ЛАБОРАТОРИЯ ТЕОРЕТИЧЕСКОЙ ФИЗИКИ

ОБЪЕДИНЕННЫЙ ИНСТИТУТ ЯДЕРНЫХ ИССЛЕДОВАНИЙ

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THE INTEGRAL EQUATION FOR THE LOW ENERGY πN -SCATTERING

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Abstract

A set of coupled integral equations for the low energy pion-nucleon S- and P-wave scattering amplitudes is derived by using the forward and the backward scattering dispersion relations only together with the unitary condition. The contribution from the cut in the unphysical region is taken into account without using analytic continuation by the Legendre expansion. The $N-\tilde{N}$ annihilation reaction amplitudes appear in the integral equations and represent explicitly the effect of the $\pi\pi$ -interaction.

1. Introduction

Recently, Chew and others [(1-3)] made new attempts to solve the problems of the strong interaction at low energy based on the two dimensional dispersion relation proposed by Mandelstam[(4-6)]. The integral equation originally given by Mandelstam involves two continuous independent variables, which is difficult to handle mathematically. Mandelstam's theory is based on the assumption, that only two particles intermediate states make significant contributions to the unitary conditions, which can then only be valid for the low energy phenomena. It is therefore convenient to subject the integral equation to a transformation, which transforms the two continuous independent variables denoting the energy and the momentum transfer respectively, into one continuous and one discrete variable, denoting such as the energy and the angular momentum. Since only states of small angular momentum are important in the low energy phenomena, all amplitudes of large angular momentum can be neglected in consistent with the approximation already made with regard to the unitary condition. The Mandelstam's integral equation of two independent variables is thereby transformed into a finite system of coupled integral equations of one independent variable, which is easier to handle mathematically.

In the works[(1-3)] mentioned above, the dispersion relations for the partial wave amplitudes are written down without difficulty. However, the unitary conditions in the unphysical region are obtained by analytic continuation with the help of Legendre expansion, which begins to break down at the boundary of the spectral functions. Even in the region before reaching the boundary of the spectral functions there is a substantial part, which is as far from the physical region as it is from the boundary of the spectral functions. Whether the higher partial wave amplitudes in the Legendre expansion can be neglected in this region is doubtful. The contribution to the integral equation from the cut in the unphysical region is therefore not as accurately represented as that from the cut in the physical region.

The circumstance is especially unfavourable in the problem of πN scattering, where the boundary of the spectral functions reaches down quite near the physical region. With the help of the two dimensional dispersion relation and the boundary of the spectral functions given by Mandelstam[6)] it can be shown, that the Legendre expansion begins to break down at $k^2 = -14.5$. k represents here the momentum of the π -meson in the center of mass-system. The mass of the π -meson is taken as unity. The neglect of the higher partial wave amplitudes can be justified only at a still narrower limit. On the other hand, a comparatively much higher cut off momentum is needed in the physical region in order to obtain the correct location of the (3.3) resonance energy. While the behaviour of the (3.3) amplitude is governed mainly by the contribution from the cut in the physical region, the influence of the contribution from the cut in the unphysical region on the other amplitudes is by no means inessential. It is therefore interesting to derive the integral equations in such a way, that the errors due to the approximations are reduced and are more evenly distributed between the physical and the unphysical region, and that the contribution from the cut in the unphysical region is more accurately taken into account.

In this paper, the integral equations for the S- and P-wave amplitudes of the πN -scattering are derived by using the forward and the backward scattering dispersion relations only together with the unitary condition. The forward and the backward scattering dispersion relations have the particular advantage, that the scattering amplitudes in the unphysical region can be expressed directly in terms of the amplitudes of the crossing reactions with no need of analytic continuation. The problem of the break down of the Legendre expansion does not occur. The integral equations are thus expected to take the contribution from the cut in the unphysical region into account more adequately. The backward scattering dispersion relation introduces the amplitudes of the annihilation reaction $N+\tilde{N} \rightarrow \pi+\pi$ into the integral equations and takes thereby the influence of the $\pi\pi$ interaction into account. On the other hand, the integral equations thus obtained have to be solved simultaneously with the integral equations for the $N-\tilde{N}$ annihilation. The method can be easily extended to treat other problems of the strong interaction.

