ОБЪЕДИНЕННЫЙ ИНСТИТУТ ЯДЕРНЫХ ИССЛЕДОВАНИЙ

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PROBLEMS OF MAJORIZATION OF FEYNMAN DIAGRAMS

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Introduction

The investigation of the analytical properties of the vertex part[1] and of the nucleon-nucleon and K-meson-nucleon scattering amplitudes[2,3], has shown that for these processes it is impossible to obtain the dispersion relations starting only from the principles of covariance, causality and spectrality. Since the perturbation theory series reflects the notions of the particle interaction in more detail, a considerable attention has been recently drawn to the study of the analytical properties of the terms of this series. Some progress has been achieved in this direction.[4 -12] It were the papers by Nambu[4] and Symanzik[5] in which such investigations have been started. To establish the dispersion relations an interesting method of majorization of graphs has been suggested in these papers. This method allowed Symanzik[5] to show that the information contained in the terms of the perturbation theory series for the vertex part and for the nucleon-nucleon scattering is sufficient for obtaining the corresponding dispersion relations.

This paper[x)] is devoted to the extension of the majorization method. A means has been worked out for finding the so-called primitive diagrams, such that any diagram is majorated by one of the primitive diagrams. By expressing the

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x) A part of the results of this paper is published.[17]

quadratic form of the diagram in terms of the incidintness matrix a generalization of Symanzik's theorem is obtained. This permitted to establish that some primitive diagrams majorize the rest ones. The method developed here is applied to the vertex part and to the scattering processes involving nucleons and scalar mesons.

The application of the obtained results to the theory of dispersion relations will be treated in subsequent paper.

Part 1. Primitive Diagrams

In the first part the notion of primitive diagrams will be introduced. The primitive diagrams of any process possess such a property that every graph of this process is majorized by one of the primitive diagrams. Here a method of obtaining the primitive diagrams will be worked out. It will be illustrated on examples of the vertex part and of some scattering processes. For the sake of simplicity it is assumed that three (and only three) lines meet in every vertex of the graph. The number of the primitive diagrams of each of the processes treated is finite. It is likely that this fact is general.

1. Regularized Expression of the Matrix Element in the Perturbation Theory

1. As is well-known, the contribution to the scattering matrix from a Feynman graph $\mathcal{D}$ may be written in the

α -representation as

$$T_{\mathcal{D}} = \delta\left(\sum_{i=1}^{n} p_i\right) J_{\mathcal{D}}$$

$$J_{\mathcal{D}}(p_1,\dots,p_n) = \lim_{\varepsilon\to+0} \int_0^\infty\dots\int_0^\infty d\alpha_1\dots d\alpha_\ell \; \mathcal{F}(p_1,\dots,p_n;\alpha_1,\dots,\alpha_\ell)\cdot \tag{1.1}$$

$$\cdot \exp\left\{ i\sum_{a,b=1}^{n-1} A_{ab}(\alpha)p_a p_b - i\sum_{\nu=1}^{\ell}\alpha_\nu m_\nu^2 - \varepsilon\sum_{\nu=1}^{\ell}\alpha_\nu \right\}$$

where p_a is the external momentum entering the vertex a , n is the number of the vertices of the graph, ℓ is the number of internal lines; the function $\mathcal{F}(p_1,\dots,p_n;\alpha_1,\dots,\alpha_\ell)$ is a polynomial in p_1, p_2, ..., p_n and a rational function of α_1 , ... , α_ℓ with poles at the points α_i =0. The matrix $A_{ab}(\alpha)$ is positive definite, if all α_i are positive.

Expression (1.1) is, generally speaking, a divergent integral due to the singularities of the function $\mathcal{F}(p;\alpha)$ at α_i =0 (the ultraviolet catastrophe). Bogolubov and Parasiuk/13/ have found the regularized expression of integral (1.1) to be

$$J_{\mathcal{D}}^{reg}(p_1,\dots,p_n) = \lim_{\varepsilon\to+0} \int_0^1\dots\int_0^1 d\tau_1\dots d\tau_z \int_0^\infty\dots\int_0^\infty d\alpha_1\dots d\alpha_\ell\,\cdot$$

$$\cdot\sum_S \mathcal{F}_S(p;\alpha;\tau)\exp\left\{ i\lambda_S^2(\tau_1,\dots,\tau_z)\sum_{a,b=1}^{n-1} A_{ab}(\alpha)p_a p_b - i\sum_{\nu=1}^{\ell}\alpha_\nu m_\nu^2 - \varepsilon\sum_{\nu=1}^{\ell}\alpha_\nu\right\} \tag{1.2}$$

where

$$0 \le \lambda_s^2(\tau_1,\dots,\tau_z) \le 1, \quad |\mathcal{F}_S(p;\alpha;\tau)| \le \frac{C(p;\alpha;\tau)}{\left[\prod_\nu \alpha_\nu\right]^{\frac{2\ell-1}{2\ell}}}$$

$C(p;\alpha;\tau)$ – is a polynomially bounded expression with respect to the variables p, α, τ.

2. The form

$$Q_{\mathcal{D}}(\alpha,p) = A_{\mathcal{D}}(\alpha,p) - M_{\mathcal{D}}^2(\alpha) \tag{1.3}$$

where
$$A_{\mathcal{D}}(\alpha,p) = \sum_{\alpha,\beta=1}^{n-1} A_{\alpha\beta}(\alpha) p_\alpha p_\beta, \quad M_{\mathcal{D}}^2(\alpha) = \sum_{\nu=1}^{\ell} \alpha_\nu m_\nu^2,$$

entering (1.1), may be obtained in the following way. Let the 4-momenta K_ν on the internal lines of the diagram be chosen so that in each of its vertices the law of momentum conservation is fulfilled. This means that the momenta K_ν satisfy a system of $n-1$ inhomogeneous linear equations. For a connected graph the number of independent "internal" momenta $t_1, \dots, t_f$ equals $f = \ell - n + 1$. The momenta K_ν are linear combinations of the "external" momenta p and of the momenta t. Introduce the function

$$\mathcal{K}_{\mathcal{D}}(\alpha,p,t) = \sum_{\nu=1}^{\ell} \alpha_\nu (K_\nu^2 - m_\nu^2) = \sum_{i,j=1}^{f} a_{ij} t_i t_j - 2\sum_{i=1}^{f} b_i t_i + c. \tag{1.4}$$

The form $Q_{\mathcal{D}}(\alpha,p)$ is obtained from (1.4), if instead of t we substitute the solution of the system of linear equations

$$\frac{1}{2}\frac{\partial \mathcal{K}_{\mathcal{D}}}{\partial t_i} \equiv \sum_{j=1}^{f} a_{ij} t_j - b_i = 0, \quad i = 1, \dots, f. \tag{1.5}$$

If the determinant $|a_{ij}|$ does not vanish, then

$$Q_{\mathcal{D}}(\alpha, \rho) = c - \sum_{ij=1}^{f} a'_{ij} b_i b_j = \frac{\begin{vmatrix} a_{ij} & b_i \\ b_j & c \end{vmatrix}}{|a_{ij}|} \tag{1.6}$$

where a'_{ij} are the matrix elements of $(a_{ij})^{-1}$.

3. The diagram is called connected, if any two of its points may be connected by a continuous path passing through its lines and vertices. Otherwise the diagram is called disconnected. Let a disconnected diagram $\mathcal{D}$ consist of ρ connected graphs $\mathcal{D}_1, \dots, \mathcal{D}_\rho$ of lower processes. Then

$$T_{\mathcal{D}} = T_{\mathcal{D}_1} \cdot T_{\mathcal{D}_2} \cdots T_{\mathcal{D}_\rho} \quad . \tag{1.7}$$

Hence, the study of disconnected graphs in a trivial way reduces to the study of connected ones.

4. A connected diagram is called weakly connected if there exists such an internal line, after whose cut the diagram decomposes into two parts. Otherwise a connected diagram is called strongly connected.

According to the definition, in a weakly connected diagram $\mathcal{D}$ there is an internal line L, after whose cut the diagram decomposes into two parts $\mathcal{D}_1$ and $\mathcal{D}_2$.

The momentum K_L, corresponding to this line, is uniquely determined by the external momenta. The integral $T_{\mathcal{D}}$ of such a diagram is equal to

$$T_{\mathcal{D}} = J_{\mathcal{D}_1} \cdot J_{\mathcal{D}_2} \frac{P(K_L)}{K_L^2 - m_L^2} \delta\left(\sum_{i=1}^{n} p_i\right) \tag{1.8}$$

where $P(K_L)$ is a well-known polynomial with respect to K_L. Thus, the line L gives a pole in $T_{\mathcal{D}}$. The other singularities of the diagram $\mathcal{D}$ are determined by those of the diagrams $\mathcal{D}_1$ and $\mathcal{D}_2$. Therefore, in order to find the branch points it is sufficient to consider the strongly connected diagrams of the given process, as well as the strongly connected diagrams of the corresponding lower processes.

5. We shall solve the following problem. Let in the diagram $\mathcal{D}$ there be such two lines L_1 and L_2, that the diagram does not decompose into two parts after the cut of any of these lines and decomposes into two parts after the cut of both of them (Fig.1). It is required to find $Q_{\mathcal{D}}(\alpha,p)$ in the case when all α except α_1 and α_2, corresponding to the lines L_1 and L_2, vanish.

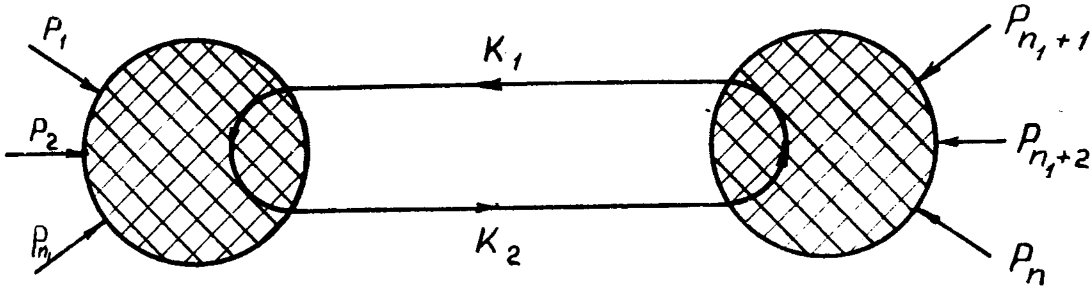

Fig.1

In this case the condition

$$K_2 - K_1 = P_1 + P_2 + \cdots + P_{n_1} \tag{1.9}$$

is imposed on the momenta K_1 and K_2, corresponding to the lines L_1 and L_2. In the diagram $\mathcal{D}$ there exists a closed loop, passing through the lines L_1 and L_2. Let $K_1^\circ, K_2^\circ, \ldots, K_\ell^\circ$ be a set of "internal" momenta of the diagram $\mathcal{D}$ satisfying the momentum conservation in each of its vertices Then K_1° and K_2° satisfy equation (1.9). We orientate the above-mentioned loop and add an arbitrary momentum t to each momentum corresponding to a line from this loop. The remaining momenta of the diagram are left unchanged. This procedure does not affect the momentum conservation in each vertex of the graph. Thus, t is an independent momentum.

The form $\mathcal{K}_{\mathcal{D}}$ is equal to

$$\mathcal{K}_{\mathcal{D}}(\alpha, p, t) = \alpha_1 (\kappa_1^2 - m_1^2) + \alpha_2 (\kappa_2^2 - m_2^2) =$$

$$= (\alpha_1 + \alpha_2) t^2 + 2t (\alpha_1 \kappa_1^\circ + \alpha_2 \kappa_2^\circ) + \sum_{i=1}^{2} \alpha_i (\kappa_i^2 - m_i^2). \tag{1.10}$$

From the condition

$$\frac{1}{2} \frac{\partial \mathcal{K}_{\mathcal{D}}}{\partial t} = (\alpha_1 + \alpha_2) t + \alpha_1 \kappa_1^\circ + \alpha_2 \kappa_2^\circ = 0 \tag{1.11}$$

follows

$$t = - \frac{\alpha_1 \kappa_1^\circ + \alpha_2 \kappa_2^\circ}{\alpha_1 + \alpha_2} . \tag{1.12}$$

By substituting (1.12) into (1.10), we get

$$Q_{\mathcal{D}}(\alpha, p) = \frac{(p_1 + \dots + p_{n_1})^2 \alpha_1 \alpha_2}{\alpha_1 + \alpha_2} - \alpha_1 m_1^2 - \alpha_2 m_2^2 . \tag{1.13}$$

6. If the momenta are such that their scalar products are real and the inequality

$$Q_{\mathcal{D}}(\alpha, p) = \sum_{a,b=1}^{n-1} A_{ab}(\alpha) p_a p_b - \sum_{v=1}^{\ell} \alpha_v m_v^2 < - h \sum_{v=1}^{\ell} \alpha_v \tag{1.14}$$

holds then (1.14) may be reduced to the form

$$J_{\mathcal{D}}^{reg}(p_1, \dots, p_n) = \int_0^1 \cdots \int_0^1 d\tau_1 \dots d\tau_z \int_0^\infty \cdots \int_0^\infty d\beta_1 \cdots d\beta_\ell \cdot \tag{1.15}$$

$$\cdot \sum_s \mathcal{F}_s (p; -i\partial; \tau) \exp\left\{ \lambda_s^2(\tau_1, \dots, \tau_z) \sum_{a,b=1}^{n-1} A_{ab}(\beta) p_a p_b - \sum_{v=1}^{\ell} \beta_v m_v^2 \right\}$$

Let us denote by $G_h(\mathcal{D})$ the domain of the values of the set of external momenta with the real scalar products determined by the condition (1.14). The sum of the regions $\bigcup_{h>0} G_h(\mathcal{D})$ is denoted by $G(\mathcal{D})$. Integral (1.15) is absolutely convergent, ifthe set of external momenta takes the values from the region $G(\mathcal{D})$.

7. For the diagram $\mathcal{D}$ treated in 5, in the region $G(\mathcal{D})$ the inequality

$$p^2 = (p_1 + \cdots + p_{n_1})^2 < (m_1 + m_2)^2. \tag{1.16}$$

is fulfilled. Indeed, in the region $G(\mathcal{D})$ for arbitrary non-negative values α such that $\sum_{\nu=1}^{\ell} \alpha_\nu > 0$ the form $Q_{\mathcal{D}} < 0$. Therefore, in this region expression (1.13) must be also negative. In order that expression (1.13) would be negative for all non-negative α_1 and α_2, it is necessary to fulfil inequality (1.16).

2. Majorization of Feynman diagrams

8. To establish the dispersion relations for the scattering amplitude it is not necessary to know the domain $G(\mathcal{D})$ for each graph $\mathcal{D}$, but only the intersection of the regions $G_R = \bigcap_{\mathcal{D}\in R} G(\mathcal{D})$ of the diagrams forming a certain set R. This is because according to the perturbation theory the amplitude of any process is represented by a sum of integrals corresponding to all the various graphs of this

process. The kind of diagrams enterning this sum dependents on the type of interaction

Let R be a set of connected diagrams of a definite process. If, for two graphs $\mathcal{D} \in R$ and $\mathcal{D}' \in R$, it is known that $G(\mathcal{D}) \subseteq G(\mathcal{D}')$ then in finding the intersection of the regions $G_R = \bigcap_{\mathcal{D} \in R} G(\mathcal{D})$ among the two graph $\mathcal{D}$ and $\mathcal{D}'$ it is sufficient to take into account only the graph $\mathcal{D}$. In this case we say that the graph $\mathcal{D}$ majorizes the graph $\mathcal{D}'$ or that the graph $\mathcal{D}'$ is majorized by the graph $\mathcal{D}$.

It follows from (1.14) that if

$$\max_{\alpha} \frac{A_{\mathcal{D}}(\alpha, P)}{M^2_{\mathcal{D}}(\alpha)} \geqslant \max_{\alpha'} \frac{A_{\mathcal{D}'}(\alpha', P)}{M^2_{\mathcal{D}'}(\alpha')}, \tag{2.0}$$

then $G(\mathcal{D}) \subseteq G(\mathcal{D}')$, i.e. the graph $\mathcal{D}$ majorized the graph $\mathcal{D}'$. The inverse is also true: if $G(\mathcal{D}) \subseteq G(\mathcal{D}')$ then (2.0) is fulfilled.

9. Let us consider the connected graphs with $\ell_N = 2\lambda$ external nucleon and ℓ_M external π-meson lines, in each vertex of which three and only three lines converge: an even number (2 or 0) of baryon lines and an odd number (1 or 3) of meson-lines. The set of all such graph is designated by $\mathcal{R}$.

In each diagram $\mathcal{D} \in \mathcal{R}$ there are λ open polygons formed by lines having a baryon charge. Since the π-meson mass is the smallest in $\mathcal{D}$, and the nucleon mass is the smallest among the baryon masses, then in replacing all the lines of the above-mentioned open polygons by the nucleon ones, and the rest lines of the graph - by π -meson ones, $M^2_{\mathcal{D}}$ decreases, and, hence, $Q_{\mathcal{D}}$ increases.

Thus, the graph $\mathcal{D}^*$ obtained after the substitution of the masses majorizes the initial graph $\mathcal{D}$. Therefore, $G_{\mathcal{R}} = G_{\mathcal{R}^*}$ where $\mathcal{R}^*$ is the sub set of graph $\mathcal{R}^* \subset \mathcal{R}$, in which there are λ nucleon open polygons, whereas all the rest lines are π -meson ones.

10. Further we will not treat the general case of real s scalar products of the external momenta p_α , but restrict ourselves to the particular case[x)] , when

$$p^2 = \Big(\sum_a A_a p_a\Big)^2 \geqslant 0 \tag{2.1}$$

for any real A_α . The space of all the vectors of the form $p = \sum_a A_a p_a$ is denoted by P . In case (2.1) from (1.4), (1.5), and (2.1) follows immediately

<u>LEMMA 1</u>. The form $Q_{\mathcal{D}}$ is equal to the smallest value of the form $\mathcal{K}_{\mathcal{D}}$ on condition that the vectors $\kappa_v \in P$ satisfy the law of momentum conservation in each vertex of the graph[/9/].

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x) In[/4,5,7-11/] the vectors satisfying condition (2.1) are called Euclidean. Note that this definition of the Euclidean space coincides with that generally adopted, if only the zero vectors has a zero length. The space of vectors satisfying only condition (2.1) is usually called semi-Euclidean.

Note that the statement of the basic theorem of paper /7/ follows immediately from this lemma.

11. Considered as an example the following graph (see Fig.2).

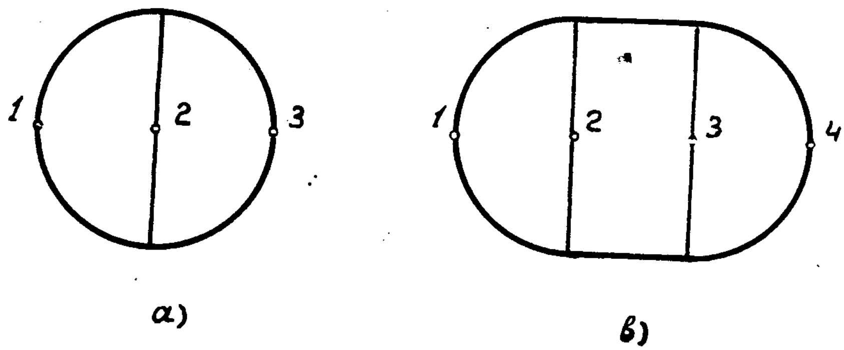

Fig.2

Denote by p_i the external momentum outgoing from the vertex i (the sum of all external momenta is equal to zero). Put the momenta on the internal lines meeting in the vertex i, equal to $\frac{1}{2} p_i$. Then

$$4\mathcal{K}_{\mathcal{D}_1} = (\alpha_1+\alpha_1')(p_1^2-4m^2) + (\alpha_2+\alpha_2')(p_2^2-4m^2) + (\alpha_3+\alpha_3')(p_3^2-4m^2) \tag{2.2a}$$

for the first graph and

$$4\mathcal{K}_{\mathcal{D}_2} = (\alpha_1+\alpha_1')(p_1^2-4m^2) + (\alpha_2+\alpha_2')(p_2^2-4m^2) + (\alpha_3+\alpha_3')(p_3^2-4m^2) + \tag{2.2b}$$

$$+ (\alpha_4+\alpha_4')(p_4^2-4m^2) + (\alpha_5+\alpha_5')\left[(p_1+p_2)^2-4m^2\right]$$

for the second graph.

If

$$p_i^2 < 4m^2 \quad , \quad i = 1,2,3 \tag{2.3a}$$

then expression (2.2a) is negative, and hence, by the lemma the corresponding $Q_{\mathcal{D}_1}$ is also less than zero.

By analogy, if

$$p_i^2 < 4m^2 , \quad i = 1,2,3,4 , \quad (p_1 + p_2)^2 < 4m^2 \tag{2.3b}$$

then expression (2.2b) is negative, and, hence, by the lemma, the corresponding $Q_{\mathcal{D}_2}$ is also less than zero. In 7 it was proved that in the region $G(\mathcal{D}_1)$ the conditions (2.3a), are fulfilled, and in the region $G(\mathcal{D}_2)$ the conditions (2.3b) are fulfilled. Thus, by condition (2.1), the region $G(\mathcal{D}_1)$ is determined by the inequalities (2.3a), whereas the region $G(\mathcal{D}_2)$ is determined by the inequalities (2.3b).

12. This lemma allows to prove the following two theorems which play the main part in the majorization of graphs.

In order to formulate the first theorem, we determine the notion of subdiagram . If as a result of removing from a diagram $\mathcal{D} \in R$ some internal lines and internal vertices[x)] a graph $\mathcal{D}' \in R$ is obtained, then we shall call

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x) The vertex is called external, if an external line enters. it. Otherwise it is called internal.

the graph $\mathcal{D}'$ subdiagram of the graph $\mathcal{D}$ (with respect to R). This definition needs a further explanation.

If in a certain vertex α of the graph $\mathcal{D}$ ℓ_α lines meet then in removing all the ℓ_α lines it is necessary to remove the vertex α inself. The vertex α may be removed also in the case when $\ell_\alpha - 2$ lines are removed, whereas the remaining two lines are united into one. In no other case the vertex α is removed.

