baryons in intermediate states and must disappear if baryon masses tend to infinity.

It will be shown below that these results of the perturbation theory also follow from the theory of dispersion relations.

Process (1) has been treated in /5/ by means of double dispersion relations, where a homogeneous equation has been obtained for it,and its solution has been found in the approximation of sharp pion resonance.This solution depends on the indefinite constant and has a resonance character when the width of the pion resonance(with its finite height) is zero. From the physical standpoint it is clear that if the pion scattering amplitude is everywhere zero except one point where it is finite, then process of scattering must display nowhere.

In the foregoing we shall start from the usual (one-dimensional) dispersion relation in the observable region, assuming that it is valid without subtractions. In §3 a physical consideration is given from which it follows that while for scattering the dispersion relations with one subtraction are valid (like in[2]), for the photoproduction, due to gauge invariance, dispersion relations are valid without subtractions. In order to obtain from the dispersion relation an inhomogeneous equation having a nonzero solution it is necessary to take into account for singularities, first of all, the singularity corresponding to the nucleon-antinucleon pair in the intermediate state in the unitarity condition* .

For the equation obtained from the dispersion relation a solution has been found in §4 in the explicit form.

This solution is unique if the pion phase-shift is vanishing at the infinity. If the phase-shift tends to π , then the solution is not unique. However, one of the solutions at the infinity tends to an inhomogeneous term most quickly. This solution has such a property that all the contribution from the pion scattering disappears from it if the width of the pion resonance (at the finite height)is tends to zero. It is this solution which choose as a physical one.

The graphs of solutions have been constructed for two pion resonance models different by the behaviour of the phase-shift at the infinity.

2. Kinematics

Let k and e_δ be the momentum and polarization vector of the photon, $q_1\alpha$ and $q_2\beta$, $q_3\gamma$ be the momenta and charge numbers of the initial and final pions respectively. The matrix element of process (1) is of the form[5]

$$\langle\pi\pi|S|\pi\gamma\rangle = (2\pi)^4\,\delta(k+q_1-q_2-q_3)\,\frac{\varepsilon_{mnrs}\,q_1^m q_2^n q_3^r e_\delta^s}{4(q_1^0 q_2^0 q_3^0 k^0)^{1/2}}\,\varepsilon_{\alpha\beta\gamma}\,\mathcal{F}(s,\bar{s},t)\,, \qquad (2)$$

where ε are completely antisymmetrical tensors, whereas $\mathcal{F}$ is the completely symmetrical function of the invariants

$$s=(q_2+q_3)^2\,,\quad \bar{s}=(q_1-q_3)^2\,,\quad t=(q_1-q_2)^2\;;\qquad s+\bar{s}+t=3\mu^2 \qquad (3)$$

---

* Analogous for singularities (for πK scattering with a charge exchange) were taken into account in the paper of Anselm and Shekhter which were reported at the Conference on Dispersion Relations held at Dubna (May, 1960).

In the center-of-mass system

$$\langle \pi\pi|S|\pi\gamma\rangle = (2\pi)^4\delta(k+q_1-q_2-q_3)\frac{\vec{e}_\gamma[\vec{q}\vec{k}]}{2\sqrt{\omega_k k}}\varepsilon_{\alpha\beta\gamma}\mathcal{F}(s,\bar{s},t) \tag{4}$$

$$\begin{aligned} s &= (k+\omega_k)^2 = 4\omega_q^2 \\ \bar{s} &= \mu^2 - 2k\omega_q - 2kq\cos\theta \\ t &= \mu^2 - 2k\omega_q + 2kq\cos\theta\,, \end{aligned} \tag{5}$$

where μ is the pion mass, $k=|\vec{k}|$, $\vec{q}=\vec{q}_3$, $\omega_k=(k^2+\mu^2)^{1/2}$, $\cos\theta=(\vec{k}\vec{q})/kq$,

$$\mathcal{F}(s,\cos\theta) = \sum_{l=0}^{\infty} f_{2l+1}(s)\,P'_{2l+1}(\cos\theta)\,. \tag{6}$$