Quite recently Efremov, Meshcherykov and Shirkov[7)] have obtained an interesting set of integral equations for the πN scattering under the assumption, that the $N-\tilde{N}$ annihilation reaction is dominated by the S- and P- states in the low energy region. They also exploited the advantage of the dispersion relation for the backward scattering[8)]. Their set of integral equations has the interesting advantage, that it can be solved without the know-ledge of the $N-\tilde{N}$ annihilation amplitudes, once the $\pi\pi$ scattering phase-shifts are given. The integral equation derived in this paper is suitable for the case, where the real part of the higher angular momentum states is not negligible in comparison with those of the S- and P-states of the $N-\tilde{N}$ annihilation.

In the second section of this paper, the location of the singularities of the partial wave scattering amplitudes and the range of validity of the Legendre expansion are discussed. In the third section, the dispersion relations for the forward and the backward scattering are written down. The connection between the S- and P-wave scattering amplitudes on the one hand and the forward and backward scattering amplitudes on the other are given. In the fourth section the integral equations for the S- and P-wave scattering amplitudes are derived. The results obtained are compared with that of Chew, Goldberger, Low and Nambu[9].

II. The Region of Validity of the Legendre Expansion

The location of the singularities and cuts of the π-N partial wave scattering amplitudes has been studied by MacDowell[10]. For the investigation of the range of validity of the Legendre expansion, it is more useful to give the location of the singularities a two dimensional representation, so that their relative positions with respect to the boundary of the spectral functions can be surveyed easily. The notations to be employed in this paper are those in the current use. However, for convenience's sake they are explained in the next paragraph.

As is well known, the following three processes

$$
\begin{array}{lllll}
\text{I.} & \pi(p_1,\alpha) + N(p_3) & \longrightarrow & \pi(-p_2,\beta) + N(-p_4) & \\
\text{II.} & \pi(p_2,\beta) + N(p_3) & \longrightarrow & \pi(-p_1,\alpha) + N(-p_4) & \qquad (1) \\
\text{III.} & N(p_3) + \tilde{N}(p_4) & \longrightarrow & \pi(-p_1,\alpha) + \pi(-p_2,\beta) &
\end{array}
$$

are described by one single Green function. π represents the π-meson, while N and $\tilde{N}$ represent the nucleon and the antinucleon respectively. The p's inside the brackets denote the corresponding momentum four vectors; α and β are the isotopic spin indices. In the momentum representation, the Green function T is of the following form:

$$
\begin{aligned}
T = \delta_{\beta\alpha} \left\{ -A^{+}(s,\bar{s},t) + \tfrac{i}{2}(\hat{p}_1 - \hat{p}_2)\, B^{+}(s,\bar{s},t) \right\} \qquad (2) \\
+ \tfrac{1}{2}[\tau_\beta, \tau_\alpha] \left\{ -A^{-}(s,\bar{s},t) + \tfrac{i}{2}(\hat{p}_1 - \hat{p}_2)\, B^{-}(s,\bar{s},t) \right\}
\end{aligned}
$$

where

$$
s = -(p_1 + p_3)^2 = -(p_2 + p_4)^2
$$

$$\begin{aligned} \bar{s} &= -(p_1 + p_4)^2 = -(p_2 + p_3)^2 \\ t &= -(p_1 + p_2)^2 = -(p_3 + p_4)^2 \\ s + \bar{s} + t &= 2m^2 + 2 \end{aligned} \tag{3}$$

Only two of the three variables s , $\bar{s}$, t are independent. m denotes here the mass of the nucleon. The invariant scattering functions $A^{\pm}$ and $B^{\pm}$ satisfy the following crossing relations.

$$\begin{aligned} A^{\pm}(s, \bar{s}, t) &= \pm A^{\pm}(\bar{s}, s, t) \\ B^{\pm}(s, \bar{s}, t) &= \mp B^{\pm}(\bar{s}, s, t) \end{aligned} \tag{4}$$