13. Let us give some example of subdiagrams. Consider the set of graphs $\mathcal{R}^*$.

a) Choose in the graph $\mathcal{D}' \in \mathcal{R}^*$ two lines ab and cd (see Fig.3a).

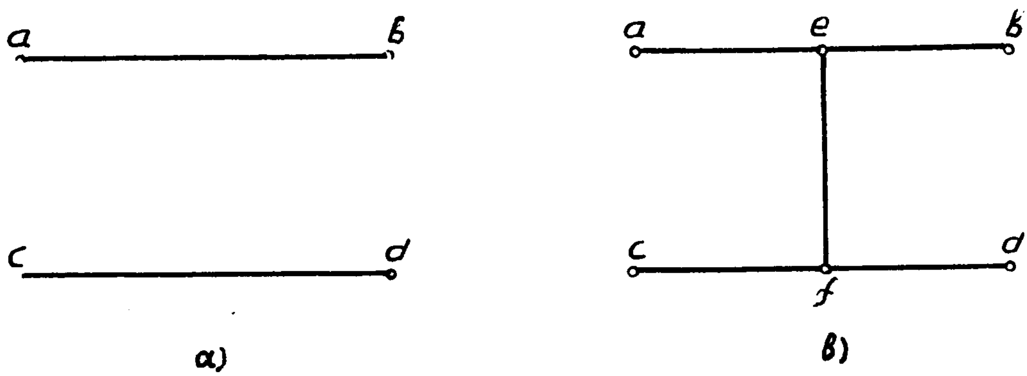


Fig.3

On these lines we add the vertices e and f and connect these vertices by a meson line (see Fig.3b). As a result we obtain a new graph $\mathcal{D} \in \mathcal{R}^*$. The initial diagram $\mathcal{D}'$ is a subdiagram of the graph $\mathcal{D}$.

b) Let in the graph $\mathcal{D} \in \mathcal{R}^*$ between the vertices a and b there be a self-energy part, nucleon or meson (Fig.4). In virtue of the fact that in each vertex of the graph an even number (0 or 2) of nucleon lines converges (that ensures the conservation of the nucleon charge), in the nucleon self-energy part there exists a continuous path ab, connecting the vertices a and b, and passing only along nucleon lines. In the meson self-energy part there exists a continuous way ab, connecting the vertices a and b, and passing only along meson lines. We remove all the lines of the self-energy part along which the path ab does not follow, and all its vertices except a and b. The way ab turns into one line ab nucleon or meson, depending upon the type of the self-energy part.

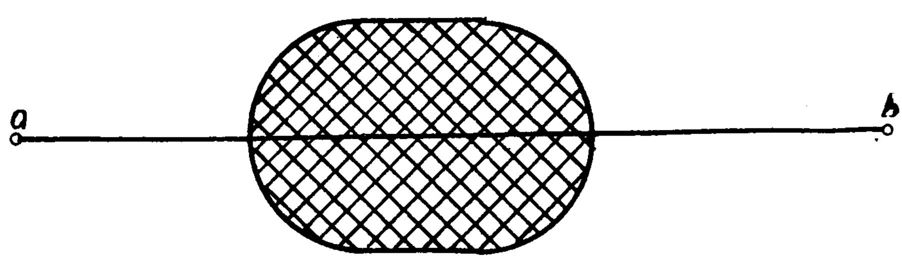


Fig.4

The graph $\mathcal{D}'$ obtained from the graph $\mathcal{D}$ by substituting the self-energy part by the corresponding line, is strongly connected and is a subgraph of the graph $\mathcal{D}$.

c) Any connected graph constructed only of meson lines and vertices is called a star, and its external lines - prongs.

Let in the graph $\mathcal{D} \in \mathcal{R}^*$ there be a certain star Z, all prongs of which are ending on the same polygon at the points S_1, S_2, ..., S_{k-1}, S_k, the points S_2, ..., S_{k-1} lying between the points S_1 and S_2. In view of connectivity in the star Z exists a continuous way L, connecting the points S_1 and S_k. Let us remove all the lines of the star except those along which the way L follows, and all its vertices, as well as the vertices S_2, ..., S_{k-1}. Then the way L turns into a meson line connecting the points S_1 and S_k, and we get a new graph $\mathcal{D}' \in \mathcal{R}^*$. The graph $\mathcal{D}'$ is a subdiagram of the diagram $\mathcal{D}$.

Having applied to each such star the above-described process of substituting the star by the meson line, we get a new subdiagram of the graph $\mathcal{D}$.

d) Consider the graph $\mathcal{D} \in \mathcal{R}^*$ of the form (see Fig.5)

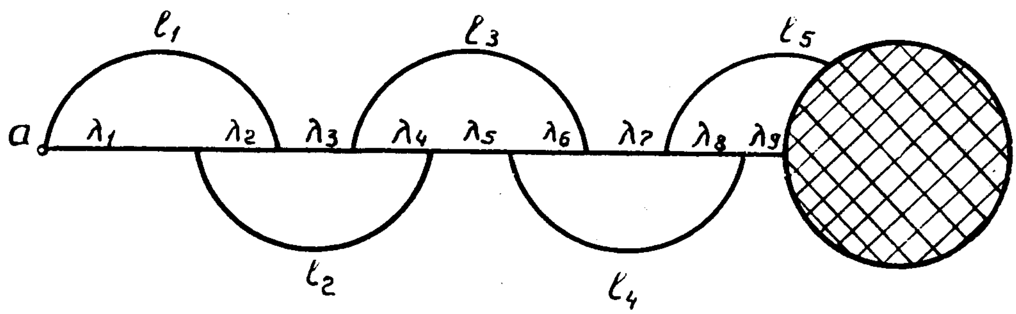


Fig.5

in which the lines λ_1 , λ_2 , ..., $\lambda_{2\nu+1}$ and ℓ_1 , ℓ_2 , ..., $\ell_{\nu+1}$ are meson ones. Having removed the lines λ_2 , λ_4 , ..., $\lambda_{2\nu}$ together with the vertices they connect we obtain the subdiagram (see Fig.6)

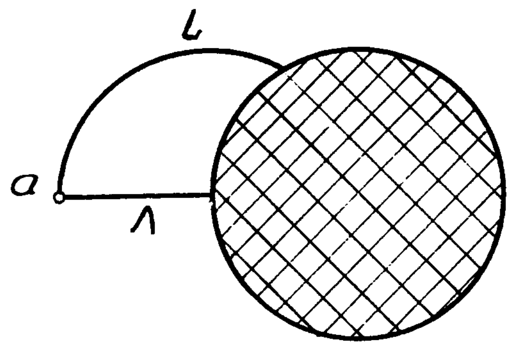


Fig.6

e) Consider the graph $\mathcal{D} \in \mathcal{R}^*$ of the form (see Fig.7)

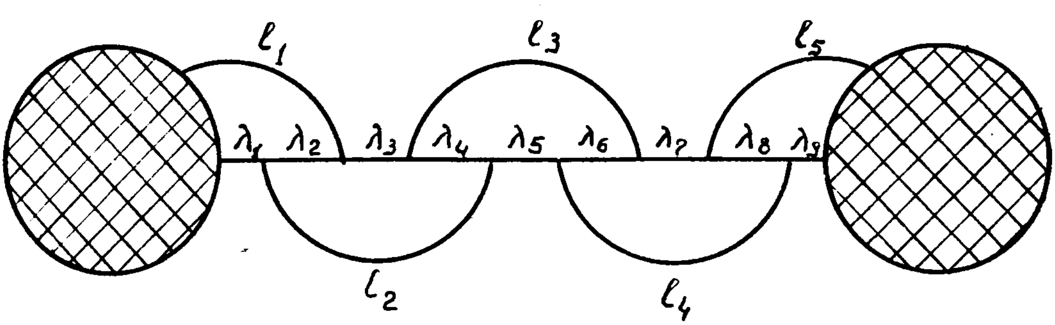


Fig.7

in which the lines λ_1, λ_2, ..., $\lambda_{2\nu+1}$ and ℓ_1, ℓ_2, ..., $\ell_{\nu+1}$ are meson ones. Having removed the lines λ_2, λ_4, ..., $\lambda_{2\nu}$ together with the vertices they connect, we get the subdiagram (see Fig.8)

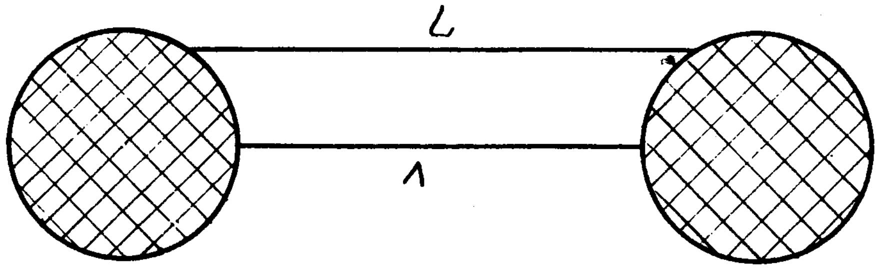


Fig.8

14. THEOREM 1. Each diagram is majorized by any of its subdiagrams.

Proof. Let some lines L_{zs}, $s=1, \ldots, 1+n_z$, $n_z \geqslant 0$ of the graph be united into one line L_z of the subgraph. We have $m_{zs}=m_z$, $s=1, \ldots, 1+n_z$. Let κ_ν be the momenta on the internal lines of the subdiagram satisfying the conservation law in each of its vertices. If on each internal line of the graph subject to a removal, the momentum is put equal to 0 and each line L_{zs} the momentum κ_{zs} is put equal to κ_z corresponding to the line L_z, then the law of momentum conservation will be fulfilled in each vertex of the graph as well.

Let us denote the Feynman parameter of the line L_{zs} by α_{zs}, and the parameters of the removed lines by β_ν.

Then

$$\mathcal{K}_{\mathfrak{D}} = -\sum_\nu \beta_\nu m_\nu^2 + \sum_z \alpha_z (\kappa_z^2 - m_z^2) \tag{2.4}$$

where

$$\alpha_z = \sum_{s=1}^{1+n_z} \alpha_{zs} .$$

Since

$$\mathcal{K}_{\mathfrak{D}'} = \sum_z \alpha_z (\kappa_z^2 - m_z^2) \tag{2.5}$$

then $\mathcal{K}_{\mathfrak{D}} \leq \mathcal{K}_{\mathfrak{D}'}$, and, hence, by lemma 1 $Q_{\mathfrak{D}} \leq Q_{\mathfrak{D}'}$ too. Thus, $G(\mathfrak{D}) \supseteq G(\mathfrak{D}')$ and the theorem is proved.

15. THEOREM 2. Let the graph $\mathfrak{D}$ contain a closed loop with $n+1$ vertices, to n sides of which the mass M corresponds, and to one side- the mass $m \leq M$. Change the masses on these sides in the following way: $M \to m$, $m \to M$. As a result a new graph $\mathfrak{D}'$ is obtained, which majorizes the graph $\mathfrak{D}$.

Proof. Let $\kappa_1, \dots, \kappa_n, \kappa_{n+1} \in P$ - be the momenta on the sides of the closed loop orientated in a definite way. If an arbitrary momentum t is added to each of these momenta, and the other momenta of the graph are left unchanged, then the law of momentum conservation in each vertex of the graph will not be violated. The form $\mathcal{K}_{\mathfrak{D}'}$ of the graph $\mathfrak{D}'$ is equal to

$$\mathcal{K}_{\mathcal{D}'} = \alpha'_{n+1}\left[(\kappa_{n+1}+t)^2 - M^2\right] + \sum_{i=1}^{n}\alpha'_i\left[(\kappa_i+t)^2 - m^2\right] + \sum_{v}\beta_v(q_v^2 - m_v^2) \tag{2.6}$$

where α'_i are the Feynman parameters of the sides of the loop β_v are the parameters of the remaining lines of the graph. The smallest value of the form $\mathcal{K}_{\mathcal{D}'}$ for $t \in P$ is

$$\overline{\mathcal{K}}_{\mathcal{D}'} = \frac{\sum_{i=2}^{n+1}\sum_{j=1}^{i-1}\alpha'_i\alpha'_j(\kappa_i-\kappa_j)^2}{\alpha'_1+\alpha'_2+\ldots+\alpha'_{n+1}} - m^2(\alpha'_1+\ldots+\alpha'_n) - M^2\alpha'_{n+1} + \tag{2.7}$$

$$+\sum_{v}\beta_v(q_v^2 - m_v^2).$$

If we put

$$\alpha'_i = \frac{\varkappa\alpha_i}{m^2},\quad i=1,2,\ldots,n,\quad \alpha'_{n+1} = \frac{\varkappa\alpha_{n+1}}{M^2},\quad \varkappa = \frac{M^2(\alpha_1+\ldots+\alpha_n)+m^2\alpha_{n+1}}{\alpha_1+\alpha_2+\ldots+\alpha_{n+1}}, \tag{2.8}$$

then

$$\overline{\mathcal{K}}_{\mathcal{D}'} = \overline{\mathcal{K}}_{\mathcal{D}} + \frac{M^2-m^2}{m^2}\,\frac{\sum_{i=2}^{n}\sum_{j=1}^{i-1}\alpha_i\alpha_j(\kappa_i-\kappa_j)^2}{\alpha_1+\alpha_2+\ldots+\alpha_{n+1}}. \tag{2.9}$$

Hence

$$\overline{\mathcal{K}}_{\mathcal{D}} \leq \overline{\mathcal{K}}_{\mathcal{D}'}$$

and, therefore, by lemma 1

$$Q_{\mathcal{D}} \leq Q_{\mathcal{D}'}$$

Thus,

$$G(\mathcal{D}') \subseteq G(\mathcal{D}) .$$

Note, that if the graph $\mathcal{D}$ belongs to a certain set R, then the graph $\mathcal{D}'$, generally speaking, does not belong to R. However, its subdiagram may belong to R. In this oase such a subdiagram majorizes the graph $\mathcal{D}$.

16. Consider the application of theorem 2 to the graph of the following two forms.

a) Let the graph $\mathcal{D} \in \mathcal{R}^*$ be of the form (see Fig.5), the lines $\lambda_1, \dots, \lambda_{2\nu+1}$ are nucleon ones, whereas the lines $\ell_1, \dots, \ell_{\nu+1}$ are meson ones.

Applying the operation of theorem 2 to the triangle with sides $\ell_1, \lambda_1, \lambda_2$, and removing then the line together with the vertices it connects, we a the graph of the same form, with ν replaced by $\nu-1$.
After ν such operations we get a graph (see Fig.6) on which the line L is a meson one, and the line Λ is a nucleon one. This diagram majorizes the initial graph $\mathcal{D}$.

b) Let the graph $\mathcal{D} \in \mathcal{R}^*$ be of the form (see Fig.7), the lines $\lambda_1, \lambda_2, \dots, \lambda_{2\nu+1}$ being nucleon ones, and the lines $\ell_1, \ell_2, \dots, \ell_{\nu+1}$ meson ones.

Applying the operation of theorem 2 to the quadrangle with sides $\ell_2, \lambda_2, \lambda_3, \lambda_4$, and removing then the lines λ_2 and λ_4 together with the vertices they connect, we get a graph of the same form, ν being replaced by $\nu-2$.

If $\nu = 2\mu - 1$ is odd, then after μ such operations we obtain the graph drawn on Fig.8 on which the line L is a meson one, whereas the line Λ is a nucleon one.

If $\nu = 2\mu$ is even, then after μ such operations we obtain the graph drawn on Fig.9.

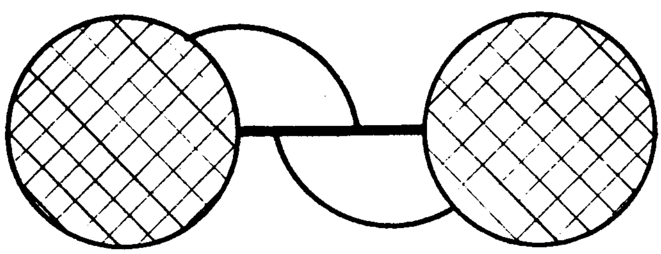


Fig.9

Depending upon the parity of ν, the initial graph is majorized by the graph of Fig.8 or Fig.9.

17. The system of lines λ_1, λ_2, ..., $\lambda_{2\nu+1}$ and ℓ_1, ℓ_2, ..., $\ell_{\nu+1}$ of Fig.5 will be designated by the sign Γ of Fig.10a, and the system of lines of Fig.7 by the sign Π of Fig.10b. If the lines

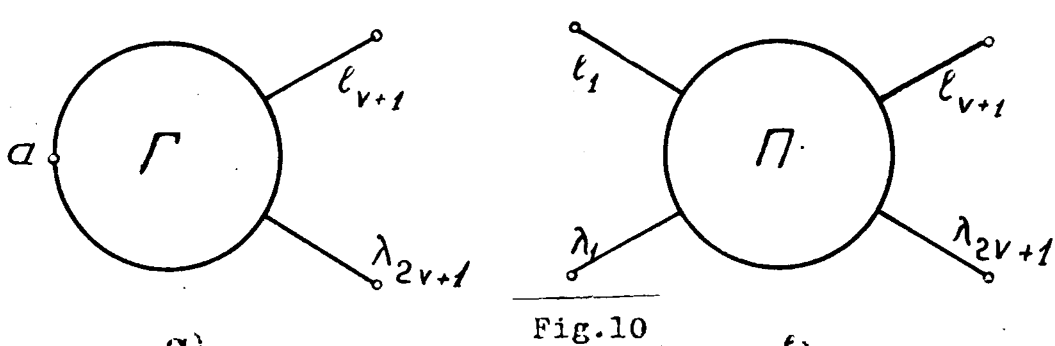


Fig.10

λ and ℓ are meson ones, then the letters Γ and Π will be provided by the index m, if the lines ℓ are meson ones, and λ are nucleon ones, then these letters will be provided by the index n. In according with what has been proved earlier if the graph $\mathcal{D} \in \mathcal{R}^*$ has a part Γ or Π, then it is majorized by the graph $\mathcal{D}'$ which is obtained from $\mathcal{D}$ by the following substitution (see Fig.11).

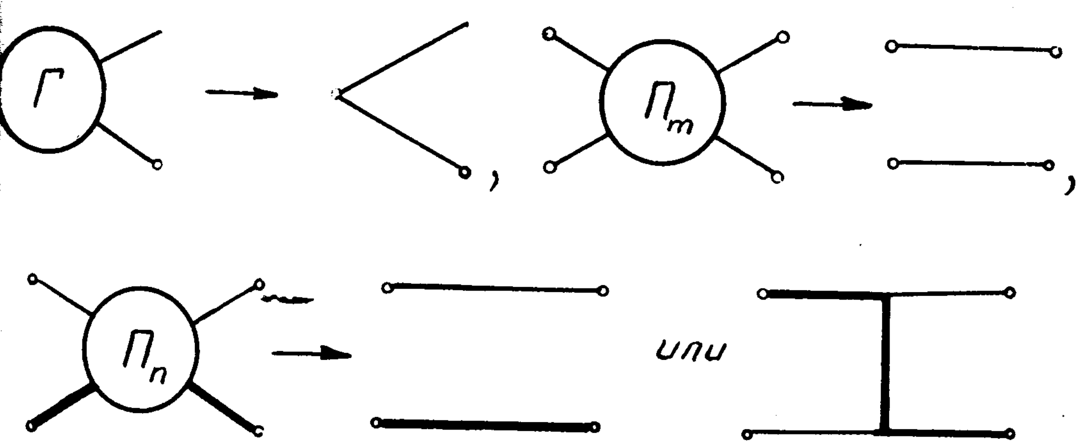


Fig.11

18. We call the diagram $\mathcal{D}$ of $\mathcal{R}^*$ primitive in $\mathcal{R}^*$, if with the aid of theorems 1 and 2 it is impossible to find a majorizing diagram in $\mathcal{R}^*$. For instance, a closed loop with $\ell_M + \ell_N$ sides from whose vertices ℓ_M external meson and ℓ_N external nucleon lines go out is a primitive diagram. The only exception is a closed loop with two external and $\ell_M + 1$ internal nucleon lines when $\ell_M > 0$.

Denote by $\mathcal{R}_0$ the set of all primitive in $\mathcal{R}^*$ diagrams. As each of the graph of $\mathcal{R}^*$ has a finite number of vertices and lines, it is majorized by at least one graph from $\mathcal{R}_0$.

Therefore,

$$G_{\mathcal{R}} = G_{\mathcal{R}^*} = G_{\mathcal{R}_0}$$

In the next paragraphs the set $\mathcal{R}_0$ with $\ell_M + \ell_N \leq 4$ will be found.

3. One-Particle Green Function

19. The nucleon self-energy part ($\ell_M = 0$, $\ell_N = 2$).
Any graph $\mathcal{D} \in \mathcal{R}^*$ of the nucleon self-energy part, according to 13c, has a subgraph which consists of a nucleon open polygon $\mathcal{P}$ and some meson lines joined to it. The open polygon $\mathcal{P}$ connects the external vertices a and b of the graph $\mathcal{D}$.

Let the meson line going out from the vertex a come back to the open polygon $\mathcal{P}$ at the point a'. If a' does not coincide with b, then among the meson lines which emanate from the vertices lying on the segment aa' of the open polygon $\mathcal{P}$ we choose that one which comes to the polygon $\mathcal{P}$ into the farthest from a point of $\mathcal{P}$. Denote this point by a_1', the other end of this line - by a_1. If a_1' does not coincide with b, then among meson lines, which emanate from the vertices lying on the segment $a'a_1'$, we choose that one which comes to the open polygon $\mathcal{P}$ into the farthest from a point of the polygon. The ends of this line are designated by a_2 and a_2'. Repeating this process until the point a_ν' coincides with b, we shall single out the system of lines drawn on Fig.12.