It follows from the unitarity condition that at low energies

$$f_l = |f_l|e^{i\delta_l^1}\,, \tag{7}$$

where δ_l^1 is the pion-pion scattering phase-shift corresponding to the state with an angular momentum l and isotopic spin I. The differential cross section is equal to

$$\frac{d\sigma}{d\Omega} = \frac{1}{8}kq^3\sin^2\theta\left|\frac{\mathcal{F}}{4\pi}\right|^2 \tag{8}$$

3. Dispersion Relation

Postulate now the behaviour of F at fixed t and $s\to\infty$. It follows from (8) that at fixed t and $s\to\infty$

$$\frac{d\sigma}{d\Omega} = const\,|\sqrt{s}\,\mathcal{F}|^2 \tag{9}$$

If this equality had related to elastic process, then it would have followed from it that at fixed t and $s\to\infty$ $\mathcal{F}$ behaves as the total cross section for this process. There are theoretical considerations (Gribov), that the total cross section must decrease as $1/\ln^2 s$ at the infinity. Therefore, $\mathrm{Im}\,\mathcal{F}$ in (9) must have decreased in the same manner for elastic process. We assume that this holds also for the process under consideration.

If we assume in general that the forward differential cross sections for the photoproduction and pion scattering at the infinity have equal degree of increasing, then the invariant ampliudes for the photoproduction processes must decrease quicker than for scattering, since there must be an energy factor between the invariant amplitude and the matrix element of the photoproduction which is absent in the case of scattering. At the same time, if for scattering the dispersion relations are valid with one subtraction, then for photoproduction they are valid without any subtractions. Thus, we assume that one-dimensional dispersion relations strictly proved in /6/ for the process we are considering are valid without subtractions:

$$\mathcal{F}(s,t) = \frac{1}{\pi}\int_{4\mu^2}^{\infty} \frac{Im\,\mathcal{F}(s',t)}{s'-s-i\varepsilon}\,ds' + (s \to \bar{s}) \qquad (10)$$

If in this relation we confine ourselves to the consideration of the nearest singularities, i.e., we take into account only the two-pion intermediate state in the unitarity condition, then for the photoproduction amplitude we obtain a homogeneous equation which must have a zero solution (the requirement of uniqueness is formulated below, 4). Thus, the process under consideration depends essentially upon the far singularities.

The following singularities correspond to 4,6 etc pions in the intermediate states and their account introduces the photoproduction amplitudes of many pions into (10). At the present time we cannot write the system of equations for these amplitudes. However, it is clear that since all these amplitudes have no "polar" singularities, such a system must be homogeneous, and its solution must be a zero one.

The following singularity corresponds to the koan-antikoan pair ($K\bar{K}$). If non-strong interactions are not taken into account, then the amplitude for the process $\gamma\pi \to K\bar{K}$ has also no polar singularity. It is not difficult to write a dispersion relation for it (without subtractions, like (10)), which has a zero solution, if the amplitude for pion photoproduction is equal to zero.

The nearest singularity which introduces inhomogenuity into (10) corresponds to the nucleon-antinucleon pair in the intermediate state since the dispersion relations for the amplitude of the process $\gamma\pi \to N\bar{N}$ have an inhomogeneous polar term.

After the inhomogenuity is introduced into the equations under consideration, the photoproduction amplitudes for 4 etc pions may be neglected in the region of low energies we are considering.

Thus, in $Im\,\mathcal{F}$ in (10) we take into account two terms

$$Im\,\mathcal{F} = (Im\,\mathcal{F})_{\pi\pi} + (Im\,\mathcal{F})_{N\bar{N}} \tag{11}$$

where $(Im\,\mathcal{F})_{\pi\pi}$ is expressed in terms of the amplitudes for the processes $\gamma\pi \to \pi\pi$ and $\pi\pi \to \pi\pi$ (formula 7), whereas $(Im\,\mathcal{F})_{N\bar{N}}$ - in terms of the amplitudes for the processes $\gamma\pi \to N\bar{N}$ and $\pi\pi \to N\bar{N}$.