According to Mandelstam[4], they also satisfy the following two dimensional dispersion relations:

$$\begin{aligned} A^{\pm}(s, \bar{s}, t) = {} & \frac{1}{\pi^2}\int_{(m+1)^2}^{\infty} ds' \int_{(m+1)^2}^{\infty} d\bar{s}' \frac{a_{12}^{\pm}(s', \bar{s}')}{(s'-s)(\bar{s}'-\bar{s})} + \frac{1}{\pi^2}\int_{(m+1)^2}^{\infty} ds' \int_{4}^{\infty} dt' \frac{a_{13}^{\pm}(s', t')}{(s'-s)(t'-t)} \\ & + \frac{1}{\pi^2}\int_{(m+1)^2}^{\infty} d\bar{s}' \int_{4}^{\infty} dt' \frac{a_{23}^{\pm}(\bar{s}', t')}{(\bar{s}'-\bar{s})(t'-t)} \end{aligned}$$

$$\begin{aligned} B^{\pm}(s, \bar{s}, t) = {} & \frac{g^2}{m^2 - s} \mp \frac{g^2}{m^2 - \bar{s}} + \frac{1}{\pi^2}\int_{(m+1)^2}^{\infty} ds' \int_{(m+1)^2}^{\infty} d\bar{s}' \frac{b_{12}^{\pm}(s', \bar{s}')}{(s'-s)(\bar{s}'-\bar{s})} \\ & + \frac{1}{\pi^2}\int_{(m+1)^2}^{\infty} ds' \int_{4}^{\infty} dt' \frac{b_{13}^{\pm}(s', t')}{(s'-s)(t'-t)} + \frac{1}{\pi^2}\int_{(m+1)^2}^{\infty} d\bar{s}' \int_{4}^{\infty} dt' \frac{b_{23}^{\pm}(\bar{s}', t')}{(\bar{s}'-\bar{s})(t'-t)} \end{aligned} \tag{5}$$

$a^{\pm}$ and $b^{\pm}$ are the spectral functions. g is the renormalized and rationalized πN coupling constant. For the convenience of the discussion, the following variables in the center of mass system are introduced: k and ϕ denote the momentum and the scattering angle of the reaction (1) $\bar{k}$ and $\bar{\phi}$ denote those of the reaction (II); p, q and θ denote the momentum of the nucleon and antinucleon, the momentum of the π -meson and the reaction angle of the reaction(III) respectively. The following relations exist then for the reaction (1):

$$s = m^2 + 1 + 2k^2 + 2\sqrt{(m^2+k^2)(1+k^2)}$$
$$\bar{s} = m^2 + 1 - 2k^2 x - 2\sqrt{(m^2+k^2)(1+k^2)} \qquad (6)$$
$$t = -2k^2(1-x), \qquad x \equiv \cos\phi$$

The corresponding relations for the reaction (II) are:

$$s = m^2 + 1 - 2\bar{k}^2 \cdot \bar{x} - 2\sqrt{(m^2+\bar{k}^2)(1+\bar{k}^2)}$$
$$\bar{s} = m^2 + 1 + 2\bar{k}^2 + 2\sqrt{(m^2+\bar{k}^2)(1+k^2)} \qquad (7)$$
$$t = -2\bar{k}^2(1-\bar{x}), \qquad \bar{x} \equiv \cos\bar{\phi}$$

Those for the reaction (III) are

$$s = -p^2 - q^2 + 2pqz$$
$$\bar{s} = -p^2 - q^2 - 2pqz \qquad (8)$$
$$t = 4(m^2+p^2) = 4(1+q^2)$$
$$Z \equiv \cos\theta$$

The physical region of the reaction(I) is represented by the region I in Fig. la. It is limited by the conditions $s \geq (m+1)^2$, $-1 \leq x \leq 1$. It is therefore to the right of the straight line $S = (m + I)^2$ and lies between the boundaries.

$$t = 0, \qquad \bar{s}s = (m^2-1)^2 \qquad (9)$$

The physical region of the reaction (11) is denoted by II in Fig.la. It is limited by the conditions $\bar{s} \geq (m+1)^2$, $-1 \leq \bar{x} \leq 1$. It also lies between the boundaries (9), but situates to the left of the straight line $\bar{s} = (m + 1)^2$. The physical region of the reaction (111) is denoted by III in Fig. la. It is limited by the conditions $t \geq 4m^2$ and $-1 \leq z \leq 1$. Its lower boundary is therefore that part of the ourve

$$\bar{s}s = (m^2-1)^2 \qquad (10)$$

whioh lies above the straight line $t = 4m^2$. Thus the straight line t=0 connects the forward soattering of the reaction (l) with the forward scattering of the reaction (11);

the curve $\bar{s}s = (m^2-1)^2$ besides connecting the backward scattering of the reaction (1) with the backward scattering of the reaction (11), also connects the backward scattering of the reaction (1) with the forward and the backward scattering of the reaction (111). For $t \geq 4m^2$ and $t \leq 4$ the equation $\bar{s}s = (m^2-1)^2$ describes the two branches of one hyperbola shown in Fig.1a. For $4 \leq t \leq 4m^2$, s and $\bar{s}$ become complex, the equation describes an ellipse touching the hyperbola at both ends. This ellipse is shown in Fig.1c.