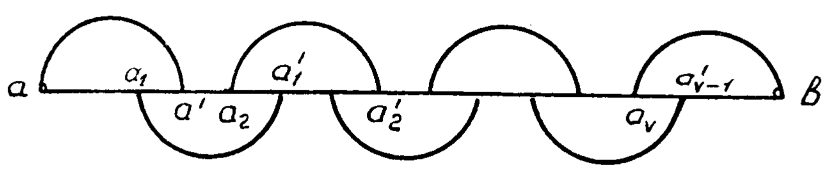
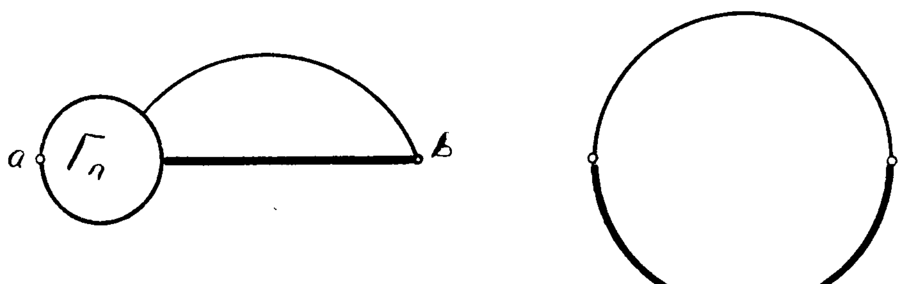


Fig.12

The remaining meson lines are removed together with the vertices they connect. As a result we get the subdiagram of the graph $\mathcal{D}$ drawn on Fig.13a.



a) b)

Fig.13

According to 17 this subgraph is majorized by the graph of Fig.13b. Thus, any graph $\mathcal{D} \in \mathcal{R}^*$ is majorized by the primitive diagram of Fig.13b.

20. The meson self-energy part ($\ell_M =2$, $\ell_N =0$). Since $\ell_N =0$, all the lines of any graph $\mathcal{D} \in \mathcal{R}^*$ of the meson self-energy part are meson ones. In view of the connectivity in the graph $\mathcal{D}$ exists a continuous path L connecting the external vertices a and b . All the stars whose

prongs are ending on the path L are replaced by meson lines connecting their extreme vertices (see 13c). Repeating further the considerations of 19, we are led to the subdiagram of Fig.14a

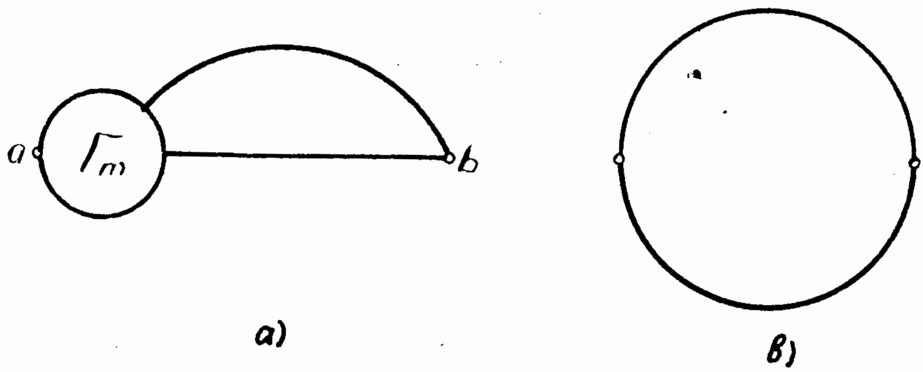


Fig.14

According to 17 this subdiagram is majorized by the primitive diagram of Fig.14b. Thus, any graph $\mathcal{D} \in \mathcal{R}^*$ is majorized by the diagram of Fig.14b.

4. Meson-Nucleon Vertex Part ($\ell_M = 1$, $\ell_N = 2$).

21. Any graph $\mathcal{D} \in \mathcal{R}^*$ of the meson-nucleon vertex part has one external meson vertex c and one nucleon open polygon connecting the external vertices a and b .

a) Let the vertex c lie on the nucleon open polygon ab . Any meson star with all its prongs ending on the nucleon open polygon ab is replaced by a meson line connecting its extreme vertices on this polygon. As a result we obtain a subdiagram in which every meson line connects two points of the nucleon open polygon. In an analogous manner one can show that in such a graph there is at least one of the following two subgraphs (see Fig.15).

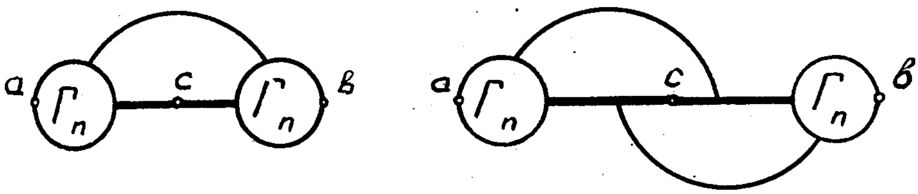


Fig.15

According to 17 these subgraphs are majorized by the graph of Fig,16.

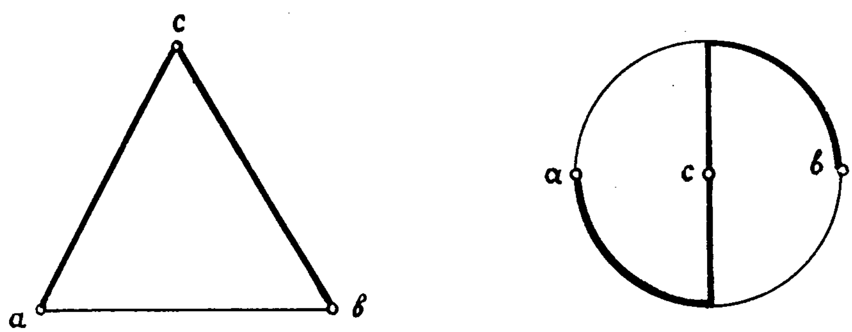


Fig.16

Applying to graphs of Fig.16 the operation of theorem 2, we get the following primitive diagrams (see Fig.17).

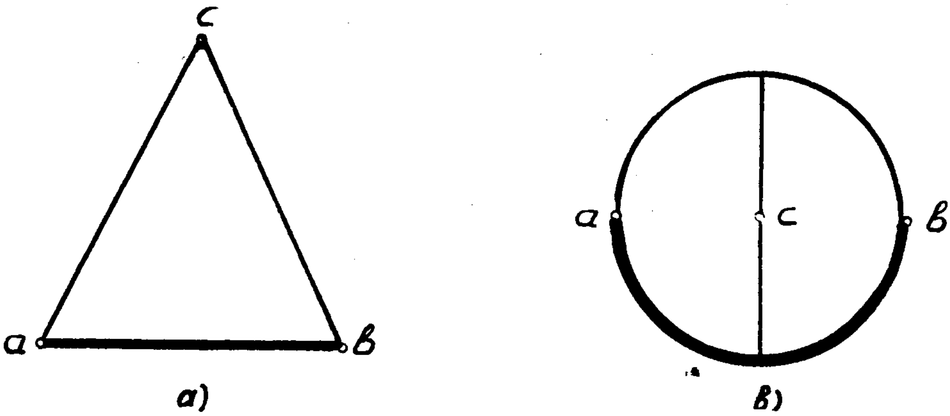


Fig.17

Thus, if in the graph $\mathcal{D}\in\mathcal{R}^*$ the vertex c lies on the nucleon open polygon, then the graph $\mathcal{D}$ is majorized by one of the primitive diagrams of Fig.17.

Further we prove that any graph $\mathcal{D}\in\mathcal{R}^*$ as also majorized by one of the graphs of Fig.17.

b) Let the vertex c in the graph $\mathcal{D}\in\mathcal{R}^*$ lie outside the nucleon polygon ab. In virtue of the connectivity there exists a meson way from at least one point of the nucleon open polygon ab to the point c. Among all such points the nearest to a is called characteristic point $\tilde{a}$. In view of the strong connectivity $\tilde{a}$ does not coincide with b. Each star connected only with the nucleon open polygon is replaced by a meson line connecting its extreme vertices on this polygon. Thus, we single out in the graph $\mathcal{D}$ a certain subgraph $\mathcal{D}'$. In view of strong connectivity of the graph $\mathcal{D}'$ among all the meson lines going out from the segment $\tilde{a}b$ of the nucleon open polygon there may be found at least one which comes to the segment $a\tilde{a}$ of this polygon. Let us denote this line by l_1, and its ends by a_1

and a_1' , $a_1 \in a\tilde{a}$, $a_1' \in \tilde{a}b$.

Let the point a_1 coincide with the point a (see Fig.18a).

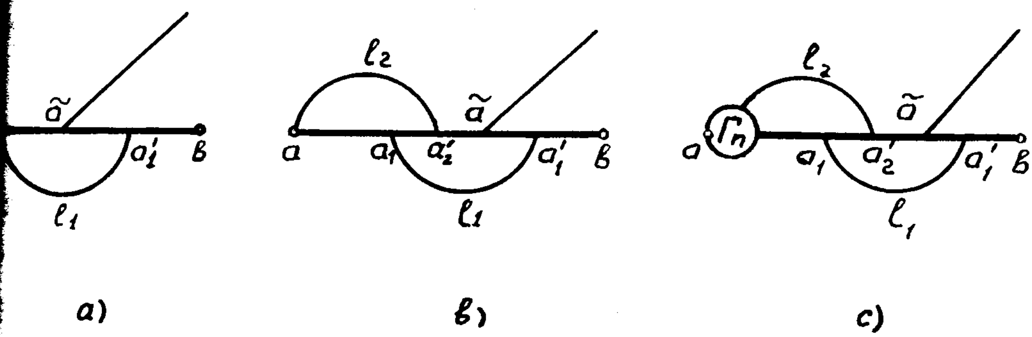


Fig.18

Having replaced the nucleon lines of the segment aa_1' by meson ones, and the meson line ℓ_1 - by a nucleon line, we obtain a graph with a new nucleon open polygon $a\ell_1 a_1' b$, on which the point a will be characteristic. By theorem 2 this graph majorizes the initial graph. Let the points a_1 and a do not coincide. In such a case, in view of strong connectivity among the meson lines going out from the segment $a_1\tilde{a}$ of the open polygon ab there exists at least one which comes to the segment aa_1 of this polygon. Among such lines is chosen that one which comes to the nearest point to a . Designate this line by ℓ_2 , and its ends by a_2 and a_2' , $a_2 \in aa_1$, $a_2' \in a_1\tilde{a}$. Let the point a_2 coincide with the point a (see Fig.18b). Having replaced the nucleon lines of the segment $a_1 a_1'$ by meson ones, and the meson line ℓ_1 by a nucleon one, we get a graph with a new nucleon open polygon $a a_1 \ell_1 a_1' b$, on which the point a will be characteristic. If the point a_2

does not coincide with a , we continue this process until the point a_n coincides with a . All the meson lines except ℓ_1 , ... , ℓ_n , ending on the segment aa_1' of the nucleon open polygon , are removed together with the vertices they connect. As a result we get the subdiagram drawn on Fig.18c. According to 17 it is majorized by the graph of Fig.18b. Therefore, if in the graph $\mathfrak{D} \in \mathcal{R}^*$ the point c does not lie on the nucleon open polygon, then it is majorized by a diagram in which there exists a meson path from the point a to the point c . Thus, there remains to consider the graphs in which there is a continuous path cab (open polygon), the segment ca of which is a meson one, whereas the segment ab is a nucleon one. Each star of such a graph whose prongs are ending on the open polygon cab is replaced by a meson line connecting its extreme vertices on this polygon. As a result we obtain a subgraph in which all meson lines end on the path cab . Now it is not difficult to prove that in such a graph exists at least one of the subdiagrams drawn in Fig.19.

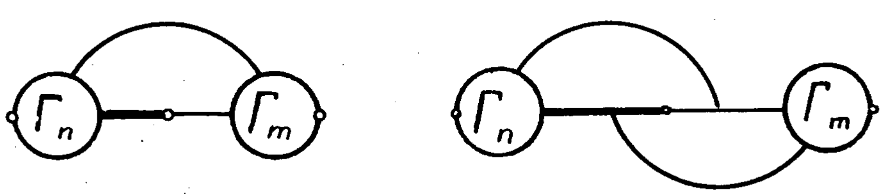
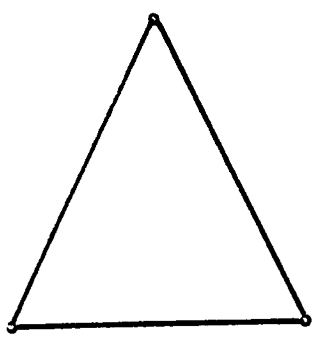


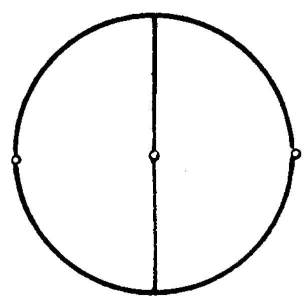
Fig.19

According to 17 these graphs are majorized by the primitive diagrams drawn in Fig.17. Thus, it is established that any graph $\mathcal{D} \in \mathcal{R}^*$ of the meson-nucleon vertex part is majorized by at least one of the two primitive diagrams of Fig.17.

22. One can prove by analogous considerations that in any graph $\mathcal{D} \in \mathcal{R}^*$ of the meson vertex part ($\ell_M = 3$, $\ell_N = 0$) there is at least one of the two subdiagrams (see Fig.20). Both these graphs are primitive diagrams



a)



б)

Fig.20

5. Nucleon-Nucleon Scattering ($\ell_M = 0$, $\ell_N = 4$)

23. Any graph of the class $\mathcal{R}^*$ of the nucleon-nucleon scattering is majorized by at least one of the following primitive diagrams (see Fig.21).

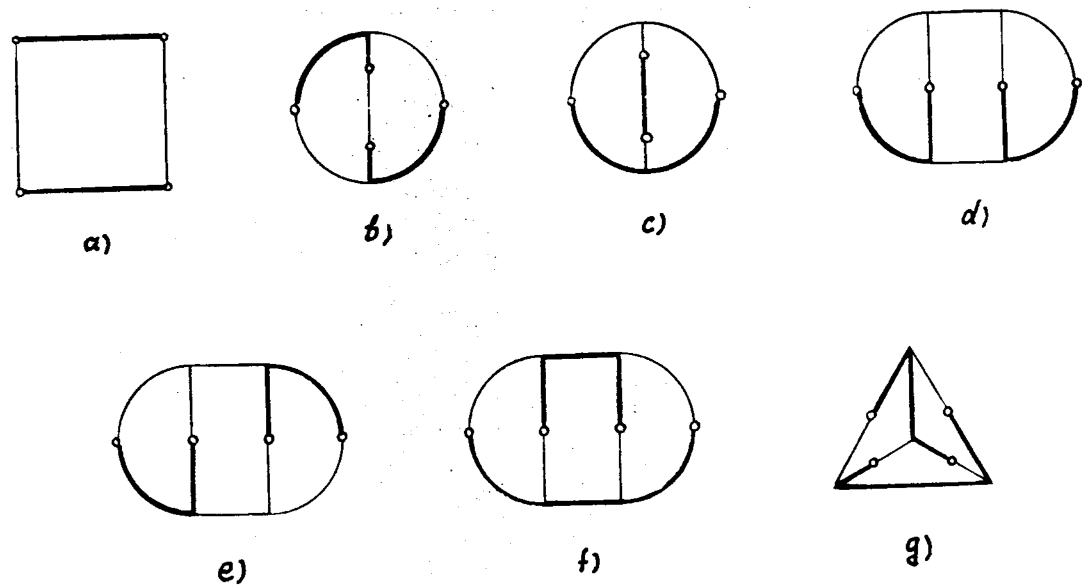


Fig.21

Proof. Let the external vertices a and b of the graph $\mathcal{D}\in\mathcal{R}^*$ lie on the nucleon open polygon ab , and two other external vertices c and d – on the nucleon open polygon cd . It is sufficient to consider the graph in which from the point a there exists a meson path to the polygon cd . The proof of this statement is analogous to 21 b. This meson path may come both to an external vertex, say, to the vertex c , and to any internal point e of the polygon cd (see Fig.22).

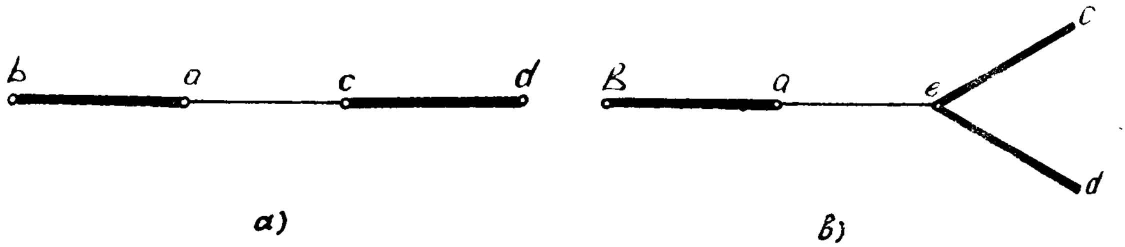


Fig.22

a) Consider the first case. Each star all prongs of which end on the open polygon $bacd$ is replaced by a meson line connecting its extreme vertices on this polygon. As a result we get a subgraph in which each meson line which does not belong to the path ac , connects two points of the open polygon $bacd$. Repeating further the considerations of 19 it is easy to prove that the graph $\mathcal{D}$ has at least one of the graph drawn in Fig.23. According to 17 the graphs A and E are majorized by the graph of Fig.21a, B and F – by the graph of Fig.21c, $\mathcal{D}$ and G – by the graph of Fig.21d; finally, the graph C is majorized by the graph of Fig.21b.

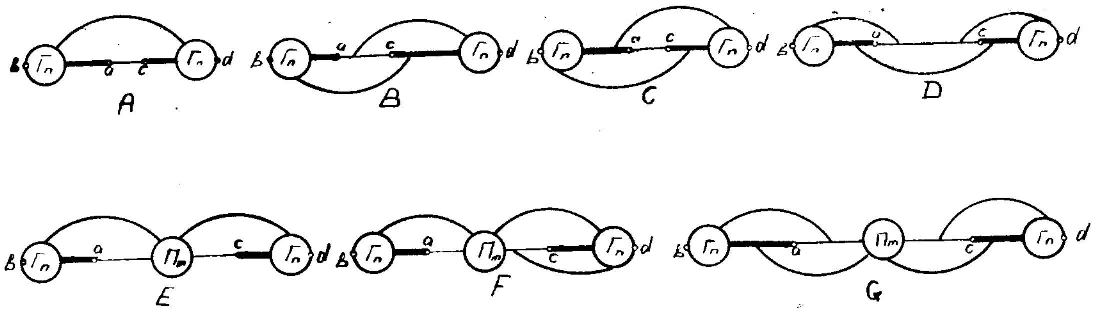


Fig.23

b) We shall call the open polygon of the type drawn in Fig.22a *NMN* polygon. Now we have only to consider such graphs from $\mathcal{R}^*$ in which no *NMN* polygon exists, but does exist the system of lines of Fig.22b which further will be denoted by Y.

Let in the graph $\mathcal{D}\in\mathcal{R}^*$ exist such a system Y. In view of strong connectivity on the open polygon bae at least one point exists from which starts a path to the open polygon ced without common linear sections with the system Y. Among such points the nearest to b is called a characteristic point of the open polygon bae and denoted by $\bar{b}$. The corresponding path from $\bar{b}$ to the open polygon ced is designated by L. Each star connected with the open polygon bae is replaced by a meson line connecting its extreme vertices on this polygon. Repeating further the considerations analogous to 21b, we prove that if $\bar{b}\in ba$

the graph $\mathcal{D}$ is majorized by one of the graphs of Fig.24.

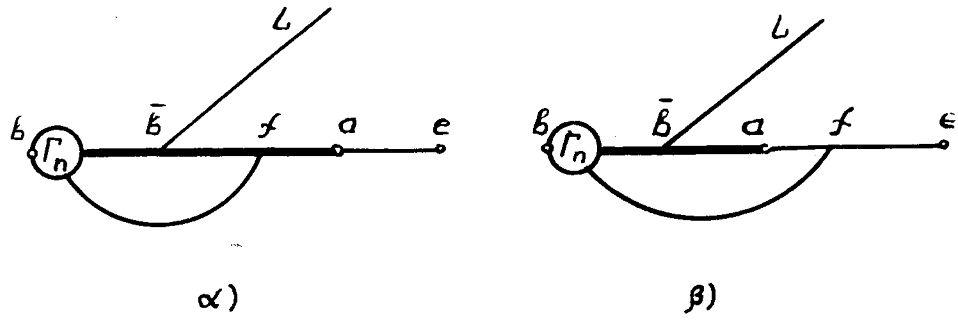


Fig.24

whereas at $\bar{b} \in ae$ - by one of the graphs of Fig.25.

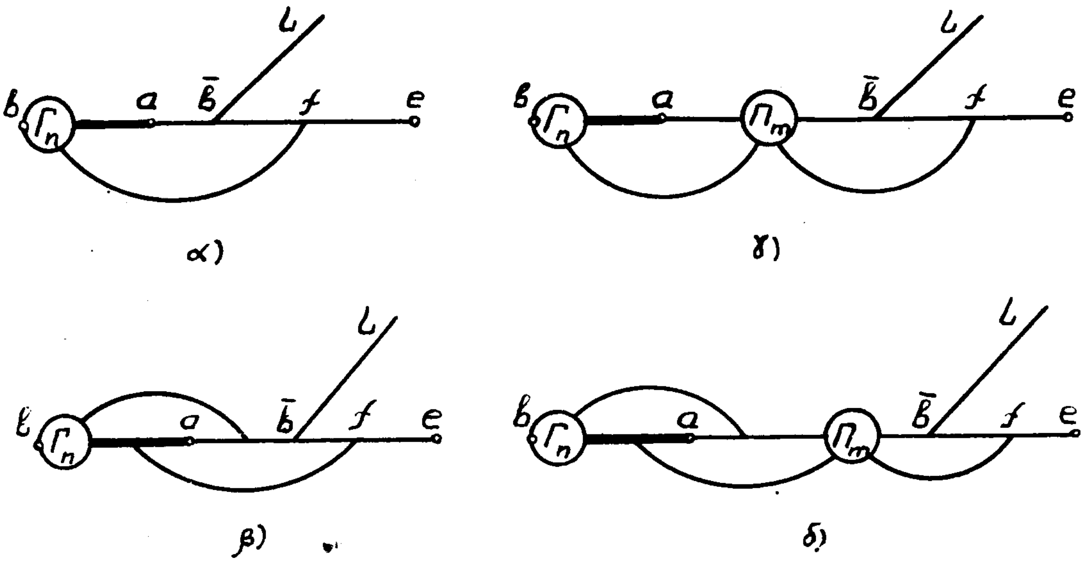


Fig.25

According to 17 the graph of Fig.25 γ) is majorized by the graph of Fig.25 α) , whereas the graph of Fig.25 δ) - by the graph of Fig.25 β) . In accordance with the same section 17 in the graphs α) and β) of Figs.24 and 25 the vertex part Γ is replaced by the vertex b. After this in graph 24α) all the nucleon lines of the segment $b\bar{b}f$ are replaced by meson ones, and the meson line bf – by nucleon one. According to theorem 2 the new graph thus obtained majorizes the graph of Fig.24α). In this new diagram the point b is a characteristic point of the open polygon $bfae$.