$(Im\,\mathcal{F})_{\pi\pi}$ in (10) contains the region of nonobservable angles at low energies. The nonobservable region is absent at

$$t = -\frac{\mu^2}{2} \tag{12}$$

and relation (10) in this case has the form

$$\mathcal{F}(s, \cos\theta = q/k) = \frac{1}{\pi}\int_{4\mu^2}^{\infty} \frac{(Im(s', \cos\theta = q'/k'))_{\pi\pi}}{s' - s}\, ds' + \\ + \frac{1}{\pi}\int_{4m^2}^{\infty} \frac{(Im\,\mathcal{F}(s', \cos\theta = q'/k'))_{N\bar{N}}}{s' - s}\, ds' + (s \to \bar{s} = \frac{5}{2}\mu^2 - s), \tag{13}$$

here m is the nucleon mass. Obviously, in the integrals with lower limit $4m^2$ one may put $\cos\theta = 1$ and neglect s, $\bar{s}$ in comparison with s'.

4. Integral Equation and its Solution

At low energies in the observable region in expansion (6) it is sufficient to take into account the lowest partial waves, by neglecting F wave and the waves having higher angular momenta

$$\mathcal{F}(s, \cos\theta) = f_1(s) \equiv f(s) \tag{14}$$

where f is the amplitude of the P wave. By denoting $\nu = q^2/\mu^2$, we obtain from (13)

$$f(\nu) = \Lambda + \frac{1}{\pi}\int_0^{\infty} Im\,f(\nu')\left(\frac{1}{\nu' - \nu - i\varepsilon} + \frac{1}{\nu' + \nu + 9/8}\right) d\nu'. \tag{15}$$

Here

$$\Lambda = \frac{2}{\pi}\int\limits_{4m^2}^{\infty}\frac{(\mathrm{Im}\,\mathcal{F}(s',1))_{NN}}{s'}\,ds' \tag{16}$$

$$f(\nu) = |f(\nu)|\,e^{i\delta(\nu)}\ , \quad \delta \equiv \delta_1^{\ 1} \tag{17}$$

Equation (15) is crossing-symmetrical (in the substitution $\nu \to -\nu - 9/8$) and an exact solution may be written for it (see/7/,/8/).

1)Let for $\nu \to \infty$

$$\delta(\nu) \to 0 \qquad (\delta(\nu) \to C\nu^{-\alpha}\ ,\ \alpha > 0) \tag{18}$$

Let us denote

$$\Delta(z) = \frac{1}{\pi}\int\limits_0^{\infty}\delta(\nu)\left(\frac{1}{\nu - z} + \frac{1}{\nu + z + 9/8}\right)d\nu \tag{19}$$

This crossing-symmetrical function is holomorph over all the plane with the cut from $-\infty$ up to $-9/8$ and from 0 up to ∞ . Its limiting values on the cut from upper (+) and lower (-) half-planes are equal to

$$\Delta^{\pm}(\nu) = \rho(\nu) \pm i\delta(\nu)\ , \tag{20}$$

where

$$\rho(\nu) = \frac{1}{\pi}P\int\limits_0^{\infty}\delta(x)\left(\frac{1}{x-\nu} + \frac{1}{x+\nu+9/8}\right)dx\ . \tag{21}$$

When $z \to \infty$

$$\Delta(z) \to 0\ . \tag{22}$$

Consider the function

$$\Psi(z) = \frac{1}{\pi}\int\limits_0^{\infty}e^{\rho(\nu)}\sin\delta(\nu)\left(\frac{1}{\nu - z} + \frac{1}{\nu + z + 9/8}\right)d\nu \tag{23}$$

It has the same symmetry, the same region of analyticity and the same cuts as $\Delta(z)$, and its jump on the cut is equal to

$$\Psi^{+} - \Psi^{-} = 2ie^{\rho}\sin\delta = (e^{\Delta})^{+} - (e^{\Delta})^{-} \tag{24}$$

Therefore, the function $\Psi(z)$ must coincide with $e^{\Delta(z)}$ with an accuracy up to a polynomial.