The boundary of the spectral functions is also schematically shown in Fig.1a as dashed curves. The region, in which $a_{12}^{\pm}$, $b_{12}^{\pm}$ do not vanish, is marked 12, those corresponding to $a_{13}^{\pm}$, $b_{13}^{\pm}$ and $a_{23}^{\pm}$, $b_{23}^{\pm}$ are marked by 13 and 23 respectively.

The singularities of the partial wave scattering amplitudes of the reaction (1) come from two sources. The first source is the functional dependence of $\bar{s}$, t on s, $x \equiv \cos\phi$. From (6) follows

$$k^2 = \frac{1}{4s}\{s-(m+1)^2\}\{s-(m-1)^2\} \tag{11}$$

It gives a singularity at S = 0, which is one of the asymptotes of the hyperbola $\bar{s}s = (m^2-1)^2$. It is easily seen from Fig.1a, that one end of this asymptote approaches the backward scattering boundary of the reaction (11), while the other end approaches the forward reaction boundary of the reaction (111).

The second source of singularities is the vanishing of the various denominators in the dispersion relations (5). The vanishing of the denominator $\frac{1}{s'-s}$ gives rise to the region of singularities lying to the right of the line $S = (m+1)^2$. It is denoted by a in Fig.1b and is identical with the whole physical region of the reactions (1). The singularity coming from the first pole term $\frac{g^2}{m^2-s}$ is designated as b in Fig.1b. It is a small segment from the line $s = m^2$. The region of singularities arising from the vanishing of the denominator $\frac{1}{\bar{s}'-\bar{s}}$ situates to the left of the line $\bar{s} = (m+1)^2$. It consists of two parts designated by c and d respectively in Fig.1b. C is a part of the physical region of the reaction (11), but d lies entirely in the unphysical region. Its largest portion is covered by the area, where the spectral functions $a_{23}^{\pm}$ and $b_{23}^{\pm}$ do not vanish and the Legendre expansion fails. The singularities coming from the second pole term $\mp\frac{g^2}{m^2-\bar{s}}$ also consist of two parts, designated as e and f in Fig.1b. The region of singularities arising from the vanishing of the denominator $\frac{1}{t'-t}$ is shown in Fig.1c. It consists of two parts g and h. g is a part of the cylinder surface

$$s^*s = (m^2-1)^2 \tag{12}$$

one end of which is limited by $t = 4$, while the other end is limited by the ellipse $\bar{s}\, s = (m^2-1)^2$ mentioned above. There is a substantial part of the cylinder surface, where the Legendre expansion fails due to the limitation from the spectral functions $a_{23}^{\pm}$ and $b_{23}^{\pm}$. The region h is limited by the boundary $t = 4$, S=0 and the upper branch of the hyperbola $\bar{s}\, s = (m^2-1)^2$. The largest part of the region h is covered by the area, where $a_{23}^{\pm}$ and $b_{23}^{\pm}$ do not vanish and the Legendre expansion fails.

The singularities and cuts of the partial wave scattering amplitudes of the reaction (1) in the complex S-plane can be obtained by projecting Fig.1b and 1c on the complex S-plane. They are shown in Fig.2. The correspondence between Fig.2 and Fig.1b,c is obvious and needs no explanation.

The most stringent limitation on the validity of the Legendre expansion occurs on the cut g in Fig.2, which corresponds to the cylinder surface g in Fig.1c. The imaginary parts of the invariant scattering amplitudes on g are to be obtained by analytic continuation with the help of Legendre expansion from the reaction amplitudes of the reaction (111). According to the theorem of Heine[11], the region of validity of the expansion at a fixed value of t is the inside of an ellipse, which will be termed as the Lehmann ellipse in the following to avoid confusion with the ellipse described by $\bar{s}\, s = (m^2-1)^2$. The Lehmann ellipse passes through the boundary of the spectral functions and possesses foci at $z \equiv \cos\theta = \pm 1$, which lie on the curve $\bar{s}\, s = (m^2-1)^2$ as shown above. Brief calculations gives the equation of the Lehmann ellipse in the region $4 \leq t \leq 4m^2$ as:

$$\frac{x^2}{\xi^2} + \frac{y^2}{\eta^2} = 1 \tag{13}$$

where

$$x = \operatorname{Im} s, \qquad y = m^2 + 1 - \frac{t}{2} - \operatorname{Re} s \tag{14}$$
$$\xi^2 = \eta^2 - 4p^2q^2, \qquad \eta = \bar{s}_0(t) + \frac{t}{2} - m^2 - 1$$