As a result, if in the graph $\mathcal{D} \in \mathcal{R}^*$ there is a system Y, then it is majorized by the graph in which there exists at least one of the four systems of lines of Fig.26.

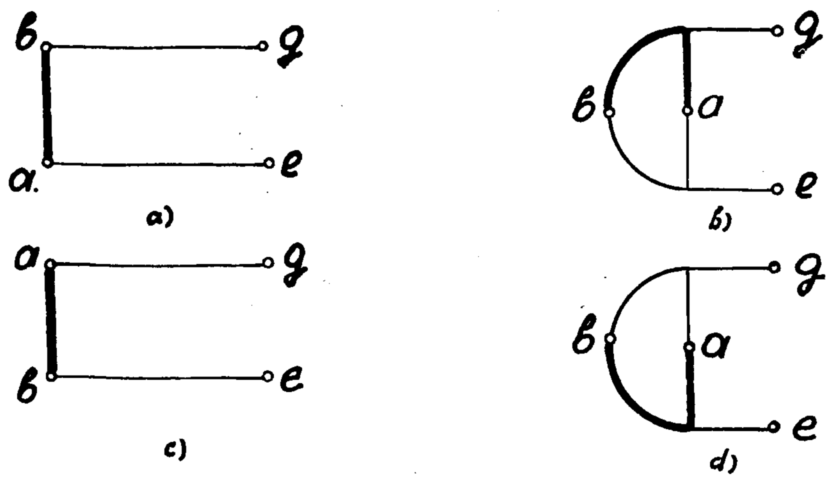


Fig.26

In this figure the point g is the end of the path L. This point may be both an internal point of the nucleon open polygon cd, and its boundary point. In the latter case it is sufficient to consider only the system of Fig.26b, since if in the graph there is a system a), c) or d) of Fig.26, then there would exist also a NMN polygon . The system of Fig.26b together with the nucleon open polygon $c\alpha$ in this case forms the following part of the graph (see Fig.27).

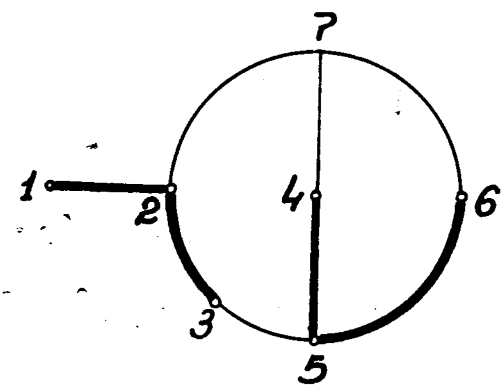


Fig.27

If g is an internal point of the nucleon open polygon $c\alpha$, the systems of Fig.26 together with the nucleon polygon $c\alpha$ form the following two systems (see Fig.28).

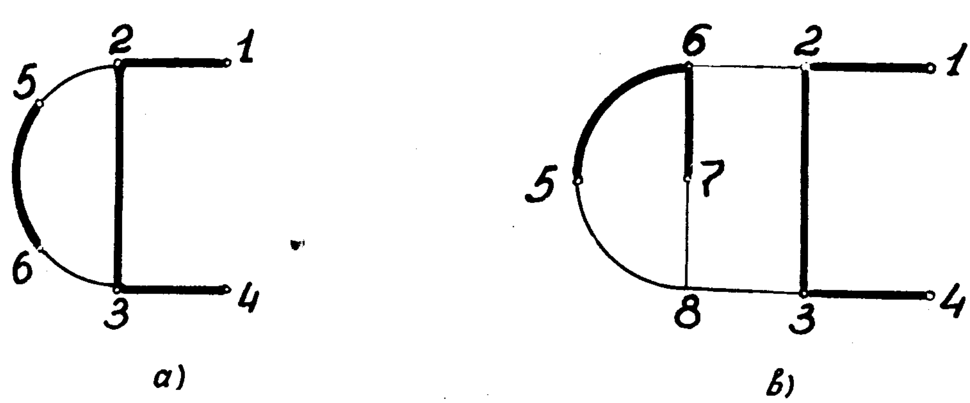


Fig.28

Thus, it is established that if in a graph $\mathcal{D} \in \mathcal{R}^*$ exists a system Y, but there is no NMN polygon, then it is majorized by a graph in which there is at least one of the three systems of Figs.27 and 28. It is these graphs which we have to consider. By analogy to 21b one can prove that it is sufficient to consider graphs containing the system of Figs.27, 28a, or 28b, in which there is a meson path L from the point 1 to the part of this system, complementary to the segment (1,2).

c) Let in the graph $\mathcal{D} \in \mathcal{R}^*$ be no NMN polygon, but exist a system of lines of Fig.27. In the following this system will be denoted by I. In such a graph the path L together with figure I forms one of the following three figures (see Fig.29).

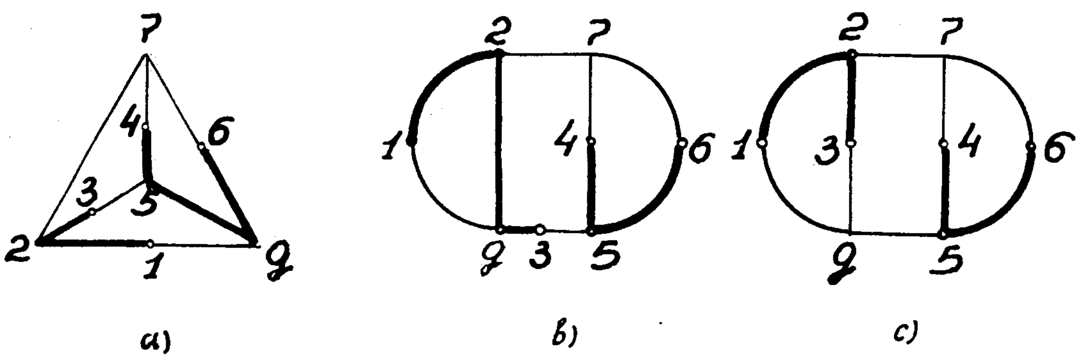


Fig.29

Hence, in the graph $\mathcal{D}$ there is at least one of the three subgraphs drawn in Fig.29. In the graph 29a we remove the line (2,7) together with vertices 2 and 7. As a result we get the diagram drawn in Fig.30. In this subdiagram we

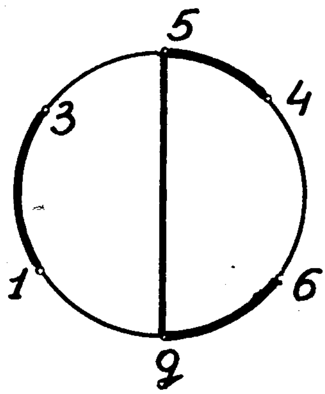


Fig.30

replace the nucleon lines (4,5), (5,g) and (g,6) by meson ones, whereas the meson line (4,6) by a nucleon one. After this we remove the line (5,g) together with the vertices 5 and g. As a result we obtain the primitive diagram of Fig.21a. In the graph of Fig.29b we replace the nucleon lines (1,2) and (2,g) by meson ones, and the meson line (1,g) - by a nucleon one. After this we remove the line (2,g) together with the vertices 2 and g. As a result we get the primitive diagram of Fig. 21c. The diagram of Fig.29c is primitive. It is drawn in Fig.21e.

Thus, in the case under consideration the graph $\mathcal{D}$ is majorized by one of the primitive diagrams a), c), e) of Fig.21.

d) Let in the graph $\mathcal{D} \in \mathcal{R}^*$ exists no NMN polygon, but exists the system of lines of Fig.28a. Further we shall denote this system by II. If the way L from vertex 1 leads to vertex 4, then the graph $\mathcal{D}$ has the subgraph of

Fig.30, and, hence, it is majorized by the primitive diagram of Fig.21a. The path L cannot lead to the segments (2,5) and (3,6), since otherwise there would exist a NMN polygon. If the path L leads to segments (5,6), (2,3) or (3,4), then the figure II together with the path L makes one of the following three, respectively (see Fig.31).

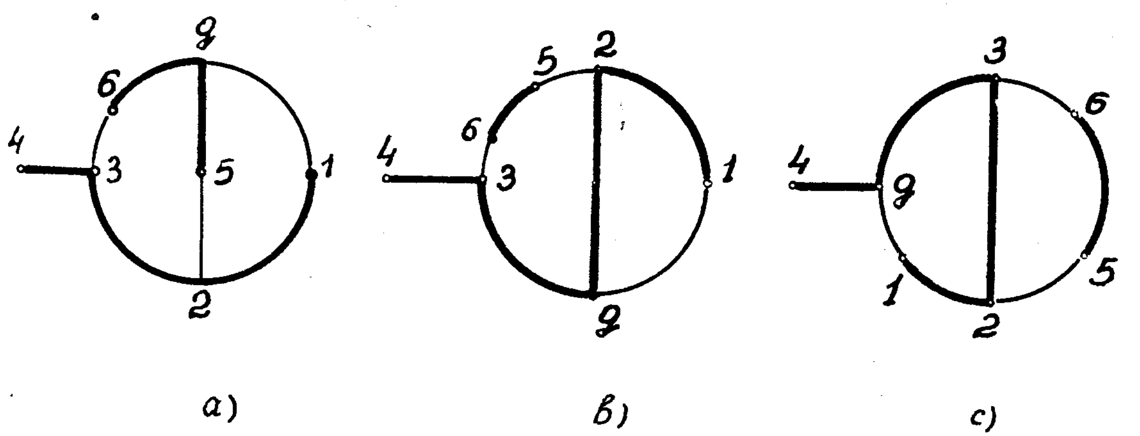


Fig.31

These figures are denoted by IIa, IIb, IIc.

e) Let in the graph $\mathcal{D} \in \mathcal{R}^*$ no NMN polygon exist, but exists the system of Fig.28b. This system will be designated by III. The way L cannot lead from point I to the segments (3,8), (5,8) or (7,8) , since otherwise there would exist a NMN polygon. If this path leads to the segment (2,6), (6,7) or (5,6) , then the graph $\mathcal{D}$ is majorized by at least one of the primitive diagrams a), c), e) of Fig.21, since in these cases there exists figure I in it.

If the way L leads

to vertex 4, then the graph $\mathcal{L}$ has a subgraph drawn in Fig.32. Apply the operation of theorem 2 to the quadrangle

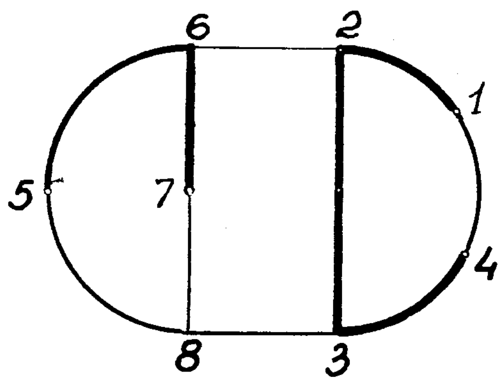


Fig.32

1234 of this subgraph and remove then line (2,3) together with vertices 2 and 3. As a result we obtain the primitive diagram of Fig.21c, majorizing in this case the graph $\mathcal{D}$.

If the path L leads to segment (2,3) or (3,4) , then figure III together with this path forms one of the following two system of lines, respectively (see Fig.33).

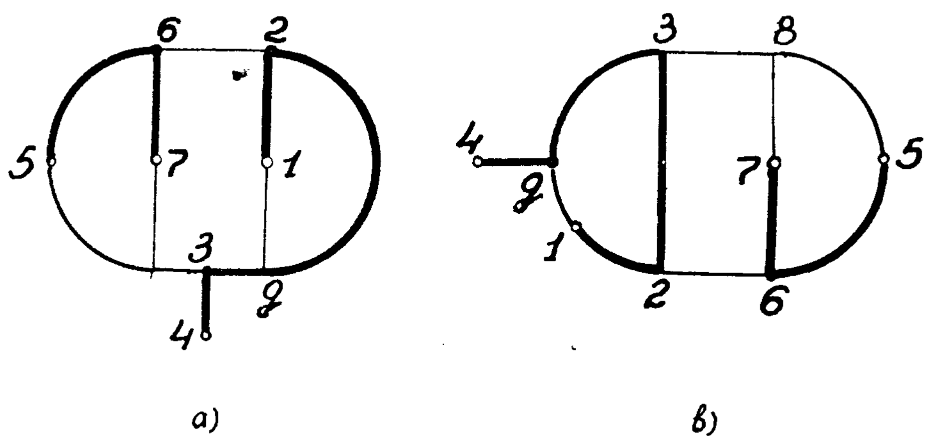
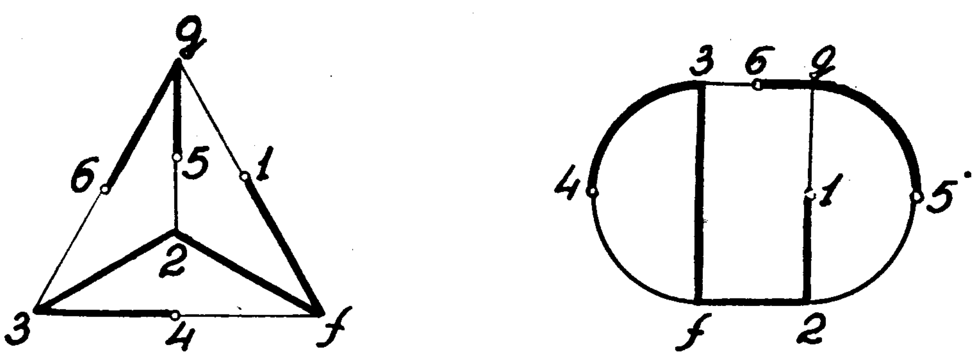


Fig.33

These systems will be denoted by IIIa, IIIb.

So, there remains only to consider the graphs which do not contain a NMN polygon, but contain at least one of the five figures IIa, IIb, IIc, IIIa, IIIb. As before, one can prove that it is sufficient to restrict oneself to the graphs in which there exists a meson path from point 4 to the system complementary to the segment (4,3) (or (4,g)).

f) Let in the graph $\mathcal{D} \in \mathcal{R}^*$ be no NMN polygon, but exists figure IIa. The path L from point 4 cannot lead to segments (3,6) and (2,5) since otherwise there would exist a NMN polygon. If this path leads to the region (g ,1), then the graph $\mathcal{D}$ has subgraph of Fig.30. Hence, it is majorized by the primitive diagram of Fig.21a. If the path L leads to the region (1,2) or to (2,3), then in the graph there is a subgraph of Fig.34a or Fig.34b, respectively.



a) б)

Fig.34

Having applied the operation of theorem 2 to the quadrangle 432f of the graph of Fig.34a and to the triangle 43f of the graph of Fig.34b, we get new graphs, in each of them there is a primitive subdiagram of Fig.21b. The latter one in the given case majorizes the graph $\mathcal{D}$. If the way L comes to the segment ($6, g$), or ($5, g$), then in the graph there is, accordingly, a primitive subgraph f) or g) of Fig.21.

g) Suppose that in the graph $\mathcal{D} \in \mathcal{R}^*$ there is no NMN polygon, but there exists the figure IIb. The path L cannot lead from point 4 to the regions (3,6) and (2,5), since otherwise there would exist a NMN polygon. If the path from point 4 leads to one of segment (5,6), ($3, g$), ($2, g$) or (1,2), then the graph $\mathcal{D}$ has a subgraph, in which the loop ($12g$ I) satisfies the condition of theorem 2. After applying the operation of theorem 2 to this loop the subgraph will turn into the graph containing a NMN polygon. Thus, in this case the graph $\mathcal{D}$ is majorized by the graph treated in 23a. If the path L leads from point 4 to the segment ($1, g$), then the graph $\mathcal{D}$ has a subgraph of Fig.30, and, therefore, it is majorized by the primitive diagram of Fig.21a.

h) Let in the graph $\mathcal{D} \in \mathcal{R}^*$ be no NMN polygon, but exist the figure IIc. If the path L leads from point 4 to any segment of this figure, except ($1, g$), then the graph $\mathcal{D}$ has a subgraph in which the closed loop ($123g$ 1) satisfies the condition of theorem 2. After applying the operation of theorem 2 to this loop this subgraph will turn into a graph containing a NMN polygon. Thus, in this case the graph $\mathcal{D}$

is majorized by the graph treated in 23a. If the path L leads from point 4 to segment $(1, g)$, then the graph has a subgraph of Fig.30 and is majorized by the primitive diagram of Fig.21a.

i) Let in the graph $\mathcal{D} \in \mathcal{R}^*$ be no NMN polygon, but exist figure IIIa. If the path L leads from point 4 to any segment of this figure, except segment $(1, g')$ then such a graph has a subgraph in which the closed loop $(12g\ 1)$ satisfies the condition of theorem 2. After applying the operation of theorem 2 to this loop this subgraph turns into a graph containing figure I. Thus, in this case the graph $\mathcal{D}$ is majorized by the graph treated in 23c. If the path L leads from point 4 to segment $(1, g)$, then the graph $\mathcal{D}$ has a subgraph of Fig.35. Apply to the loop (12341) of this subgraph the operation of theorem 2 and remove then line (2,3) together with vertices 2 and 3. As a result we get the primitive diagram of Fig.21c, which in the given case majorizes the graph $\mathcal{D}$.

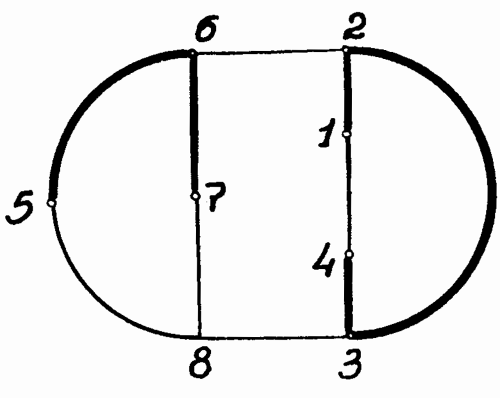


Fig.35

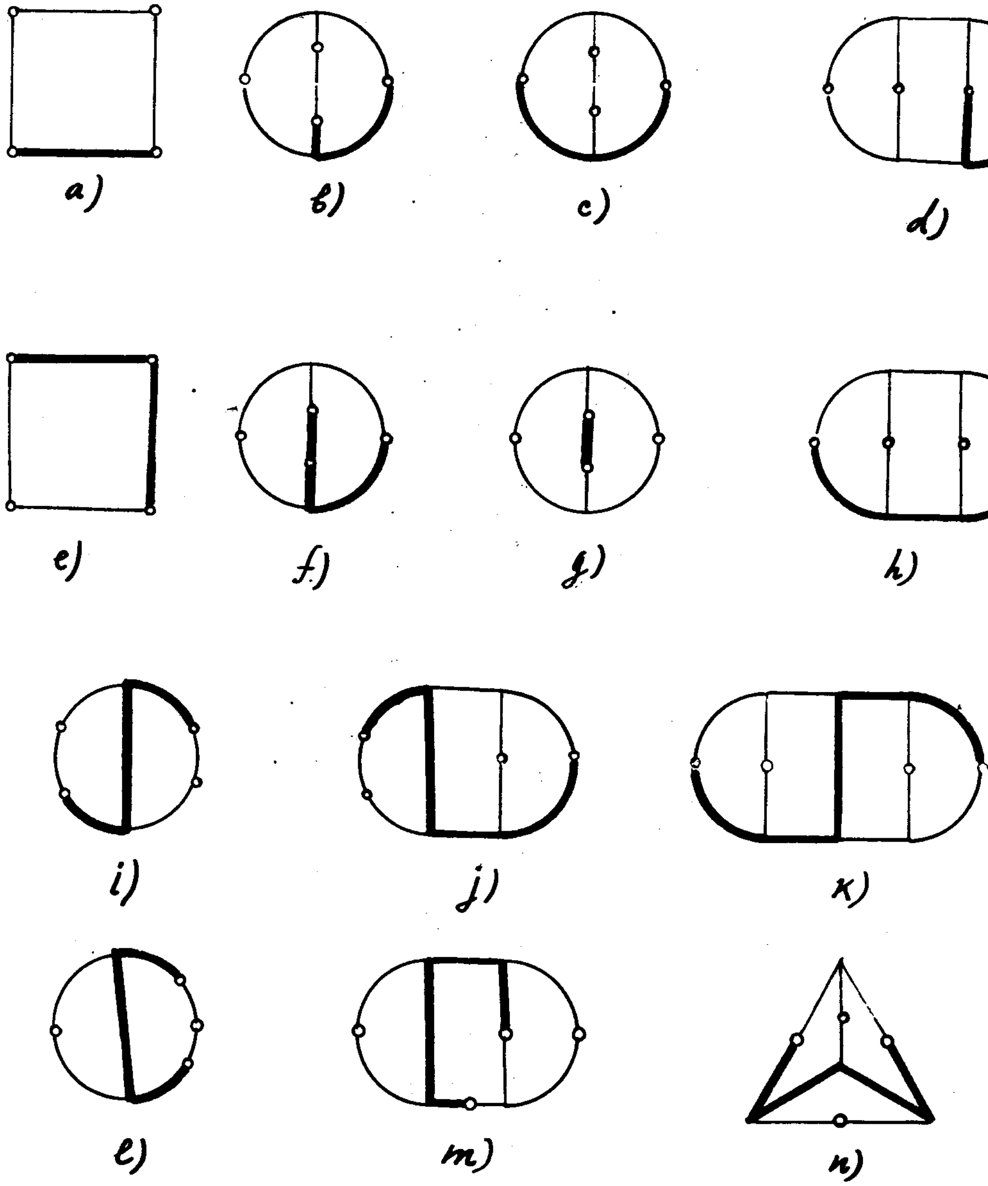


Fig.36

Proof. In the graph $\mathcal{D} \in \mathcal{R}^*$ of meson-nucleon scattering the nucleon lines make up only one nucleon open polygon. Let a and b be the external vertices situated at the ends of this polygon, whereas c and d – the other two external vertices.

j) Let, finally, in the graph $\mathcal{D} \in \mathcal{R}^*$ be no NMN polygon, but system IIIb. If the path L leads from point 4 to any segment of this system, except the segment $(1, g)$, then the graph has a subgraph in which the closed loop $(123g\,1)$ satisfies the condition of theorem 2. After applying the operation of theorem 2 to this loop, this subgraph will turn into the graph in which there is a NMN polygon. Thus, in this case the graph $\mathcal{D}$ is majorized by the graph treated in 23a. If the path L leads from point 4 to the segment $(1, g)$, then in the graph $\mathcal{D}$ there is a subgraph drawn in Fig.35, and, hence, it is majorized by the primitive diagram of Fig.21c.