Since at $x\to\infty$ $\Psi\to 0$ and $e^{\Delta}\to 1$, then this polynomial is equal to -1 :

$$\Psi(z) = e^{\Delta(z)} - 1 \tag{25}$$

It follows from (23), (25) and (20) that

$$f(\nu) = \Lambda\, e^{\rho(\nu) + i\delta(\nu)} \tag{26}$$

is the solution of equation (15). This solution is unique, as the general solution of the corresponding homogeneous equation has the form $P\,e^{\rho+i\delta}$, where P is a polynomial (crossing symmetrical); at the infinity it must tend to 0, i.e. $P \equiv 0$.

2) Let now for $\nu\to\infty$

$$\delta(\nu) \to \pi \qquad (\delta(\nu) \to \pi - c\,\nu^{-\alpha}\,,\ \alpha > 0) \tag{27}$$

Denote

$$\Delta(z) = \frac{z + 9/16}{\pi} \int_0^\infty \frac{\delta(\nu)}{\nu + 9/16} \left(\frac{1}{\nu - z} - \frac{1}{\nu + z + 9/8} \right) d\nu \tag{28}$$

This function has the same properties as function (19), but for $z\to\infty$

$$\Delta(z) \to const - 2\ln(z + 9/16) \tag{29}$$

$$e^{\Delta(z)} \to \frac{const}{(z + 9/16)^2} \tag{30}$$

Like in the previous case, it is easy to show that

$$f^{(1)}(\nu) = \Lambda\, \frac{1}{A} (\nu + 9/16)^2\, e^{\rho+i\delta} \tag{31}$$

where

$$\rho(\nu) = \frac{\nu + 9/16}{\pi}\, P \int_0^\infty \frac{\delta(x)}{x + 9/16} \left(\frac{1}{x-\nu} - \frac{1}{x + \nu + 9/8} \right) dx \tag{32}$$

$$A = \left[\nu^2\, e^{\rho+i\delta}\right]_{\nu=\infty} \tag{33}$$

is the solution of equation (15). However, this solution is not unique, since we can add the general solution of a homogeneous equation which in this case is equal to

$$C\, e^{\rho+i\delta} \tag{34}$$

where C is an arbitrary constant (C cannot be a polynomial of the first degree according to the requirement of crossing - symmetry).

Thus, all the solutions of equation (15) in the case under consideration are of the form

$$f^{(1)}(\nu) + C e^{\rho+i\delta}. \tag{35}$$

All of them tend to Λ as $1/\nu^2$ at the infinity, since for $\nu \to \infty$

$$\begin{aligned} f^{(1)}(\nu) &= \Lambda\left(1 + \frac{\alpha}{\nu^2}\left(1 - \frac{9}{8\nu}\right) + \frac{\beta}{\nu^4} + \cdots\right), \\ e^{\rho+i\delta} &= \frac{\alpha_0}{\nu^2}\left(1 - \frac{9}{8\nu}\right) + \frac{\beta_0}{\nu^4} + \cdots \end{aligned} \tag{36}$$

And only the unique solution for which

$$C\alpha_0 = -\Lambda\alpha \tag{37}$$

is tending to Λ as $1/\nu^4$ at the infinity. We shall consider as physical that solution of equation (15) which at the infinity is tending to an inhomogeneous term more quickly than any solution of the corresponding homogeneous equation is tending to zero.