$\bar{s}_0(t)$ is here the boundary of the spectral functions $a_{23}^{\pm}$ and $b_{23}^{\pm}$. The equation (13) for the cylinder surface can be rewritten as:

$$x^2 + \left(y + \frac{t}{2} - m^2 - 1\right)^2 = (m^2-1)^2 \tag{15}$$

The limit of the region of validity of the Legendre expansion is determined by the inter-section of the Lehmann ellipse with the cylinder surface, which can be obtained by solving the equations (13) and (15), using the boundary of the spectral functions given by Mandelstam. The most stringent limit occurs at $k^2 = -14{,}5$ as mentioned in the introduction. The corresponding values of t and $x \equiv \cos\phi$ are t =12, ϕ =54º respectively.

Of course, the range of validity of the Legendre expansion for the real part of the invariant scattering functions is very much narrower as pointed out by Lehmann[12]. It is not determined by the boundary of the spectral functions, but by the lower limit of the integration over $d\bar{s}'$ in (5). It occurs at k^2 =-2.36, which is very stringent indeed.

III. The Forward and the Backward Dispersion Relations

The dispersion relations for the pion-nucleon partial wave scattering amplitudes have been written down by MacDowell[10]. Together with the unitary condition, they can be used as the integral equations for the πN -scattering after suitable analytic continuation, The discussion of the previous section shows however, that the integral equations thus obtained do not take the contribution from the unphysical region accurately due to the early failure of the Legendre expansion. On the other hand, it is expected, that partial waves other than that of the (3.3) state receive important influences from the contribution of the unphysical region. It is therefore interesting to investigate the possibility of avoiding this difficulty.

The previous discussion also shows, that the limit of the range of validity of the Legendre expansion varies for different scattering angles. For the forward scattering, the dispersion integral is taken along the path t=0: The scattering amplitudes of the reaction (1) in the unphysical region are directly connected with those of the reaction (11) in the physical region with $\bar{x} \equiv \cos\bar{\phi}$ =1. No analytic continuation is thus needed. For the backward scattering the dispersion integral is taken along the path $\bar{s}s = (m^2-1)^2$. Besides being connected with the backward scattering of the reaction (11) in the physical region, the scattering amplitudes of the reaction (1) in the unphysical region are also directly connected with those of the reaction (111) with $z \equiv \cos\theta = \pm 1$. Once the unitary condition for the reaction (111) is analytically continued into the region $4 \leq t \leq 4m^2$ as shown by Mandelstam[13], no further analytic continuation is needed in this case either. The most unfavourable case occurs at ϕ =54° as mentioned before, where the Legendre expansion breaks down at k^2 = -14,5. For the low energy πN scattering, it is know experimentally, that only a small number of angular momentum states are important, while all other states are negligible. To determine these small number of scattering amplitudes, only dispersion relations for a small number of angles are needed. These angles can be choosen such a way, that the Legendre expansion fails only in distant regions. In fact, in the energy region, where waves other than those of the S- and P-states are negligible only dispersion relations for two different angles are needed. The obvious choice is then $\phi = 180^\circ$ together with $\phi = 0^\circ$. The integral equations thus obtained are expected to take the contribution from the unphysical region into account more adequately.

More waves can be taken into account by employing more dispersion relations at other angles. For the case of πN scattering, it is advantageous to choose angles near the backward direction, since the angle worst for the Legendre expansion is $\phi = 54^\circ$, which is nearer to the forward direction. Recently, Efremov, Meshcherykov and Shirkov[7] have obtained an interesting set of integral equations for the πN scattering, which take into account the effect of $\pi\pi$ interaction, but in which the amplitudes of the reaction (111) do not appear. Besides assuming, that the reaction (111) is dominated by the S- and P-wave in the low energy region, they also exploited this advantage of the dispersion relation for scattering angle near the backward direction[8].