So, it is proved that any graph from $\mathcal{R}^*$ of the nucleon-nucleon scattering is majorized by at least one of the primitive diagrams of Fig.21.

6. Meson-Nucleon Scattering ($\ell_M = 2$, $\ell_N = 2$)

24. Any graph of the class $\mathcal{R}^*$ of meson-nucleon scattering is majorized by at least one of the following primitive diagrams (see Fig.36).

a) Consider first the graph $\mathcal{D} \in \mathcal{R}^*$, in which the vertices c and d are situated on the nucleon polygon. Each star in such a graph connected only with nucleon open polygon is replaced by the meson line connecting its extreme vertices on this polygon. As a result we get a subgraph in which each meson line ends on the nucleon polygon. Repeating further the considerations of 19, it is easy to prove that the graph $\mathcal{D}$ has at least one of the subgraph drawn in Fig.23. Hence, it follows by 17 that in the case under consideration the graph $\mathcal{D}$ is majorized by at least one of the graphs drawn in Fig.37.

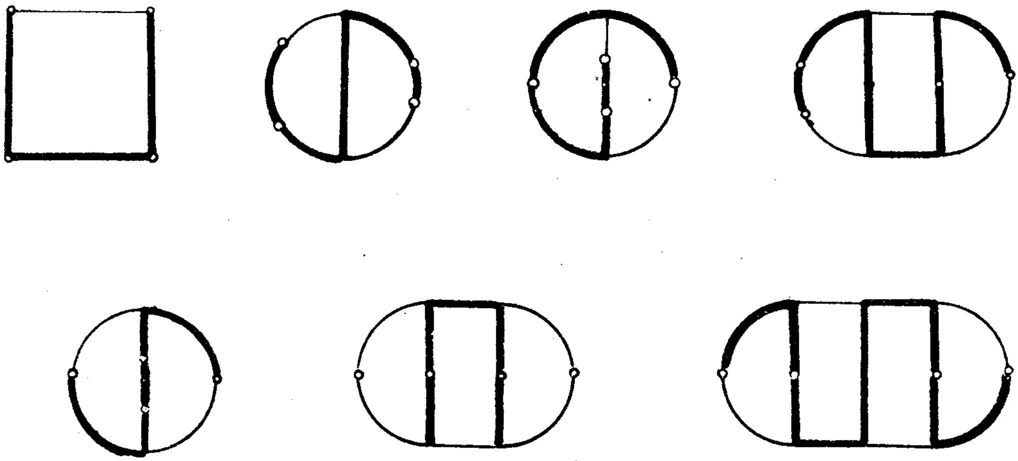


Fig.37

Having applied to these graphs the operation of theorem 2, and, then, if required, the operation of theorem I as well, we obtain the primitive diagrams a), b), c), e), f), h), k) of Fig.36.

In all these primitive graphs exists one of the open polygons drawn in Fig.38 (the external vertices are marked by numbers). Prove that any graph from $\mathfrak{R}^*$ is majorized by a graph in which exists one of these polygons. The first polygon will be called NNM polygon , and the second NM polygon.

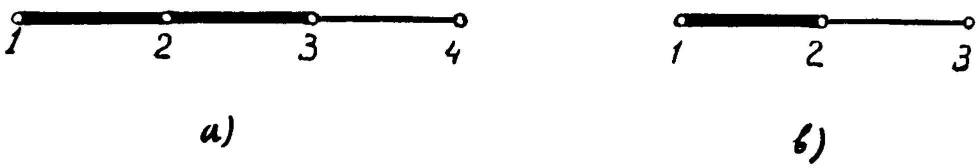


Fig.38

b) This has been already proved for the graphs in which all the external vertices lie on the nucleon open polygon.

Consider the case when three external vertices, say, a , c and b lie on the nucleon polygon, whereas the forth vertex d – outside this polygon. In view of the connectivity in the graph there is a meson path from the vertex d to the nucleon open polygon , e.g. to its segment ac . Repeating further the considerations of 21b, it can be seen that in the case under consideration the graph has at least one of the subgraph drawn in Fig.39.

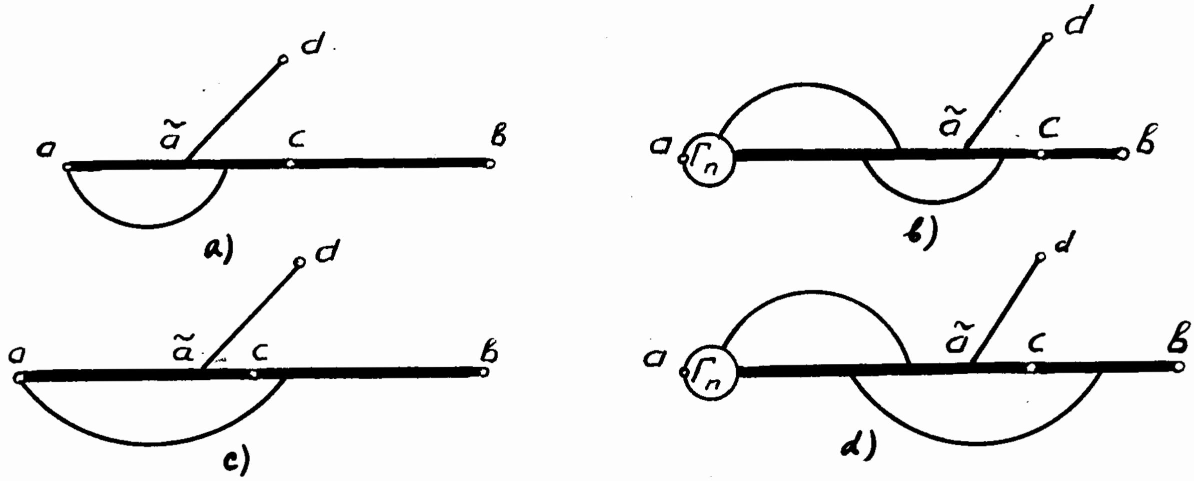


Fig.39

According to 17 in the subgraphs b) and d) the vertex part Γ is replaced by the vertex α . Applying further the operation of theorem 2 , we establish that the subgraphs a) and b) are majorized by a graph in which a NNM polygon exists, whereas the subgraphs c) and d) - by a graph in which there is a NM polygon.

c) Let now two external vertices lie outside the nucleon polygon. In view of the connectivity there exists a meson path from at least one point of the nucleon open polygon to one of the external vertices c or d , not passing through the other one. That point of the nucleon polygon from which such a path starts and which is nearest to a is called a characteristic one. Repeating further the considerations of 21b, we shall prove that in the case considered the graph is majorized by a graph with a NM polygon.

d) Let in the graph $\mathcal{D} \in \mathcal{R}^*$ exist a *NNM* polygon. Repeating the considerations of 23a, one can show that in this case the graph $\mathcal{D}$ is majorized by at least one of the graphs of Fig. 40. Applying the operation of theorem 2,

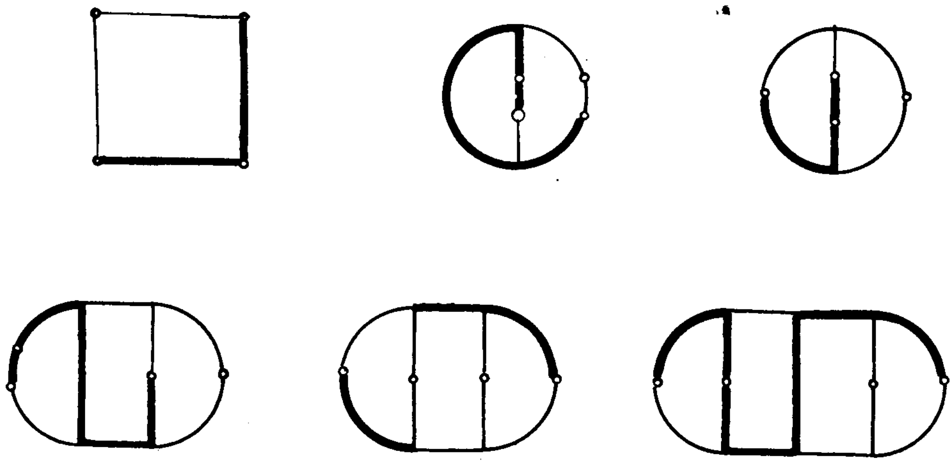


Fig.40

it is easily seen that these graphs are majorized by the primitive diagrams a), b), e), f), l), k) of Fig.36.

e) Let in the graph $\mathcal{D} \in \mathcal{R}^*$ a *NM* polygon exist. Because of connectivity on the *NM* polygon will be found at least one point from which there leads a meson path to the point α, having no common points with the *NM* polygon. Among such points the nearest to α is called characteristic point and denoted by $\bar{\alpha}$. If the point $\bar{\alpha}$ is situated on the segment ab, then the graph $\mathcal{D}$ has at least one of the four subgraphs of Fig.41.

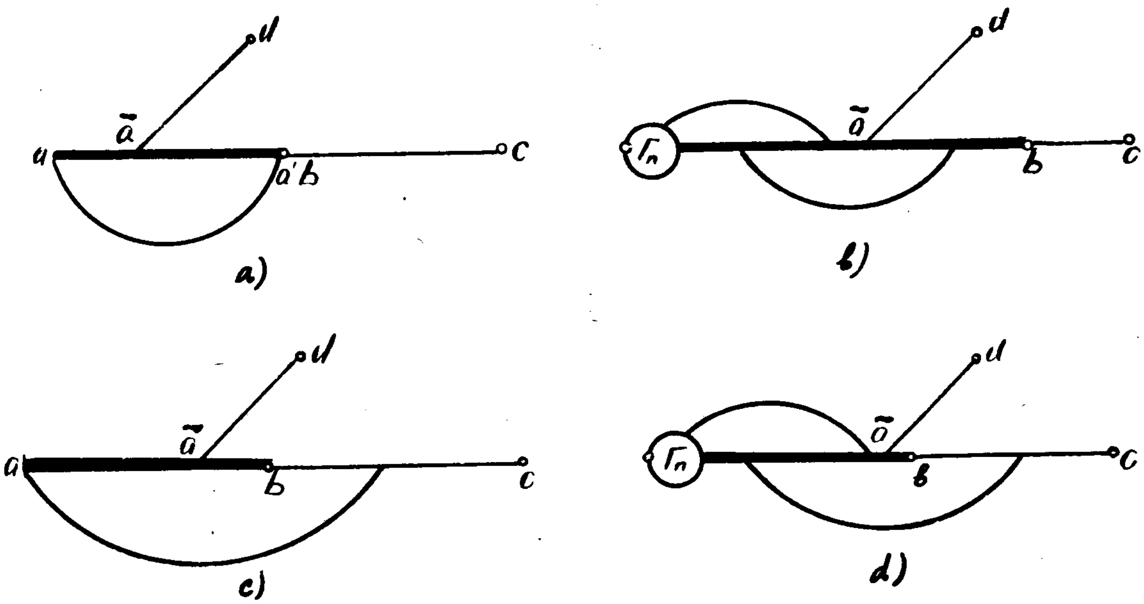


Fig.41

If it is on the segment bc , it has at least one of the four subgraphs of Fig.42.

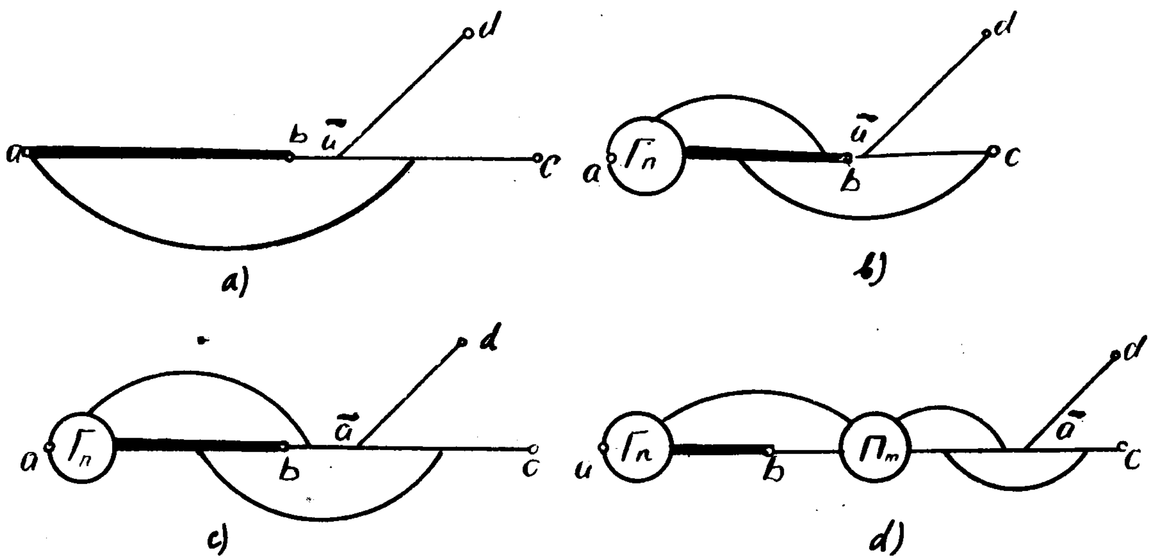


Fig.42

By 17 and theorem 2 it is easily seen that any graph in which there is a NM polygon is majorized by the graph in which there is either a MNM polygon of figure I (see Fig.43).

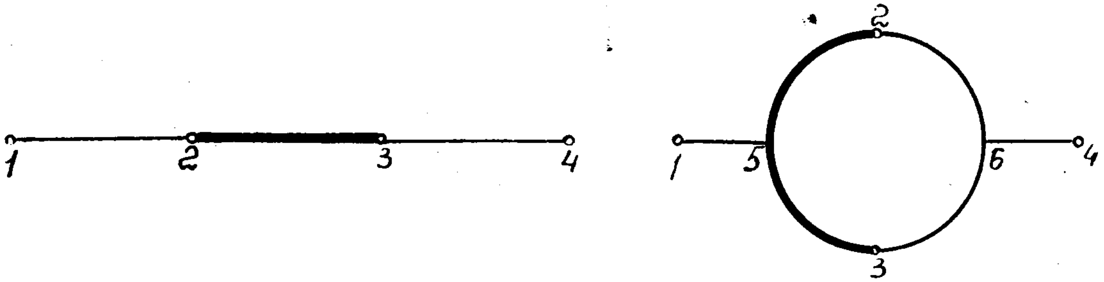


Fig.43

f) Let in the graph $\mathcal{D}$ of $\mathcal{R}^*$ there be a MNM polygon. Repeating the considerations of 23a, one can prove that in this case the graph $\mathcal{D}$ is majorized by at least one of the primitive graphs a), b), g), h), i), j), k) of Fig.36.

g) Let now in the graph be no MNM polygon but figure I. By analogy to 21b one can prove that it is sufficient to consider such graphs containing figure I, in which there is a meson path L from point 1 to the part of this figure complementary to the segment (1,5). The path L cannot lead from point 1 to the segment (2,6) and (3,6) since otherwise there would exist a MNM polygon. If this path leads to point 4, then the graph has a primitive subdiagram of Fig.36c. If the path L comes to one of the segments (2,5),(3,5) or

(4,6) it forms together with figure I one of the following two figures (see Fig.44)

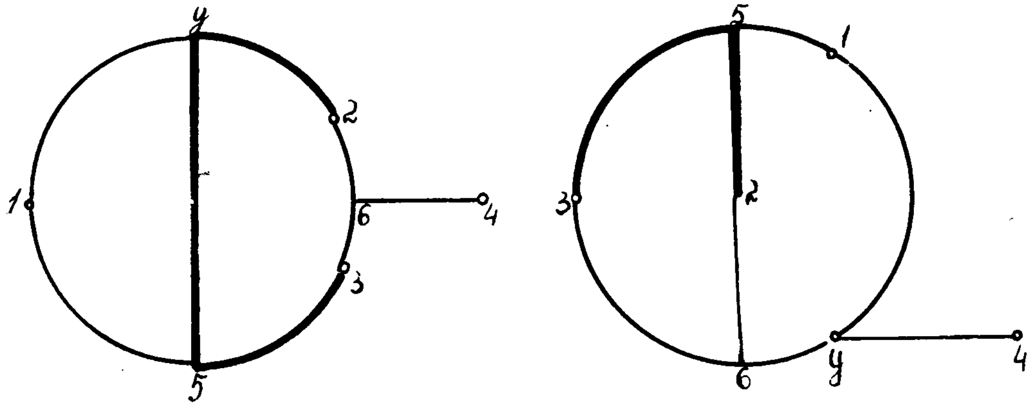


Fig.44

These figures are denoted by Ia, Ib.

Repeating the considerations of 21b, one can easily make sure that it is sufficient to cinsider only such graphs containing figures Ia or Ib, in which there is a meson way from point 4 to the part of this figure complementary to the segment (4,6) or (4, g), respectively.

h) Let the graph $\mathcal{D}$ from $\mathcal{R}^*$ have figure Ia. If the path L leads to one of the segment (1,g), (2,6), (2,g) or (5,g) then in this graph there is one of the primitive subdiagrams c), e), m), n) of Fig.36, respectively.

i) Let now the graph of $\mathcal{R}^*$ contain figure 1b. If the path L leads to segment (2,6) (1,g) or (1,5), then in this graph there is one of the primitive subdiagrams b), c), d) of Fig.36.

If the path L leads to the segment (2,5), then the graph $\mathcal{D}$ has a subgraph of Fig.45, which is reduced to the primitive graph of Fig.36 a) with the aid of the operation of theorem 2.

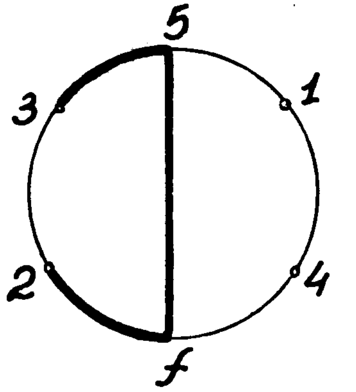


Fig.45

We do not consider the other segments since the figure is symmetrical. Thus, it is proved that any graph from $\mathcal{R}^*$ of the meson-nucleon scattering is majorized by at least one of the primitive graphs of Fig.36.

25. It is possible to prove by analogous considerations that in any graph of the class $\mathcal{R}^*$ of meson -meson scattering ($\ell_M = 4$, $\ell_N = 0$) , there is at least one of the primitive diagrams of Fig.46.

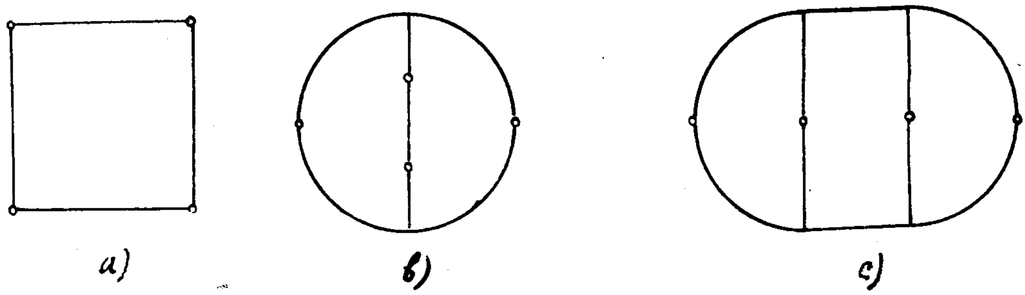


Fig.46

In the given case the proof is especially simple since in any graph from $\mathcal{R}^*$ of the meson-meson scattering there exists a polygon passing through all the external vertices. Besides, these primitive graphs may be obtained as subgraphs of the graphs of Fig.36 in which the nucleon lines are replaced by the meson ones.

It is evident that the obtained primitive graphs of Figs.36 and 46 may be also used in studying the amplitudes of the processes involving γ- quanta /14/.

Part II. Majorization of Primitive Graphs

In order to determine the region $G_{\mathcal{R}}$ it is sufficient now to restrict ourselves to the finite set of primitive diagrams. Some of the primitive graphs majorize the others. For instance, the primitive graph of Fig.20b is majorized by the primitive graph of Fig.20a, whereas the primitive graph of Fig.46c - by the primitive graph of Fig.46a.

This follows from the results obtained in 11. To compare the primitive diagrams with each other more detailed information about the quadratic forms is required. Symanzik's theorem and some its generalization prove to be an effective means for this purpose.