From (31), (35)-(37) we obtain the expression for this solution

$$f(\nu) = \Lambda\,\frac{1}{A}\left[(\nu + 9/16)^2 - A_1\right] e^{\rho+i\delta} \tag{38}$$

where A is given by formula (33) and

$$A_1 = \left[\nu^2\left(\frac{1}{A}(\nu + 9/16)^2 e^{\rho+i\delta} - 1\right)\right]_{\nu=\infty}. \tag{39}$$

At high energies the contribution of the dispersion integral to this solution is vanishing quickly. Therefore, among all solutions (35) it is least of all sensitive to the behaviour of the phase-shift δ at the infinity. Only this solution gives a physical correct result in the case of zero width of the pion resonance. Indeed, at

$$\delta = \begin{cases} 0 & \nu < \nu_R \\ \pi & \nu > \nu_R \end{cases} \tag{40}$$

$$e^{\rho(\nu)} = \left|\frac{(\nu_R + 9/16)^2}{(\nu_R - \nu)(\nu_R + \nu + 9/8)}\right|, \qquad f^{(1)}(\nu) = \Lambda\,\frac{(\nu + 9/16)^2}{(\nu - \nu_R)(\nu + \nu_R + 9/8)} \tag{41}$$

while in (38)

$$f(\nu) = \Lambda. \tag{42}$$

We give the expression for the photoproduction amplitude for two models of the pion resonance

1)
$$\delta = \begin{cases} 0 & \nu < \nu_R - b \ , \quad \nu > \nu_R + b \\ \pi/2b\,(\nu + b - \nu_R) & \nu_R - b < \nu < \nu_R \\ \pi/2b\,(\nu_R + b - \nu) & \nu_R < \nu < \nu_R + b \end{cases} \tag{43}$$

Expression (26) yields

$$f(\nu) = \Lambda e^{i\delta}\varphi(\nu)$$

$$\varphi(\nu) = \left\{ \left| \frac{(\nu - \nu_R)^2}{(\nu + b - \nu_R)(\nu_R + b - \nu)} \right|^{\frac{\nu - \nu_R}{2b}} \left| \frac{\nu_R + b - \nu}{\nu + b - \nu_R} \right|^{\frac{1}{2}} \right\} \left\{ \nu \to -\nu - \frac{9}{8} \right\} \tag{44}$$

2)
$$\delta = \begin{cases} 0 & \nu < \nu_R - b \\ \frac{\pi}{2b}(\nu + b - \nu_R) & \nu_R - b < \nu < \nu_R + b \\ \pi & \nu > \nu_R + b \end{cases} \tag{45}$$

Expression (38) yields

$$f(\nu) = \Lambda e^{i\delta}\varphi(\nu)$$

$$\varphi(\nu) = \left[(\nu_R - \nu)\left(\nu_R + \nu + \frac{9}{8}\right) + \frac{b^2}{3} \right] \left[\left\{ \frac{e}{|\nu_R + b - \nu|} \left| \frac{\nu_R + b - \nu}{\nu_R - b - \nu} \right|^{\frac{\nu + b - \nu_R}{2b}} \right\} \left\{ \nu \to -\nu - \frac{9}{8} \right\} \right] \tag{46}$$

These phase-shifts and solutions are shown in Fig.2 and 3 for the values of the resonance parameters taken from [9]: ν_R =1,5; b =0,4, . In both cases the photoproduction amplitude has a resonance character, and its resonance is somewhat displaced with respect to the pion resonance. In the first model (43) the photoproduction resonance is considerably sharper. In the second model (45) the photoproduction amplitude vanishes at the energy near the pion resonance.

The author is very grateful to G.Bialkowski, A.Urewicz, P.S.Isaev and M.I.Shirokov for useful discussions.