The forward scattering dispersion relation is well known. They are:

$$A^{\pm}(s, \cos\phi = 1) = \frac{1}{\pi}\int_{(m+1)^2}^{\infty} ds' \left\{ \frac{1}{s'-s} \pm \frac{1}{s'-\bar{s}_+} \right\} A_1^{\pm}(s', 1)$$

$$B^{\pm}(s, 1) = \frac{g^2}{m^2 - s} \mp \frac{g^2}{m^2 - \bar{s}_+} + \frac{1}{\pi}\int_{(m+1)^2}^{\infty} ds' \left\{ \frac{1}{s'-s} \mp \frac{1}{s'-\bar{s}_+} \right\} B_1^{\pm}(s', 1) \tag{16}$$

$A_1^{\pm}$ and $B_1^{\pm}$ are defined in Mandelstam's paper[4], which coincide with the imaginary parts of $A^{\pm}$ and $B^{\pm}$ in the physical region of the reaction (1). $\bar{s}_+$ is defined as

$$\bar{s}_+ \equiv 2m^2 + 2 - s \tag{17}$$

It is also straightforward to write down the dispersion relation for the backward scattering. The location of the singularities and cuts is given in Fig.1 and Fig.2. The pole terms are separated out first. The integration contour is choosen as shown in Fig.3. The backward dispersion relation are then:

$$A^{\pm}(s, \cos\phi = -1) = \frac{1}{\pi}\int_{(m+1)^2}^{\infty} ds' \frac{A_1^{\pm}(s', -1)}{s'-s} - \frac{1}{\pi}\int_{0}^{(m-1)^2} ds' \frac{A_2^{\pm}(s', -1)}{s'-s}$$

$$- \frac{1}{\pi}\left\{ \int_{g_+} + \int_{g_-} - \int_{-m^2+1}^{0} + \int_{-\infty}^{-m^2+1} \right\} ds' \frac{A_3^{\pm}(s', -1)}{s'-s} \tag{18}$$

$$B^{\pm}(s, -1) = \frac{g^2}{m^2 - s} \mp \frac{g^2}{m^2 - \bar{s}_-} + \frac{1}{\pi}\int_{(m+1)^2}^{\infty} ds' \frac{B_1^{\pm}(s', -1)}{s'-s} - \frac{1}{\pi}\int_{0}^{(m-1)^2} ds' \frac{B_2^{\pm}(s', -1)}{s'-s}$$

$$-\frac{1}{\pi}\left\{\int_{g_+}+\int_{g_-}-\int_{-m^2+1}^{0}+\int_{-\infty}^{-m^2+1}\right\}ds'\ \frac{B_3^{\pm}(s',-1)}{s'-s}$$

$A_2^{\pm}$, $B_2^{\pm}$ and $A_3^{\pm}$, $B_3^{\pm}$ are also given in Mandelstam's paper. They coincide with the imaginary parts of $A^{\pm}$, $B^{\pm}$ in the physical region of the reaction (11) and (111) respectively. The contour g_+ is the upper half circle proceeding in the clockwise direction, while the contour g_- is the lower half circle proceeding in the anti-clockwise direction as shown in Fig.3. $\bar{s}_-$ is defined as

$$\bar{s}_- \equiv \frac{1}{s}(m^2-1)^2 \tag{19}$$

The signs before $A_2^{\pm}$, $B_2^{\pm}$ and $A_3^{\pm}$, $B_3^{\pm}$ occuring in the integrals have to be determined by examining the signs of the small imaginary parts of $\bar{s}$ and t occuring in the dispersion relations (5). For the sake of illustration, the determination of the sign before $A_3^{\pm}$ in the integral along g_+ is shown in the following. For the backward scattering, $t=-4k^2$. On the upper half circle s is of the form

$$s = \rho e^{i\varphi}, \qquad 0 \leq \varphi \leq \pi \tag{20}$$

Using (11), we obtain immediately:

$$\operatorname{Im} t = \frac{1}{\rho}\left\{(m^2-1)^2-\rho^2\right\}\sin\varphi \tag{21}$$

Thus t has a negative imaginary part outside the upper half circle, and a positive imaginary part inside the upper half circle. The sign before $A_3^{\pm}$ in the integral along g_+ has to be negative, a result opposite to that of Mac-Dowell[10].