1. An Expression of the quadratic Form of the Graph in Terms of its Incidence Matrix

2. Let an arbitrary graph with n vertices and ℓ internal lines be given and let on each such line a direction be defined. Suppose that all the external lines enter the corresponding vertices. We enumerate the vertices and lines separately. The incidence matrix $E=(e_{i\nu}), i=1,\ldots,n$; $\nu=1\ldots,\ell$ of such a graph is determined in the following manner:

$$e_{i\nu}=\begin{cases} I, \text{ if the line } \nu \text{ goes out from the vertex } i, \\ -I, \text{ if the line } \nu \text{ enters the vertex } i, \\ 0, \text{ if the vertex } i \text{ does not belong to the line } \nu. \end{cases} \tag{1.1}$$

The law of momentum conservation in each vertex of the graph is written as

$$\sum_{\nu=1}^{\ell} e_{i\nu} K_\nu = P_i \quad , \quad i=1,\ldots,n, \tag{1.2}$$

where $K_1, \ldots, K_\ell$ are the momenta on the internal lines, $p_1, \ldots, p_n$ are the external momenta.

Note, that in each colomn of the matrix E two and only two elements are different from zero, one of them is equal to I, the other - to -1. If follows from here that

$$\sum_{i=1}^{n} \sum_{\nu=1}^{\ell} e_{i\nu} K_\nu = 0 . \tag{1.3}$$

Thus, for the self-consistency of Eqs. (1.2) it is necessary to fulfill the condition

$$\sum_{i=1}^{n} p_i = 0. \tag{1.4}$$

3. If the graph is connected, then the number of independent equations among Eqs.(1.2) is equal to $n-1$, and, hence, $f=\ell-n+1$ momenta may be given arbitrary. The momenta may be fixed arbitrary to such and only on such lines whose cut does not violate the connectivity of the graph. By changing, if necessary, the numeration of lines one can achieve that the first f lines possess this property. Consider first the case when all the vertices of the graph are external. We add the following equations

$$K_1 = t_1 , \ldots , K_f = t_f \tag{1.5}$$

to the system of Eqs. (1.2) at $i=1, \dots, n-1$.

As a result we obtain the system

$$L K = S \tag{1.6}$$

where

$$S_1 = t_1, \dots, S_f = t_f; \; S_{f+1} = P_1, \dots, S_\ell = P_{n-1};$$

$$L = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\ e_{11} & e_{12} & \cdots & e_{1f} & e_{1,f+1} & \cdots & e_{1\ell} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ e_{n-1,1} & e_{n-1,2} & \cdots & e_{n-1,f} & e_{n-1,f+1} & \cdots & e_{n-1,\ell} \end{pmatrix} \tag{1.7}$$

The solution of this system is unique and may be written in the form

$$K = L^{-1} \begin{bmatrix} t \\ P \end{bmatrix} = L^{-1} S . \tag{1.8}$$

4. Let us determine in the 4ℓ -dimensional vector space K the scalar product

$$(\kappa, q) = \sum_{\nu=1}^{\ell} \kappa_\nu q_\nu \tag{1.9}$$

and introduce the diagonal matrix $a_{\lambda\nu} = \alpha_\nu \delta_{\lambda\nu}$, $\alpha_\nu > 0$. Then

$$(\kappa, a\kappa) = \sum_{\nu=1}^{\ell} \alpha_\nu \kappa_\nu^2 \tag{1.10}$$

Substituting (1.8) into (1.10), we get

$$A_{\mathcal{D}}(\alpha; t, p) = (L^{-1}s, aL^{-1}s) = (s, \tilde{L}^{-1} a L^{-1} s) \tag{1.11}$$

The matrix inverse to the matrix $\tilde{L}^{-1} a L^{-1}$, will be $L a^{-1} \tilde{L}$. If the variables t and p are put in correspondance to y and z, then the quadratic form inverse to (1.11), takes on the following form

$$\overline{A}_{\mathcal{D}}(\alpha; y, z) = (x, L a^{-1} \tilde{L} x) =$$

$$= \sum_{s=1}^{f} \frac{1}{\alpha_s} \left(y_s + \sum_{i=1}^{n-1} e_{is} z_i\right)^2 + \sum_{s=f+1}^{\ell} \frac{1}{\alpha_s} \left(\sum_{i=1}^{n-1} e_{is} z_i\right)^2 \tag{1.12}$$

5. The quadratic form $A_{\mathcal{D}}(\alpha, p)$ obtained after the integration over the internal momenta is equal to (see Part I)

$$A_{\mathcal{D}}(\alpha, p) = \underset{t}{extr}\, A_{\mathcal{D}}(\alpha; t, p) \tag{1.13}$$

The following lemma (see, e.g. /5/) makes it simple to find the quadratic form $\bar{A}_{\mathcal{D}}(\alpha, z)$ inverse to the form $A_{\mathcal{D}}(\alpha, p)$.

LEMMA 2. Let the quadratic form $F(x,y)$ be defined by

$$F(x,y) = \sum_{i,j=1}^{n} a_{ij} x_i x_j + 2 \sum_{i=1}^{n} \sum_{l=1}^{m} b_{il} x_i y_l + \sum_{l,k=1}^{m} c_{lk} y_l y_k \equiv$$

$$\equiv (x, ax) + 2(x, by) + (y, cy).$$

The quadratic form, inverse to it, is denoted by $\bar{F}(\xi, \eta)$. The quadratic form $F(x) = \underset{y}{\text{extremum}}\ F(x,y)$ has an inverse quadratic form $\bar{F}(\xi)$ equal to the value of the form $\bar{F}(\xi, \eta)$ at $\eta = 0$.

According to this lemma

$$\bar{A}_{\mathcal{D}}(\alpha, z) = \bar{A}_{\mathcal{D}}(\alpha; 0, z) = \sum_{s=1}^{l} \frac{1}{\alpha_s} \Big(\sum_{i=1}^{n-1} e_{is} z_i \Big)^2. \tag{1.14}$$

It follows from here that

$$A_{\mathcal{D}}(\alpha, p) = - \frac{\begin{vmatrix} 0 & p \\ p & d \end{vmatrix}}{|d|}, \qquad \text{where}$$

$$d_{ij} = \sum_{s=1}^{l} \frac{e_{is} e_{js}}{\alpha_s} \tag{1.15}$$

Let now some of the vertices of the graph be internal. We shall consider the vertex n as an external one. Then $A_{\mathcal{D}}(\alpha, p)$ turns out to be a quadratic form of the momenta of the external vertices of the graph only. By (1.14) and Lemma 2 the quadratic form $\bar{A}_{\mathcal{D}}(\alpha, z)$ inverse to $A_{\mathcal{D}}(\alpha, p)$, is equal to

$$\bar{A}_{\mathcal{D}}(\alpha, z) = \underset{z_{int}}{extr} \sum_{s=1}^{\ell} \frac{1}{\alpha_s} \Big(\sum_{i=1}^{n-1} e_{is} z_i \Big)^2, \tag{1.16}$$

where z_{int} are the variables corresponding to the internal vertices of the graph.

Note that in the derivation of these formulae we did not make use of the assumption that the momenta are Euclidean.

In the particular case when the vectors z_i are Euclidean, the extremum in (1.16) turns into the minimum.

2. Symanzik's Theorem and Its Generalization

To formulate and prove Symanzik's theorem it is useful to give here some elementary notions about norms.x)

1. Let Z be a real N dimensional Euclidean space. The space $\mathcal{P}$ of the linear functions (p, z) defined in Z is called the space conjugated to Z, if the scalar product in $\mathcal{P}$ is determined by the formula

$$p^2 = (p, p) = \max_{z^2 \le 1} (p, z)^2 \tag{2.1}$$

---

x) Further, we omit some elementary proofs which one can find in the books on functional analysis (see, e.g.,[16], chapter XI).

The spaces $\mathcal{P}$ and Z are isomorphous and conjugated to each ther.

2. The real function $f(z)$, $z \in Z$ is called a norm if it possesses the following properties:

a) For each $z \in Z$ and any real number λ

$$f(\lambda z) = |\lambda| f(z)$$

b) The triangle condition

$$f(x+y) \leq f(x) + f(y)$$

where

$$x \in Z, \; y \in Z.$$

c) If $f(z)=0$, then $z=0$

From properties a) and b) it follows that $f(z) \geq 0$.

3. Let A be a positive definite symmetrical matrix. Then the function

$$f(z) = \sqrt{(z, Az)} \tag{2.2}$$

satisfies the conditions a), b), and c) and is an example of an norm. In particular, $|z| = \sqrt{(z,z)}$ is a norm.

There exist norms non-representable in the form (2.2). If, for instance, $\alpha_s > 0$ then the functions

$$f(z_1, \dots, z_N) = \sum_{s=1}^{N} \alpha_s |z_s|$$

$$\bar{f}(z_1, \dots, z_N) = \max_{s=1,\dots,N} \frac{|z_s|}{\alpha_s}$$

are norms.

4. Let $f(z)$ be a norm in Z, then the function

$$\bar{f}(p) = \max_{f(z)\leq 1} |(z,p)| \tag{2.3}$$

is a norm in $\mathcal{P}$. This norm is called conjugated to $f(z)$.

The norm conjugated to $\bar{f}(z)$ is equal to $f(z)$.

We give some properties of mutually conjugated norms.

a) The norm conjugated to the norm (2.2) has the form

$$\bar{f}(p) = \sqrt{(p, A^{-1}p)} \tag{2.4}$$

Proof. Since A is a positive definite symmetric matrix, then on the basis of the Cauchi-Bunyakovski inequality we have

$$(z,p)^2 = (A^{\frac{1}{2}}z, A^{\frac{1}{2}}p)^2 \leq (z, Az)(p, A^{-1}p)$$

Since in the region $(z, Az) \leq 1$ for any p there exists a non-zero vector parallel to $A^{-1}p$, then

$$\bar{f}(p) = \max_{(z,Az)\leq 1} |(z,p)| = \sqrt{(p, A^{-1}p)}$$

b) Let $\bar{f}_1(p)$ and $\bar{f}_2(p)$ be two norms in $\mathcal{P}$, $f_1(z)$ and $f_2(z)$ the conjugated norms in Z. In order that the inequality

$$\bar{f}_1(p) \geq \bar{f}_2(p), \quad p \in \mathcal{P} \tag{2.5}$$

holds, it is necessary and sufficient to fulfil the inequality

$$f_1(z) \leq f_2(z), \qquad z \in Z. \tag{2.6}$$

Proof. Let G_1 and G_2 be the sets of points z, where $f_1(z) \leq 1$ and $f_2(z) \leq 1$ respectively. By (2.6) $G_2 \subseteq G_1$ and, hence

$$\max_{z \in G_1} |(z,p)| \geqslant \max_{z \in G_2} |(z,p)|$$

c) If $f(z)$ and $\bar{f}(p)$ are mutually conjugated norms and $\lambda > 0$, then the norm $\lambda f(z)$ is conjugated to the norm $\frac{1}{\lambda} \bar{f}(p)$.

5. LEMMA 3. Let there be a family of norms $\bar{f}_\alpha(p)$, limited for any p, if α takes on all volves of a set J; let $f_\alpha(z)$ be the family of the conjugated norms.

Then the function

$$\bar{f}(p) = \max_{\alpha \in J} \bar{f}_\alpha(p) \tag{2.7}$$

is also a norm; the norm conjugated to it is the largest one satisfying the inequality

$$f(z) \leq \min_{\alpha \in J} f_\alpha(z) \tag{2.8}$$

It follows that if $\min_{\alpha\in J} f_\alpha(z)$ is a norm, then it will be conjugated to $\bar{f}(p)$, i.e.,

$$f(z) = \min_{\alpha\in J} f_\alpha(z) \tag{2.9}$$

Proof. It can be easily seen that function (2.7) satisfies the properties a), b) and c). For instance, the property b)

$$\bar{f}(p+q) = \max_{\alpha\in J} \bar{f}_\alpha(p+q) \leq \max_{\alpha\in J} \bar{f}_\alpha(p) + \max_{\alpha\in J} \bar{f}_\alpha(q) = \bar{f}(p)+\bar{f}(q).$$

So, $\bar{f}(p)$ is a norm.

The interval of values p satisfying the conditions

$\bar{f}(p) \leq 1$, $\bar{f}_\alpha(p) \leq 1$

will be denoted by $\bar{G}$ and $\bar{G}_\alpha$, respectively. From (2.7) follows that

$$\bar{G}_\alpha \supseteq \bar{G} \tag{2.10}$$

Since

$$f_\alpha(z) = \max_{p\in\bar{G}_\alpha} |(p,z)|$$

then by (2.10)

$$f(z) = \max_{p\in\bar{G}} |(p,z)| \leq \max_{p\in\bar{G}_\alpha} |(p,z)| = f_\alpha(z),$$

$$f(z) \leq f_\alpha(z) \qquad \text{for any } \alpha\in J \tag{2.11}$$

In virtue of what has been proved earlier there exists a largest norm $\varphi(z)$ satisfying the inequality (2.11)

$$f(z) \leq \varphi(z) \leq f_\alpha(z) \quad \text{for any} \quad \alpha \in J. \tag{2.12}$$

But since $\varphi(z) \leq f_\alpha(z)$, then by 4b we have

$$\bar{\varphi}(p) \geq \bar{f}_\alpha(p) \qquad \text{for any} \quad \alpha \in J \qquad ,$$

hence $\bar{\varphi}(p) \geq \max\limits_{\alpha \in J} \bar{f}_\alpha(p) = \bar{f}(p)$.

Applying again the property of 4b, we get

$$\varphi(z) \leq f(z) \tag{2.13}$$

Comparing (2.12) and (2.13), we obtain

$$f(z) = \varphi(z) \tag{2.14}$$

Thus, it follows from (2.11) and (2.14) that $f(z)$ is the largest norm satisfying the inequality

$$f(z) \leq \min_{\alpha \in J} f_\alpha(z).$$

6. Let the space Z be a direct sum of the subspaces X and Y, $z = (x, y) \in Z$, $x \in X$, $y \in Y$.

Then, if $f(z) \equiv f(x, y)$ is a norm in Z, then

$$f(x) = \min_y f(x,y) \tag{2.15}$$

is a norm in X.

The space $\mathcal{P}$ conjugated to Z decomposes into the sum of the subspaces Q and S, respectively. The following analog of lemma 2 holds

$$\bar{f}(q) = \bar{f}(q,s) \quad at \quad s=0; \quad q \in Q, \ s \in S.$$

7. By 3 the function

$$\bar{f}_{\mathcal{D}}(\alpha, p) = \sqrt{\frac{A_{\mathcal{D}}(\alpha, p)}{M^2_{\mathcal{D}}(\alpha)}}, \tag{2.16}$$

where $A_{\mathcal{D}}$ and $M^2_{\mathcal{D}}$ are given by formulae (1.15) and (1.3 p.I), is a norm in $\mathcal{P}$. According to 4 a), b) the function

$$f_{\mathcal{D}}(\alpha, z) = \sqrt{M^2_{\mathcal{D}}(\alpha) \cdot \bar{A}_{\mathcal{D}}(\alpha, z)}, \tag{2.17}$$

where $\bar{A}_{\mathcal{D}}$ is set by formula (1.16) is the norm conjugated with $\bar{f}_{\mathcal{D}}(\alpha, p)$.

8. Let us calculate the minimum with respect to α of the function

$$J_{\mathcal{D}}(\alpha, z) = \sum_{s=1}^{\ell} \alpha_s m_s^2 \sum_{\nu=1}^{\ell} \frac{1}{\alpha_\nu} \Big(\sum_{i=1}^{n-1} e_{i\nu} z_i \Big)^2. \tag{2.18}$$

Since

$$\lim_{x_s \to 0} J_{\mathcal{D}}(x, z) = \lim_{x_s \to \infty} J_{\mathcal{D}}(x, z) = \infty,$$

then the minimum is attained at an internal point of the region where α changes, in which

$$\frac{\partial J_{\mathcal{D}}}{\partial \alpha_s} = m_s^2 \sum_{v=1}^{\ell} \frac{1}{\alpha_v} \left(\sum_{i=1}^{n-1} e_{iv} z_i \right)^2 - \frac{\sum_{v=1}^{\ell} \alpha_v m_v^2}{\alpha_s^2} \left(\sum_{i=1}^{n-1} e_{is} z_i \right)^2 = 0.$$

Hence we find

$$\alpha_s = \frac{\left| \sum_{i=1}^{n-1} e_{is} z_i \right|}{m_s} \cdot \left[\frac{\sum_{v=1}^{\ell} \alpha_v m_v^2}{\sum_{v=1}^{\ell} \frac{1}{\alpha_v} \left(\sum_{i=1}^{n-1} e_{iv} z_i \right)^2} \right]^{\frac{1}{2}}. \tag{2.19}$$

Substituting (2.19) into (2.18), we get

$$\sqrt{\min_{\alpha} J_{\mathcal{D}}(\alpha, z)} = \sum_{v=1}^{\ell} m_v \left| \sum_{i=1}^{n-1} e_{iv} z_i \right| \tag{2.20}$$

One can easily make sure that (2.20) is a norm.

9. Now it is easy to find the minimum with respect to α of the fucntion $f_{\mathcal{D}}(\alpha, z)$. Indeed, by using (2.20), we get

$$f_{\mathcal{D}}(z) = \min_{\alpha} f_{\mathcal{D}}(\alpha, z) = \min_{z_{int}} \sqrt{\min_{\alpha} J_{\mathcal{D}}(\alpha, z)} = \min_{z_{int}} \sum_{v=1}^{\ell} m_v \left| \sum_{i=1}^{n-1} e_{iv} z_i \right|. \tag{2.21}$$

By 8 and 6 $f_{\mathcal{D}}(z)$ is a norm.

By lemma 3, the norm $f_{\mathcal{D}}(z)$ is conjugated with the norm

$$\bar{f}_{\mathcal{D}}(p) = \max_{\alpha} \sqrt{\frac{A_{\mathcal{D}}(\alpha,p)}{M^2_{\mathcal{D}}(\alpha)}} \tag{2.22}$$

10. Symanzik's theorem follows immediately from 4b and 9.

Theorem 3. Let $\mathcal{D}$ and $\mathcal{D}'$ be two diagrams of the same process. The necessary and sufficient condition for

$$\max_{\alpha} \frac{A_{\mathcal{D}}(\alpha,p)}{M^2_{\mathcal{D}}(\alpha)} \geq \max_{\alpha'} \frac{A_{\mathcal{D}'}(\alpha',p)}{M^2_{\mathcal{D}'}(\alpha')} \tag{2.23}$$

to hold for all $p \in \mathcal{P}$, is that for all $z \in Z$ the inequality

$$\min_{z_{int}} \sum_{v=1}^{\ell} m_v \left| \sum_{i=1}^{n-1} e_{iv} z_i \right| \leq \min_{z_{int}} \sum_{v=1}^{\ell'} m_v' \left| \sum_{i=1}^{n'-1} e_{iv}' z_i' \right| \tag{2.24}$$

is fulfilled.

Here e_{iv} and e_{iv}' are the incidence matrix of the graph $\mathcal{D}$ and $\mathcal{D}'$. The variable z and z' corresponding to the external vertices coincide.

II. In 6 (part I) the domain $G(\mathcal{D})$ of the graph $\mathcal{D}$ with a given numeration of the external vertices was defined.

If such a graph belongs to the set $\mathcal{R}$ (see 9 part I) then together with this graph the set $\mathcal{R}$ contains all the graphs obtained from the given one by permuting the numeration of the external nucleon and meson vertices separately.

Each such graph is denoted by $\mathcal{D}_{\varkappa}^{\pi}$ where

$$\pi = \begin{pmatrix} 1 & 2 & \dots & \ell_M \\ i_1 & i_2 & \dots & i_{\ell_M} \end{pmatrix}, \quad \varkappa = \begin{pmatrix} 1 & 2 & \dots & \ell_N \\ j_1 & j_2 & \dots & j_{\ell_N} \end{pmatrix}$$

are arbitrary permutations.

Determine the domain $H(\mathcal{D})$ as the intersection of the domains

$$H(\mathcal{D}) = \bigcap_{\varkappa,\pi} G(\mathcal{D}_{\varkappa}^{\pi})$$

over various π and $\varkappa$.

The region $H(\mathcal{D})$ characterizes the graph $\mathcal{D}$ irrespective of the numeration of its external vertices. The region $G_{\mathcal{R}}$ coincides with the intersection of all the regions $H(\mathcal{D})$.

12. From 4b and 9 results immediately also <u>the following generalization of Symanzik's theorem.</u>

Theorem 4. Let there be given $K+1$ diagrams $\mathcal{D}_{\sigma}$ $(\sigma = 0,1,\dots,K)$ of the same process with fixed numerations of the external vertices. Let $\bar{f}_{\mathcal{D}_{\sigma}}(p)$ be the norms corresponding to the graphs $\mathcal{D}_{\sigma}$ (2.22). In order that for all $p \in \mathcal{P}$ the inequality

$$\bar{f}_{\mathcal{D}_0}(p) \leq \max_{\sigma=1,\dots,K} \bar{f}_{\mathcal{D}_{\sigma}}(p) = \bar{f}_{1,\dots,K}(p) \qquad (2.25)$$

holds, it is necessary and sufficient that for all $z \in Z$ the conjugated norms would satisfy the inequality

$$f_{1,\dots,K}(z) \leqslant f_{\mathcal{D}_0}(z). \tag{2.26}$$

By Lemma 3 $f_{1,\dots,K}(z)$ is the largest norm, satisfying the condition

$$f_{1,\dots,K}(z) \leqslant \min_{\sigma=1,\dots,K} f_{\mathcal{D}_\sigma}(z). \tag{2.27}$$

The theorem presents a necessary and sufficient condition for the domain $G_{\mathcal{D}_0}$ to contain the intersection of the domains $G_{\mathcal{D}_\sigma}$, i.e.,

$$G_{\mathcal{D}_0} \supseteq \bigcap_{\sigma=1}^{K} G_{\mathcal{D}_\sigma}.$$

In the following we shall make use of this theorem only in the particular case when $\mathcal{D}_\sigma$ ($\sigma=1,\dots,K$) are representatives of the same graph $\mathcal{D}$, with different numerations of the external meson and nucleon vertices. In this case the theorem presents a criterion for including the region $H_{\mathcal{D}}$ (see 11) into the region $G_{\mathcal{D}_0}$.