References

1. S.Mandelstam, Phys.Rev. 112, 1344 (1958).
2. G.F.Chew and S.Mandelstam, "Theory of Low-Energy Pion-Pion Interaction", Lawrence Radiation Laboratory Reports , UCRL-8728.
3. Н.Н.Боголюбов, Д.В.Ширков "Введение в теорию квантованных полей."Гостехиздат, 1958 год.
4. A.Pais, Phys.Rev. 112, 624 (1959).
5. M.Gourdin and A.Martin "Photoproduction of pions on pions", CERN preprint.
6. А.А.Логунов. Диссертация, ОИЯИ, 1959 г.
7. Н.И.Мусхелишвили. Сингулярные интегральные уравнения. Гостехиздат, 1946г.
8. K.Blankenbechler and S.Gartenhaus, Phys.Rev. 116, 1297 (1959)
9. W.R.Frazer and J.R.Fulco, Phys.Rev. 117, 1609 (1960).

Received by Publishing Department
on August 6,1960

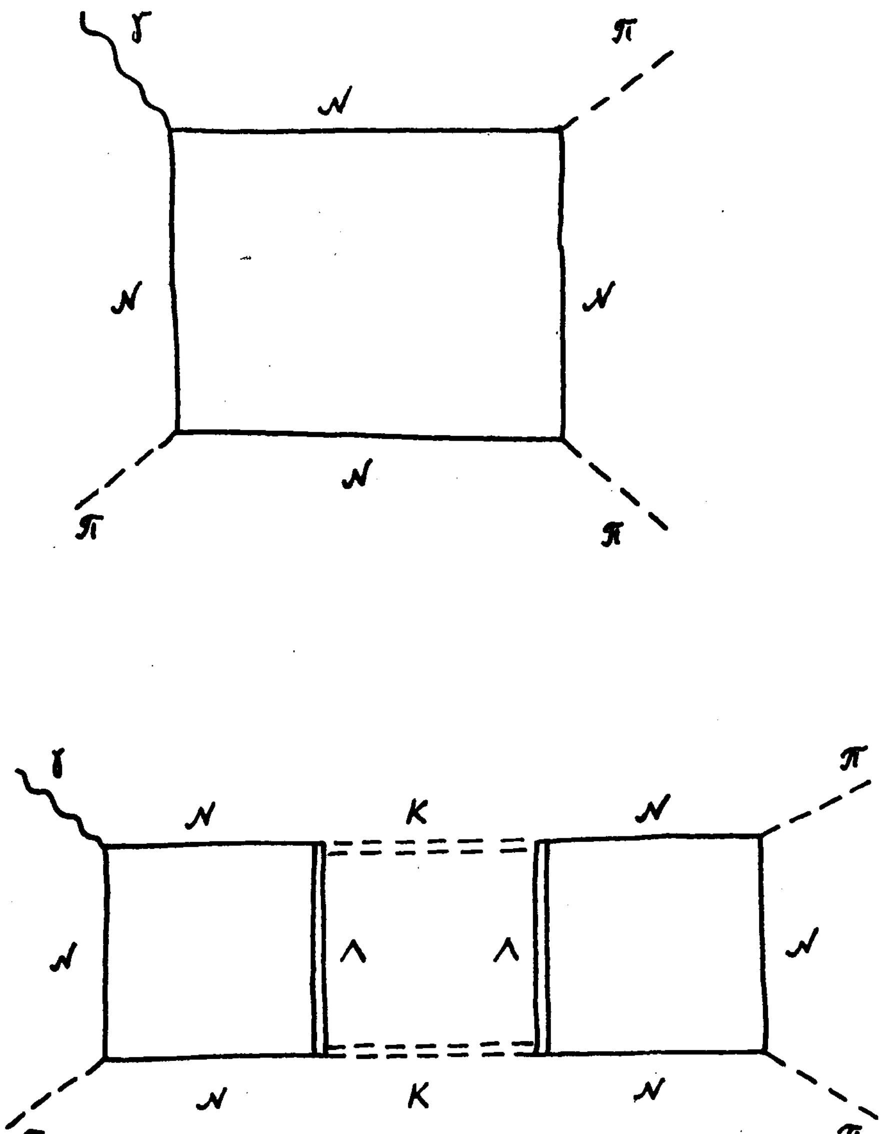

Fig. 1

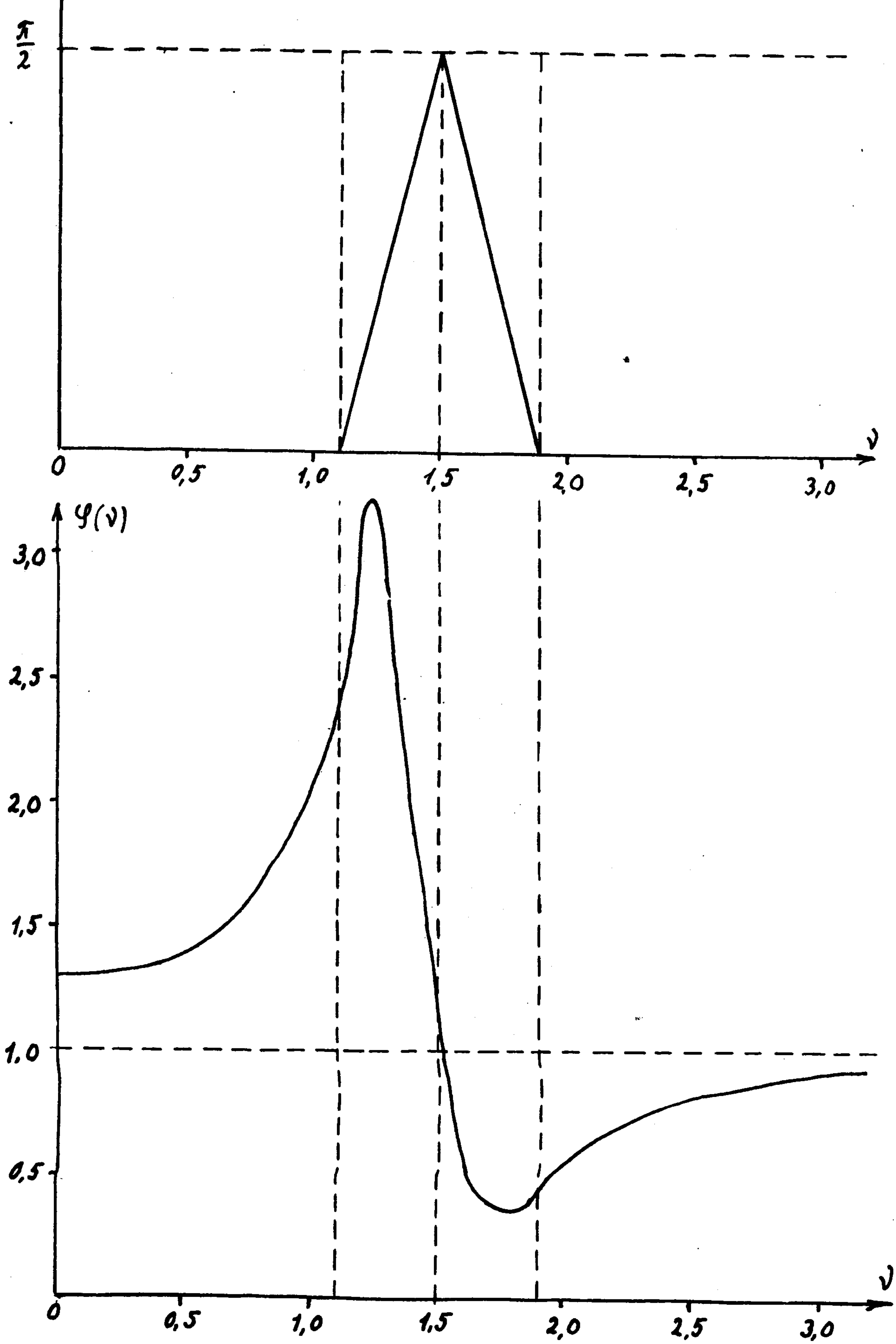

Fig. 2

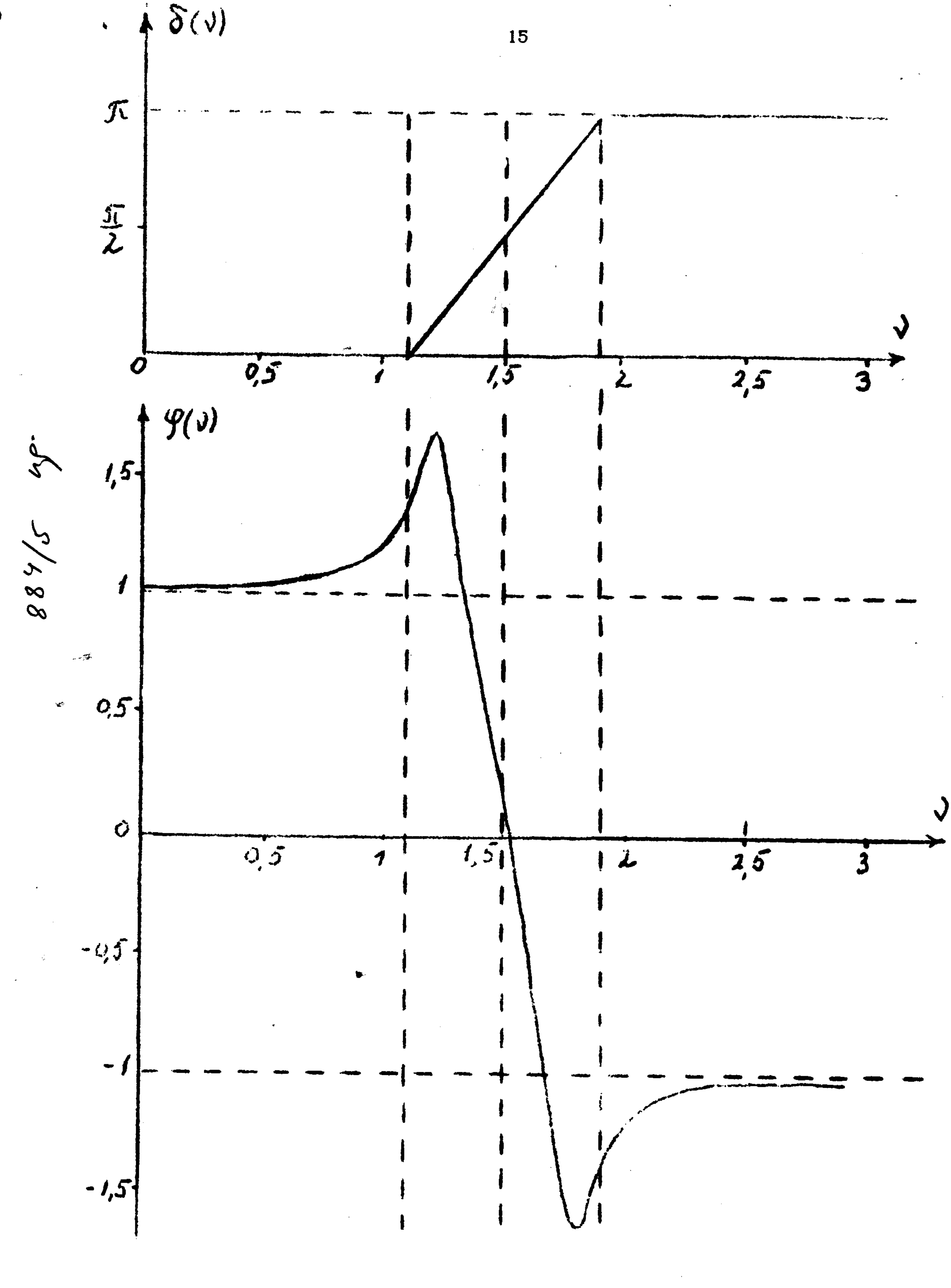

Fig. 3