In this paper we shall only make use of the following simple corollary from (2.27):

Corollary. To fulfil inequality (2.25) it is sufficient that for all $z \in Z$

$$\min_{\sigma=1,\dots,K} f_{\mathcal{D}_\sigma}(z) \leqslant f_{\mathcal{D}_0}(z). \tag{2.28}$$

Note. Let $\bar{G}(\mathcal{D}_\sigma)$ be a set of points z, in which $f_{\mathcal{D}_\sigma}(z) \leq 1$. The set of points z, satisfying the inequality

$$f_{1,\ldots,\kappa}(z) \leq 1$$

is the convex envelope of the sum sets of $\bar{G}(\mathcal{D}_\sigma)$, $\sigma = 1, \ldots, \kappa$.

3. Majorization of Primitive Diagrams

13. The vertex part.

According to §4 of part I the class $\mathcal{R}_0$ of the primitive diagrams of the meson-nucleon vertex part consists of the two graphs of Fig.17. With the aid of Symanzik's theorem we shall show that the diagram of Fig.17a majorizes the graph of Fig.17b. Indeed, the norm $f_{\Phi}(z)$ of the graph of Fig.47 in virtue of the triangle inequality

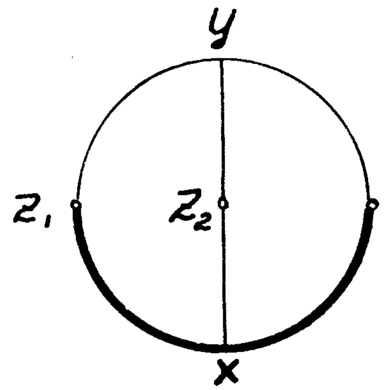


Fig. 47

is greater than the norm $f_{\Delta}(Z)$ of the graph of Fig.17 a):

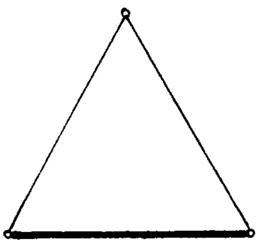
$$f_{\text{①}}(Z) = \min_{x,y} \left\{ M\left[|z_1 - x| + |x|\right] + m\left[|z_1 - y| + |y - z_2| + |z_2 - x| + |y|\right] \right\} =$$

$$= \min_{x,y} \left\{ (M-m)\left[|z_1 - x| + |x|\right] + \frac{m}{2}\left[(|z_1 - x| + |x - z_2|) + (|x| + |z_2 - x|) + \right.\right.$$

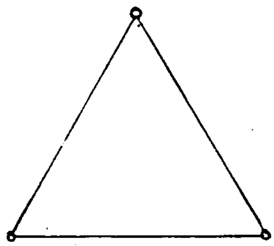
$$\left.\left. + (|z_1 - x| + |x|) + (|z_1 - y| + |y - z_2|) + (|z_2 - y| + |y|) + (|z_1 - y| + |y|) \right] \right\} \geq$$

$$\geq M|z_1| + m\left[|z_1 - z_2| + |z_2|\right] = f_{\Delta}(Z).$$

So, any graph of the meson-nucleon vertex part from $\mathcal{R}$ (9 part I) is majorized by the graph of Fig. 48 a).



a)



б)

Fig.48

It follows from here that any graph of the meson meson vertex part from $\mathcal{R}$ is majorized by the graph of Fig.48 b). This result has already been obtained earlier (see point 1 part II).

14. Meson-Meson Scattering.

According to 25 of part I the class $\mathcal{R}_0$ of the primitive diagrams of meson-meson scattering consists of three graphs of Fig.46. The primitive diagram of Fig.46 b) is majorized by the graph of Fig.46 a) (see point 1 part II). The same result follows from theorem s. Indeed, the norm $f_{\mathcal{D}_c}(z)$ of this graph (see Fig.49 b))

$$f_{\mathcal{D}_c}(z) = m \cdot \min_{x,y} \Big\{ |z_1 - x_1| + |x_1 - z_2| + |x_1 - x_2| + |z_1 - y_1| + |y_1 - z_2| + |y_1 - y_2| + |x_2 - z_3| + |z_3 - y_2| + |x_2| + |y_2| \Big\} =$$

$$= m \cdot \min_{x,y} \frac{1}{2} \Big\{ \big[|z_1 - x_1| + |x_1 - z_2| \big] + \big[|z_1 - x_1| + |x_1 - x_2| + |x_2 - z_3| + |z_1 - y_1| + |y_1 - y_2| + |y_2 - z_3| \big] + \big[|z_1 - y_1| + |y_1 - z_2| \big] + \big[|z_3 - x_2| + |x_2| + |z_3 - y_2| + |y_2| \big] + \big[|z_2 - x_1| + |x_1 - x_2| + |x_2| + |z_2 - y_1| + |y_1 - y_2| + |y_2| \big] \Big\}$$

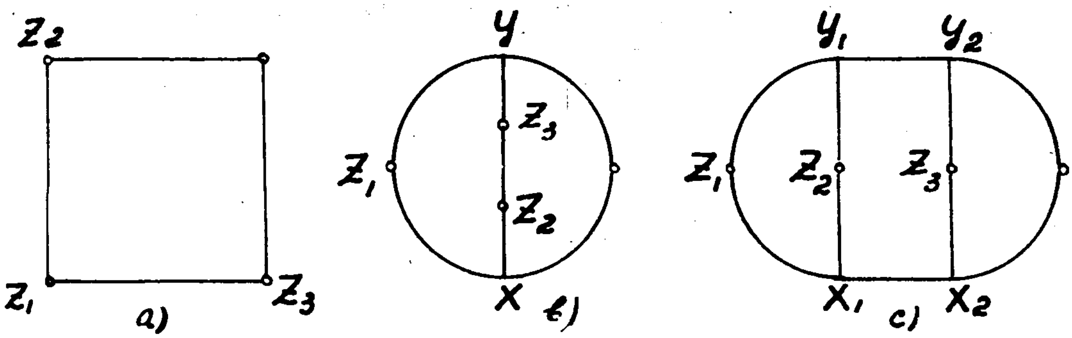


Fig.49

because of the triangle inequality is greater than the norm $f_{\mathcal{D}_a}(z_1,z_2,z_3)$ of the graph of Fig.49 a), equal to

$$f_{\mathcal{D}_a}(z_1,z_2,z_3) = m\left[|z_1-z_2|+|z_1-z_3|+|z_2|+|z_3|\right]. \tag{3.1}$$

Now show that according to theorem 4 , the graph of Fig.49 b) is majorized by a pair of the graphs of Fig. 46a) with two different numerations of the external vertices, i.e., that

$$G(\mathcal{D}_b) \supseteq H(\mathcal{D}_a).$$

Indeed,

$$f_{\mathcal{D}_b}(z) = m \cdot \min_{x,y}\Big\{|z_1-x|+|x-z_2|+|x|+|z_2-z_3|+|z_3-y|+$$

$$+|y-z_1|+|y|\Big\} =$$

$$= m|z_2-z_3| + \frac{1}{2}m \cdot \min_{x,y}\Big\{(|z_1-x|+|x|)+(|z_3-y|+|y|)+$$

$$+(|z_1-x|+|x-z_2|)+(|z_2-x|+|x|)+(|z_1-y|+|y|)+(|z_3-y|+|y-z_1|)\Big\} \geqslant$$

$$\geqslant m|z_2-z_3|+m|z_1|+\frac{1}{2}m\Big\{|z_2|+|z_3|+|z_1-z_2|+|z_1-z_3|\Big\} \geqslant$$

$$\geq m|z_2-z_3| + m|z_1| + m\cdot \min\{|z_1-z_2|+|z_3|,\ |z_1-z_3|+|z_2|\} =$$

$$= \min\{ f_{\mathcal{D}_a}(z_2,z_1,z_3),\ f_{\mathcal{D}_a}(z_3,z_1,z_2)\}$$

where $f_{\mathcal{D}_a}(z_1,z_2,z_3)$ is determined by formula (3.1).

So, any graph of meson-meson scattering from $\mathcal{R}$ is majorized by the diagram $\square$ of Fig.50.

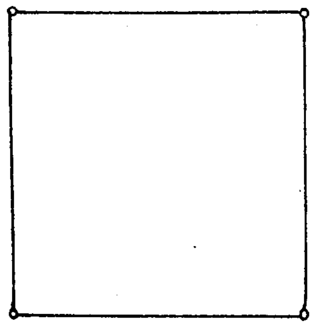


Fig.50

i.e., the domain $G_{\mathcal{R}} = G_{\mathcal{R}_0} = H(\square)$.

Note, that according to Symanzik's theorem it can be shown that the graph $\mathcal{D}_b$ is majorized by neither of the graphs $\mathcal{D}_a$ with a fixed numeration of external vertices, i.e. that the domain $G(\mathcal{D}_b)$ does not contain either of the regions $G(\mathcal{D}_a)$.

15. Nucleon-Nucleon Scattering

According to §5 of part I the class $\mathcal{R}_0$ of the primitive diagrams of nucleon-nucleon scattering consists of seven graphs of Fig.21. On the basis of Symanzik's theorem and its generalization it is possible to show that the primitive diagrams c), d), e), f), g) of this Figure are majorized by the graph a). Therefore, any graph of nucleon- nucleon scattering from $\mathcal{R}$ is majorized by at least one of the two graphs of Fig.51,

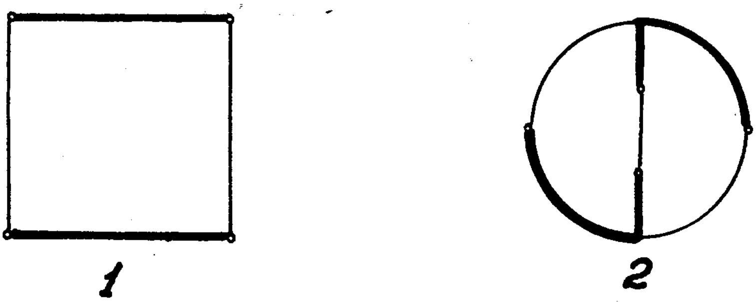


Fig.51

i.e., $G_{\mathcal{R}} = G_{\mathcal{R}_0} = H(\mathcal{D}_1) \cap H(\mathcal{D}_2)$. We proceed to the proof of this assertion. Consider first the graphs d), e) and f) of Fig.21. The graph d) (see Fig.52) has the norm $f_{\mathcal{D}_d}(z)$ equal to

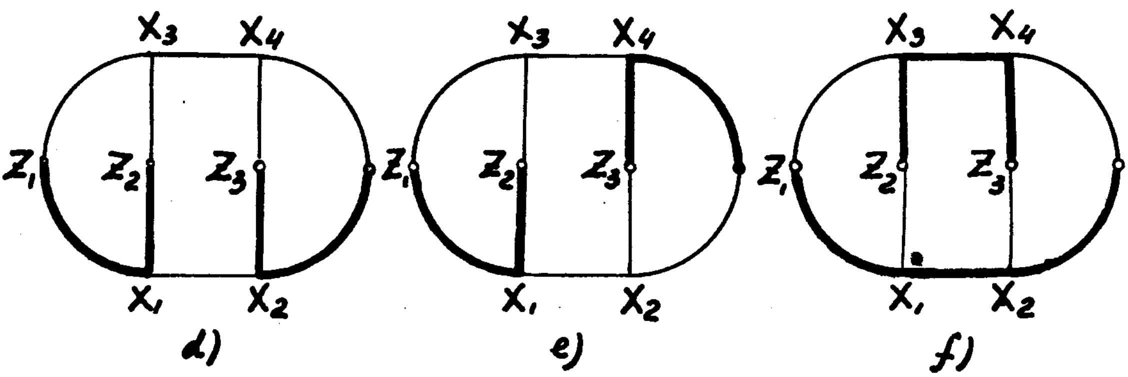


Fig. 52

$$f_{\mathcal{D}_d}(z) = \min_x \Big\{ M\left(|z_1 - x_1| + |x_1 - z_2| + |z_3 - x_2| + |x_2|\right) +$$

$$+ m\left(|x_1 - x_2| + |z_1 - x_3| + |x_3 - z_2| + |x_3 - x_4| + |x_4| + |x_4 - z_3|\right)\Big\} \geq$$

$$\geq \min_{x_1,x_2} (M-m)(|z_1-x_1|+|x_1-z_2|+|z_3-x_2|+|x_2|)+$$

$$+m\cdot \min_{x_1,x_2}\{(|z_1-x_1|+|x_1-z_2|)+(|z_3-x_2|+|x_2|)+$$

$$+(|z_1-x_1|+|x_1-x_2|+|x_2|)+(|z_2-x_1|+|x_1-x_2|+|x_2-z_3|)\}\geq$$

$$\geq f_{\mathcal{D}_a}(z_1,z_2,z_3)$$

where

$$f_{\mathcal{D}_a}(z_1,z_2,z_3)=M(|z_1-z_2|+|z_3|)+m(|z_1|+|z_2-z_3|). \tag{3.2}$$

The graph e) (see Fig.52) has the norm $f_{\mathcal{D}_e}(z)$ equal to

$$f_{\mathcal{D}_e}(z)=\min_{x}\{M(|z_1-x_1|+|x_1-z_2|+|z_3-x_4|+|x_4|)+$$

$$+m(|x_1-x_2|+|x_2|+|z_1-x_3|+|x_3-z_2|+|x_2-z_3|+|x_3-x_4|)\}\geq$$

$$\geq \min_{x_1,x_4} \ (M-m)\left(|z_1-x_1|+|x_1-z_2|+|z_3-x_4|+|x_4|\right)+$$

$$+m\cdot \min_{x_1,x_2} \left\{ \left(|z_1-x_1|+|x_1-z_2|\right)+\left(|x_2|+|x_2-z_3|\right)+\right.$$

$$\left.+\left(|z_1-x_1|+|x_1-x_2|+|x_2|\right)+\left(|z_2-x_1|+|x_1-x_2|+|x_2-z_3|\right)\right\}\geq$$

$$\geq f_{\mathcal{D}_a}(z_1,z_2,z_3)$$

where $f_{\mathcal{D}_a}(z_1,z_2,z_3)$ is determined by formula (3.2). The graph f) (see Fig.52) has the norm $f_{\mathcal{D}_f}(z)$ equal to

$$f_{\mathcal{D}_f}(z)=\min_{x}\left\{ m\left(|z_1-x_3|+|x_1-z_2|+|x_2-z_3|+|x_4|\right)+\right.$$

$$\left.+M\left(|z_1-x_1|+|x_1-x_2|+|x_2|+|z_2-x_3|+|x_3-x_4|+|x_4-z_3|\right)\right\}=$$

$$=\min_{x}\left\{(M-m)\left(|z_1-x_1|+|x_1-x_2|+|x_2|+|z_2-x_3|+|x_3-x_4|+|x_4-z_3|\right)+\right.$$

$$+m\left(|z_1-x_1|+|x_1-x_2|+|x_2|+|x_1-z_2|+|x_2-z_3|\right)+$$

$$\left.+m\left(|z_1-x_3|+|x_3-x_4|+|x_4|+|x_3-z_2|+|x_4-z_3|\right)\right\}\geq$$

$$\geq \min_{x} \ (M-m)\left(|z_1-x_1|+|x_1-x_2|+|x_2|+|z_2-x_3|+|x_3-x_4|+|x_4-z_3|\right)+$$

$$+m\cdot \min_{x_1,x_2} \left\{|z_1-x_1|+|x_1-z_2|+|z_3-x_2|+|x_2|+|z_1-x_1|+\right.$$

$$\left.+|x_1-x_2|+|x_2|+|z_2-x_1|+|x_1-x_2|+|x_2-z_3|\right\}\geq$$

$$\geq f_{\mathcal{D}_a}(z_3, z_2, z_1)$$

where $f_{\mathcal{D}_a}(z_1, z_2, z_3)$ is defined by formula (3.2).

Now proceed to the consideration of the graphs c) and g) of Fig.21. The graph c) (see Fig.53) has the norm $f_{\mathcal{D}_c}(z)$ equal to

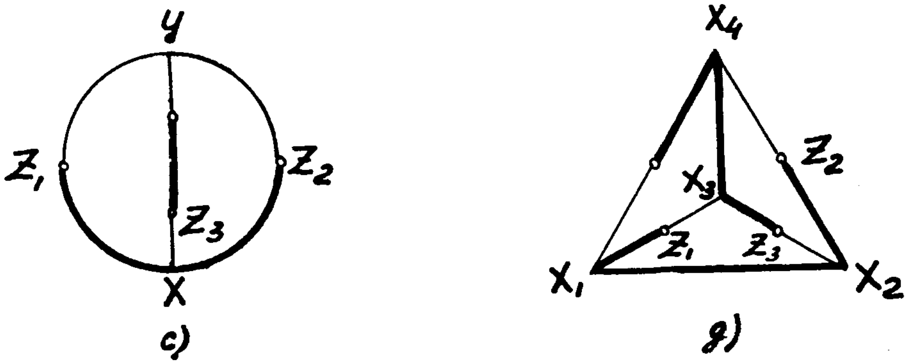


Fig.53

$$f_{\mathcal{D}_c}(z) = \min_{x,y} \Big\{ M\left(|z_1 - x| + |x - z_2| + |z_3|\right) +$$

$$+ m\left(|y| + |y - z_1| + |z_2 - y| + |x - z_3|\right)\Big\} \geq$$

$$\geq \min_{x,y} (M-m)\left(|z_1 - x| + |x - z_2| + |z_3|\right) + m|z_3| +$$

$$+ \frac{m}{2} \min_{x,y} \Big\{ (|z_1 - x| + |x - z_2|) + (|z_2 - y| + |y - z_1|) + (|z_1 - x| + |x - z_3|) +$$

$$+ (|z_2 - x| + |x - z_3|) + (|z_1 - y| + |y|) + (|z_2 - y| + |y|)\Big\} \geq$$

$$\geq M\left(|z_1-z_2|+|z_3|\right)+\frac{1}{2}\cdot m\left(|z_1-z_3|+|z_2-z_3|+|z_1|+|z_2|\right)\geq$$

$$\geq M\left(|z_1-z_2|+|z_3|\right)+m\cdot\min\left(|z_1-z_3|+|z_2|,\ |z_1|+|z_2-z_3|\right)=$$

$$=\min\left(f_{\mathcal{D}_a}(z_2,z_1,z_3),\ f_{\mathcal{D}_a}(z_1,z_2,z_3)\right)$$

from where, by theorem 4 , the graph $\mathcal{D}_c$ is majorized by the graph $\mathcal{D}_a$.

The graph g) (see Fig.53) has the norm $f_{\mathcal{D}_g}(z)$ equal to

$$f_{\mathcal{D}_g}(z)=\min_x\Big\{m\left(|z_1-x_3|+|x_1|+|x_2-z_3|+|z_2-x_4|\right)+$$

$$+M\left(|z_1-x_1|+|x_1-x_2|+|x_2-z_2|+|z_3-x_3|+|x_3-x_4|+|x_4|\right)\Big\}=$$

$$=\min_x\Big\{(M-m)\left(|z_1-x_1|+|x_1-x_2|+|x_2-z_2|+|z_3-x_3|+|x_3-x_4|+|x_4|\right)+$$

$$+\frac{1}{2}m\Big(|z_1-x_1|+|x_1-x_2|+|x_2-z_2|+|z_1-x_3|+|x_3-x_4|+|x_4-z_2|+$$

$$+|x_1-x_2|+|x_1|+|x_2-z_3|+|z_3-x_3|+|x_3-x_4|+|x_4|+|z_1-x_1|+$$

$$+|x_1|+|z_2-x_4|+|x_4|+|z_2-x_2|+|x_2-z_3|+|z_1-x_3|+|x_3-z_3|\Big)\Big\}\geq$$

$$\geq M\left(|z_1-z_2|+|z_3|\right)+\frac{1}{2}m\left(|z_1|+|z_2|+|z_2-z_3|+|z_1-z_3|\right)\geq$$

$$\geq \min \left[f_{\mathfrak{D}_a}(z_1, z_2, z_3),\ f_{\mathfrak{D}_a}(z_2, z_1, z_3) \right]$$

from where, by theorem 4 the graph $\mathfrak{D}_j$ is majorized by the graph $\mathfrak{D}_a$.

16. Meson-Nucleon Scattering

According to §6, part I, the class $\mathfrak{R}_o$ of the primitive diagrams of meson-nucleon scattering consists of the 14 graph of Fig.36. On the basis of Symanzik's theorem and its generalization it will be shown in 16 a) that the primitive diagrams c), d), g), h), j), k), and n) of this Figure are majorized by the diagram a).

It will be established in 16 b) and 16 c) that the graphs l) and i) are majorized by the graph e), while the graph m) is majorized by the graph f).

Thus, any graph of meson-nucleon scattering from $\mathfrak{R}$ is majorized by at least one of the following four graphs (see Fig.54),

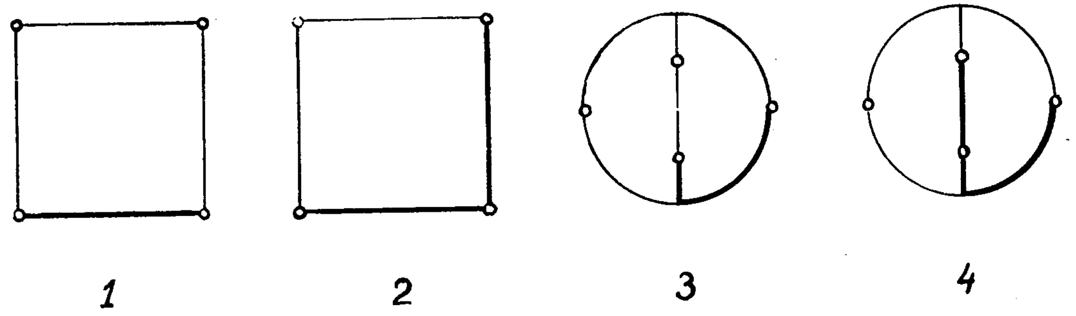


Fig.54

i.e., $G_{\mathcal{R}} = G_{\mathcal{R}_0} = \bigcap\limits_{\alpha=1}^{4} H(\mathcal{D}_\alpha)$.

16 a. Consider first the graphs d), h), j), k) of Fig.36. The graph d) (see Fig.55) has the norm $f_{\mathcal{D}_d}(\mathbf{z})$ equal to

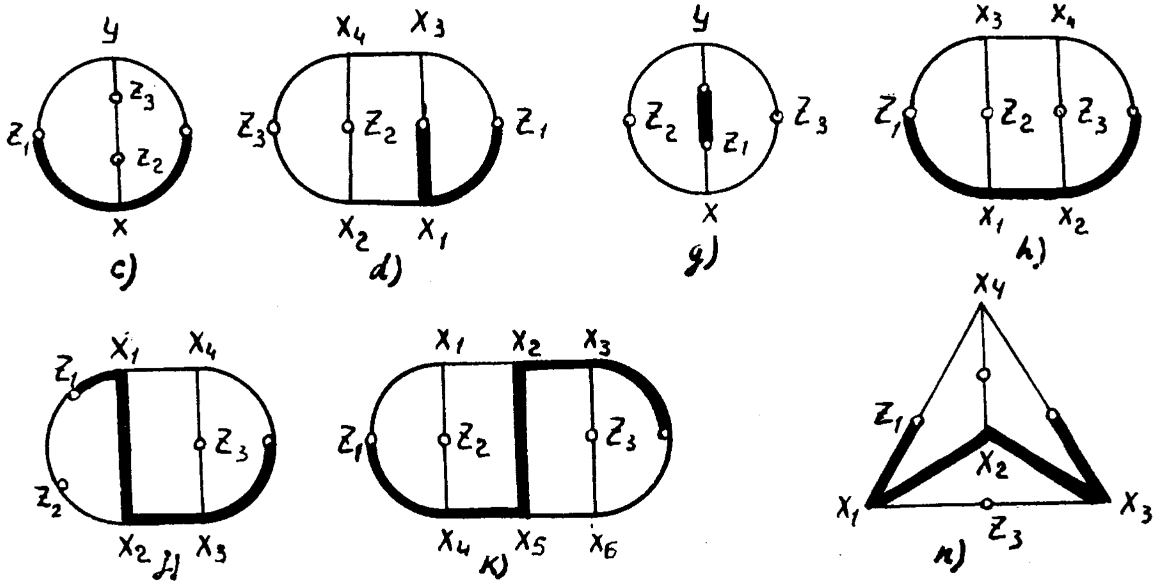


Fig.55

$$f_{\mathcal{D}_d}(z) = \min_x \Big\{ M(|z_1 - x_1| + |x_1|) +$$

$$+ m(|z_1 - x_3| + |x_3| + |x_3 - x_4| + |x_1 - x_2| + |x_2 - z_2| + |z_2 - x_4| + |z_3 - x_2| + |x_4 - z_3|\Big\} =$$

$$= \min_x \Big\{ (M-m)(|z_1 - x_1| + |x_1|) + \frac{1}{2} m \Big[(|z_1 - x_1| + |x_1| + |z_1 - x_3| + |x_3|) +$$

$$+ (|z_1 - x_1| + |x_1 - x_2| + |x_2 - z_2| + |z_1 - x_3| + |x_3 - x_4| + |x_4 - z_2|) +$$

$$+ (|z_2 - x_2| + |x_2 - z_3| + |z_2 - x_4| + |x_4 - z_3|) +$$

$$+ (|z_3 - x_4| + |x_4 - x_3| + |x_3| + |z_3 - x_2| + |x_1 - x_2| + |x_1|)\Big]\Big\} \geq$$

$$\geq f_{\mathcal{D}_a}(z_1, z_2, z_3)$$

where

$$f_{\mathcal{D}_a}(z_1, z_2, z_3) = M|z_1| + m(|z_1 - z_2| + |z_2 - z_3| + |z_3|). \quad (3.3)$$

It follows from here that the graph $\mathcal{D}_d$ is majorized by the graph $\mathcal{D}_a$. The graph h) (see Fig.55) has the norm $f_{\mathcal{D}_h}(z)$ equal to

$$f_{\mathcal{D}_h}(z) = \min_x \Big\{ M(|z_1 - x_1| + |x_1 - x_2| + |x_2|) +$$

$$+ m(|z_1 - x_3| + |x_3 - z_2| + |z_2 - x_1| + |z_3 - x_2| + |z_3 - x_4| + |x_3 - x_4| + |x_4|)\Big\} \geq$$

$$\geq \min_{x_1, x_2} (M-m)(|z_1 - x_1| + |x_1 - x_2| + |x_2|) +$$

$$+ m \cdot \min_{x_1, x_2} \{ (|z_1 - x_1| + |x_1 - x_2| + |x_2|) + (|z_1 - x_1| + |x_1 - z_2|) +$$

$$+ (|z_2 - x_1| + |x_1 - x_2| + |x_2 - z_3|) + (|z_3 - x_2| + |x_2|) \} \geq$$

$$\geq f_{\mathcal{D}_a}(z_1, z_2, z_3)$$

where $f_{\mathcal{D}_a}(z_1,z_2,z_3)$ is determined by formula (3.3)

The graph j) (see Fig. 55) has the norm $f_{\mathcal{D}_j}(z)$ equal to

$$f_{\mathcal{D}_j}(z) = \min_{x} \{ M (|z_1 - x_1| + |x_1 - x_2| + |x_2 - x_3| + |x_3|) +$$

$$+ m (|z_1 - z_2| + |z_2 - x_2| + |x_1 - x_4| + |z_3 - x_3| + |x_4 - z_3| + |x_4|) \} \geq$$

$$\geq \min \{ (M-m) (|z_1 - x_1| + |x_1 - x_2| + |x_2 - x_3| + |x_3|) +$$

$$+ \frac{1}{2} m [(|z_1 - x_1| + |x_1 - x_2| + |x_2 - x_3| + |x_3| + |z_1 - x_1| + |x_1 - x_4| + |x_4|) +$$

$$+ (|z_2 - x_2| + |x_1 - x_2| + |x_1 - x_4| + |x_4 - z_3| + |z_2 - x_2| + |x_2 - x_3| + |x_3 - z_3|) +$$

$$+ (|z_3 - x_4| + |x_4| + |z_3 - x_3| + |x_3|)] \} \geq$$

$$\geq f_{\mathcal{D}_a}(z_1, z_2, z_3).$$

Therefore the graph $\mathcal{D}_j$ is majorized by the graph $\mathcal{D}_a$.
The graph k) (see Fig. 55) has the norm $f_{\mathcal{D}_k}(z)$ equal to

$$f_{\mathcal{D}_k}(z) = \min_x \Big\{ M\,(|z_1 - x_4| + |x_4 - x_5| + |x_5 - x_2| + |x_2 - x_3| + |x_3|) +$$

$$+ m\,(|z_1 - x_1| + |x_1 - z_2| + |x_4 - z_2| + |x_1 - x_2| + |x_5 - x_6| + |x_3 - z_3| + |x_6 - z_3| + |x_6|)\Big\}$$

$$\geq (M-m)\min_x\,(|z_1 - x_4| + |x_4 - x_5| + |x_5 - x_2| + |x_2 - x_3| + |x_3|) +$$

$$+ \frac{m}{2}\cdot\min_x \Big\{ (|z_1 - x_4| + |x_4 - x_5| + |x_5 - x_2| + |x_2 - x_3| + |x_3| +$$

$$+ |z_1 - x_1| + |x_1 - x_2| + |x_2 - x_5| + |x_5 - x_6| + |x_6|) +$$

$$+ (|z_1 - x_4| + |x_4 - z_2| + |z_1 - x_1| + |x_1 - z_2|) +$$

$$+ (|z_2 - x_4| + |x_4 - x_5| + |x_5 - x_6| + |x_6 - z_3| + |z_2 - x_1| + |x_1 - x_2| + |x_2 - x_3| + |x_3 - z_3|) +$$

$$+ (|z_3 - x_6| + |x_6| + |z_3 - x_3| + |x_3|) \Big\} \geq$$

$$\geq f_{\mathcal{D}_a}(z_1, z_2, z_3).$$

Consequently, the graph $\mathcal{D}_k$ is majorized by the graph $\mathcal{D}_a$.
Consider now the graphs c), g), and n). The graph c) (see Fig. 55) has the norm $f_{\mathcal{D}_c}(z)$ equal to

$$f_{\mathcal{D}_c}(z) = \min_{x,y} \left\{ M(|z_1 - x| + |x|) + m(|z_2 - x| + |z_2 - z_3| + |z_1 - y| + |y - z_3| + |y|) \right\} \geq$$

$$\geq (M-m) \min_{x,y} (|z_1 - x| + |x|) + m|z_2 - z_3| +$$

$$+ \frac{m}{2} \cdot \min_{x,y} \left\{ (|z_1 - x| + |x| + |z_1 - y| + |y|) + \right.$$

$$+ (|z_2 - x| + |x| + |z_3 - y| + |y|) +$$

$$\left. + (|z_1 - x| + |z_2 - x| + |z_1 - y| + |y - z_3|) \right\} \geq$$

$$\geq M|z_1| + \frac{m}{2}(|z_1 - z_2| + |z_1 - z_3| + |z_2| + |z_3|) + m|z_2 - z_3| \geq$$

$$\geq \min \left[f_{\mathcal{D}_a}(z_1, z_2, z_3), f_{\mathcal{D}_a}(z_1, z_3, z_2) \right].$$

It follows, then, that the graph $\mathcal{D}_c$ is majorized by the graph $\mathcal{D}_a$.

The graph g) (see Fig.55) has the norm $f_{\mathcal{D}_g}(z)$ equal to

$$f_{\mathcal{D}_g}(z) = \min_{x,y} \left\{ M|z_1| + m(|z_1 - x| + |x - z_2| + |x - z_3| + |z_2 - y| + |y| + |y - z_3|) \right\} \geq$$

$$\geq M|z_1| + \frac{m}{2} \min_{x,y} \left\{ (|x - z_2| + |x - z_3|) + (|z_1 - x| + |x - z_2|) + \right.$$

$$\left. + (|z_1 - x| + |x - z_3|) + (|z_2 - y| + |y|) + (|y - z_3| + |y|) + (|z_2 - y| + |y - z_3|) \right\} \geq$$

$$\geq \min \left[f_{\mathcal{D}_a}(z_1, z_2, z_3), f_{\mathcal{D}_a}(z_1, z_3, z_2) \right].$$

From this follows that the graph $\mathcal{D}_g$ is majorized by the graph $\mathcal{D}_a$.

The graph n) (see Fig.55) has a norm equal to

$$f_{\mathcal{D}_n}(z) = \min_x \Big\{ M\,(|z_1 - x_1| + |x_1 - x_2| + |x_2 - x_3| + |x_3|) +$$

$$+ m\,(|z_3 - x_1| + |x_3 - z_3| + |z_1 - x_4| + |z_2 - x_2| + |z_2 - x_4| + |x_4|)\Big\} \geq$$

$$\geq \min_x\ (M-m)(|z_1 - x_1| + |x_1 - x_2| + |x_2 - x_3| + |x_3|) +$$

$$+ \frac{m}{2} \cdot \min_x \Big\{(|z_1 - x_1| + |x_1 - x_2| + |x_2 - x_3| + |x_3| + |z_1 - x_4| + |x_4|) +$$

$$+ (|z_2 - x_2| + |x_2 - x_1| + |x_1 - z_3| + |z_2 - x_2| + |x_2 - x_3| + |x_3 - z_3|) +$$

$$+ (|z_1 - x_4| + |x_4 - z_2|) + (|z_2 - x_4| + |x_4|) +$$

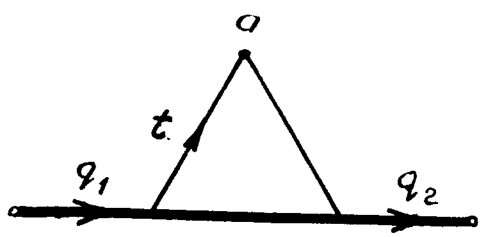
$$+ (|x_3| + |z_3 - x_3|) + (|z_1 - x_1| + |x_1 - z_3|)\Big\} \geq$$

$$\geq M|z_1| + m|z_2 - z_3| + \frac{m}{2}(|z_1 - z_2| + |z_1 - z_3| + |z_2| + |z_3|) \geq$$

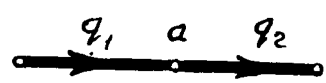
$$\geq \min\left[f_{\mathcal{D}_a}(z_1, z_2, z_3),\ f_{\mathcal{D}_a}(z_1, z_3, z_2)\right]$$

Consequently the $\mathcal{D}_n$ graph is majorized by the graph $\mathcal{D}_a$.

16b. Consider now the graphs l) and m) of Fig.36. Note that in these graphs there is a part drawn in Fig.56 a).



a)



b)

Fig.56

Prove the following lemma.

LEMMA 4. If in the graph $\mathcal{D}'$ there is a part drawn in Fig.56 a), then the domain $G(\mathcal{D}')$ of this graph contains the intersection of the domains $p^2 < 4m^2$ and $G(\mathcal{D})$, where p is the external momentum in the vertex a , $\mathcal{D}$ is the graph obtained from $\mathcal{D}'$ by substituting the part drawn in Fig. 56 a), by that drawn in Fig. 56 b).

Proof. The form $\mathcal{K}_{\mathcal{D}'}$ (see part I (1.4)) may be written as

$$\mathcal{K}_{\mathcal{D}'} = \alpha_1'(q_1^2 - M^2) + \alpha_2'(q_2^2 - M^2) + \alpha[(t+q_1)^2 - M^2] + \\ + \beta_1(t^2 - m^2) + \beta_2[(t-p)^2 - m^2] + \sum{}' \gamma_v(\kappa_v^2 - m_v^2). \tag{3.4}$$

Putting $t = \frac{p}{2}$ for $p^2 < 4m^2$ we have

$$\mathcal{K}_{\mathcal{D}'} \leq \alpha_1'(q_1^2-M^2)+\alpha_2'(q_2^2-M^2)+\alpha[(q_1+\tfrac{P}{2})^2-M^2]+\Sigma'\gamma_\nu(\kappa_\nu^2-m_\nu^2).$$

Since

$$(q_1+\frac{P}{2})^2=\frac{1}{2}(q_1+P)^2+\frac{1}{2}q_1^2-\frac{1}{4}P^2, \quad q_1+P=q_2,$$

then

$$(q_1+\frac{P}{2})^2 \leq \frac{1}{2}(q_1^2+q_2^2)$$

and

$$\mathcal{K}_{\mathcal{D}'} \leq \alpha_1(q_1^2-M^2)+\alpha_2(q_2^2-M^2)+\Sigma'\gamma_\nu(\kappa_\nu^2-m_\nu^2)=\mathcal{K}_{\mathcal{D}} \quad (3.5)$$

where

$$\alpha_1=\alpha_1'+\frac{\alpha}{2}, \quad \alpha_2=\alpha_2'+\frac{\alpha}{2}.$$

By Lemma 1 (see part I, 10) then follows that for $p^2<4m^2$ the inequality

$$Q_{\mathcal{D}'} \leq Q_{\mathcal{D}} \quad (3.6)$$

which was to be proved.

Applying the lemma proved earlier to the graphs l) and m) of Fig.36 we find that the graph l) is majorized by the graph e), while the graph m) - by the graph f), namely,

$$G(\mathcal{D}_l) \geqq H(\mathcal{D}_e), \quad G(\mathcal{D}_m) \geqq H(\mathcal{D}_f). \quad (3.7)$$

The condition $p^2<4m^2$ is taken into account automatically, according 7 of part I by changing the numeration of the meson vertices in the graphs e) and f).

16c. Consider, finally, the graph i) of Fig.36. In this graph there is a part drawn in Fig.57 a).

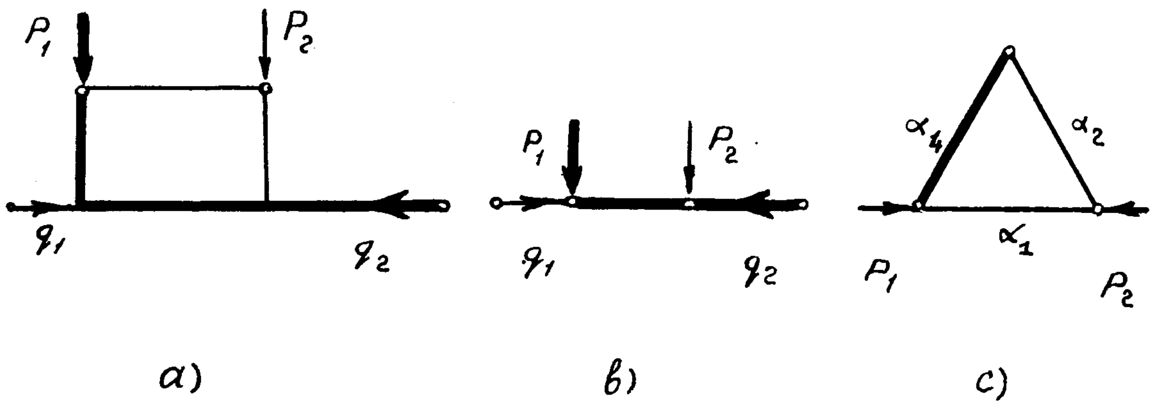


Fig.57

Prove the following lemma.

LEMMA 5. If the graph $\mathcal{D}'$ there is a part drawn in Fig.57 a), then the domain $G(\mathcal{D}')$ of this graph contains the intersection of the domains $G(\mathcal{D})$ of the graph $\mathcal{D}$ obtained from $\mathcal{D}'$ by substituting the part a) by the part b) of Fig.57, and the domain $G(\Delta)$ of the triangular drawn in Fig.57 c).

Proof. The form $\mathcal{K}_{\mathcal{D}'}$ of the graph $\mathcal{D}'$ minimized over the independent internal momentum corresponding to the loop of the part a) of Fig.57, is equal to

$$\overline{\mathcal{K}}_{\mathcal{D}'} = \frac{\alpha_1\alpha_2 p_2^2 + \alpha_1\alpha_3 (p_1+q_1)^2 + \alpha_1\alpha_4 p_1^2 + \alpha_2\alpha_3 q_2^2 + \alpha_2\alpha_4 (p_1+p_2)^2 + \alpha_3\alpha_4 q_1^2}{\alpha_1+\alpha_2+\alpha_3+\alpha_4} -$$

$$-(\alpha_1+\alpha_2)m^2 - (\alpha_3+\alpha_4)M^2 + \beta_1'(q_1^2-m^2) + \beta_2'(q_2^2-M^2) + \sum{}' \gamma_\nu (\kappa_\nu^2 - m_\nu^2)$$

Putting

$$\beta_1 = \beta_1' + \frac{\alpha_3\alpha_4}{\alpha_1+\alpha_2+\alpha_3+\alpha_4}, \quad \beta_2 = \beta_2' + \frac{\alpha_2\alpha_3}{\alpha_1+\alpha_2+\alpha_3+\alpha_4}, \quad \beta = \frac{\alpha_1\alpha_3}{\alpha_1+\alpha_2+\alpha_3+\alpha_4}$$

we write $\overline{\mathcal{K}}_{\mathcal{D}}'$ as follows

$$\overline{\mathcal{K}}_{\mathcal{D}'} = \mathcal{K}_{\mathcal{D}} + Q_\Delta \frac{\alpha_1+\alpha_2+\alpha_4}{\alpha_1+\alpha_2+\alpha_3+\alpha_4} - \frac{\alpha_3\{(\alpha_1+\alpha_2)m^2 + \alpha_3 M^2 + \alpha_4(2M^2-m^2)\}}{\alpha_1+\alpha_2+\alpha_3+\alpha_4} \tag{3.8}$$

where $\mathcal{K}_{\mathcal{D}}$ is the form of the graph $\mathcal{D}$ obtained from $\mathcal{D}'$ by substituting the part a) by the part b) of Fig.57

$$\mathcal{K}_{\mathcal{D}} = \beta_1(q_1^2-m^2) + \beta\left[(p_1+q_1)^2 - M^2\right] + \beta_2(q_2^2-M^2) + \sum{}'\gamma_\nu(\kappa_\nu^2-m_\nu^2);$$

Q_Δ - is the quadratic form of the graph drawn in Fig.57c):

$$Q_\Delta = \frac{\alpha_1\alpha_2 p_2^2 + \alpha_1\alpha_4 p_1^2 + \alpha_2\alpha_4(p_1+p_2)^2}{\alpha_1+\alpha_2+\alpha_4} - (\alpha_1+\alpha_2)m^2 - \alpha_4 M^2.$$

It follows from Lemma 1 and formula (3.8) that for $Q_\Delta < 0$

$$Q_{\mathcal{D}'} \leq Q_{\mathcal{D}}$$

what was to be proved.

Applying the Lemma proved earlier to the graph i) of Fig.36, we find that it is majorized by the graph e) of this figure, namely,

$$G(\mathcal{D}_i) \supseteq H(\mathcal{D}_e). \tag{3.9}$$

Indeed, in the case given, after substituting in the graph $\mathcal{D}_i$ the part a) by the part b) of Fig.57 we obtain the graph $\mathcal{D}_e$ (see Fig.58 a)).

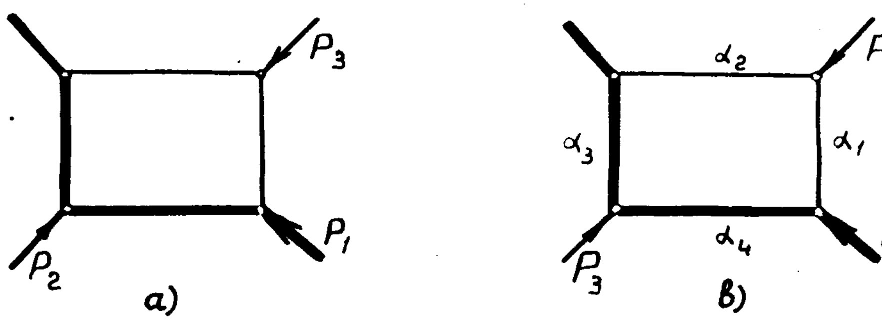


Fig.58

The change of the numeration of the external meson momenta in the graph of Fig.58 a) transforms in into the graph of Fig.58 b). Assuming $\alpha_3=0$ in the form Q of the graph of Fig.58 b) we obtain the form Q_Δ of the triangular graph of Fig.57 c).

Thus, the region $G(\Delta)$ of this triangular graph contains the region G of the graph of Fig.58 b), and, hence, (3.9) is valid.

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