The dispersion relations (18) can be put into a form similar to (16). The integrals in (20) are actually taken along the curve $\bar{s}s=(m^2-1)^2$. It is advantageous to subject some of the integrals to the transformation:

$$\bar{s} = \frac{1}{s}(m^2-1)^2 \tag{22}$$

In particular,

$$-\frac{1}{\pi}\int_0^{(m-1)^2} ds'\ \frac{A_2^{\pm}(s',-1)}{s'-s} = \frac{1}{\pi}\int_{(m+1)^2}^{\infty} d\bar{s}'\cdot\frac{\bar{s}_-}{\bar{s}'}\cdot\frac{A_2^{\pm}(s',-1)}{\bar{s}'-\bar{s}_-} \tag{23}$$

$$\frac{1}{\pi}\int_{-m^2+1}^{0} ds'\ \frac{A_3^{\pm}(s',-1)}{s'-s} = -\frac{1}{\pi}\int_{-\infty}^{-m^2+1} d\bar{s}'\cdot\frac{\bar{s}_-}{\bar{s}'}\cdot\frac{A_3^{\pm}(s',-1)}{\bar{s}'-\bar{s}_-}$$

Applying the crossing relations to (23), we obtain

$$-\frac{1}{\pi}\int_{0}^{(m-1)^2} ds' \; \frac{A_2^{\pm}(s',-1)}{s'-s} \;=\; \pm\frac{1}{\pi}\int_{(m+1)^2}^{\infty} ds' \cdot \frac{\bar{s}_-}{s'} \cdot \frac{A_1^{\pm}(s',-1)}{s'-\bar{s}_-}$$
$$\frac{1}{\pi}\int_{-m^2+1}^{0} ds' \; \frac{A_3^{\pm}(s',-1)}{s'-s} \;=\; \mp\frac{1}{\pi}\int_{-\infty}^{-m^2+1} ds' \cdot \frac{\bar{s}_-}{s'} \cdot \frac{A_3^{\pm}(s',-1)}{s'-\bar{s}_-} \tag{24}$$

Similary, we get

$$-\frac{1}{\pi}\int_{0}^{(m-1)^2} ds' \; \frac{B_2^{\pm}(s',-1)}{s'-s} \;=\; \mp\frac{1}{\pi}\int_{(m+1)^2}^{\infty} ds' \cdot \frac{\bar{s}_-}{s'} \cdot \frac{B_1^{\pm}(s',-1)}{s'-\bar{s}_-} \tag{25}$$
$$\frac{1}{\pi}\int_{-m^2+1}^{0} ds' \; \frac{B_3^{\pm}(s',-1)}{s'-s} \;=\; \pm\frac{1}{\pi}\int_{-\infty}^{-m^2+1} ds' \cdot \frac{\bar{s}_-}{s'} \cdot \frac{B_3^{\pm}(s',-1)}{s'-\bar{s}_-}$$

Furthermore, the integral over g_+ is the complex conjugate of the integral over g_- for physical value of s. (20) can then be transformed into the following form:

$$A^{\pm}(s,-1) = \frac{1}{\pi}\int_{(m+1)^2}^{\infty} ds' \left\{ \frac{1}{s'-s} \pm \frac{\bar{s}_-}{s'} \cdot \frac{1}{s'-\bar{s}_-} \right\} A_1^{\pm}(s',-1) - \frac{2}{\pi}\,\mathrm{Re}\int_{g_-} ds' \; \frac{A_3^{\pm}(s',-1)}{s'-s}$$
$$- \frac{1}{\pi}\int_{-\infty}^{-m^2+1} ds' \left\{ \frac{1}{s'-s} \pm \frac{\bar{s}_-}{s'} \cdot \frac{1}{s'-\bar{s}_-} \right\} A_3^{\pm}(s',-1)$$
$$\tag{26}$$
$$B^{\pm}(s,-1) = \frac{g^2}{m^2-s} \mp \frac{g^2}{m^2-\bar{s}_-} + \frac{1}{\pi}\int_{(m+1)^2}^{\infty} ds' \left\{ \frac{1}{s'-s} \mp \frac{\bar{s}_-}{s'} \cdot \frac{1}{s'-\bar{s}_-} \right\} B_1^{\pm}(s',-1)$$
$$- \frac{2}{\pi}\,\mathrm{Re}\int_{g_-} ds' \; \frac{B_3^{\pm}(s',-1)}{s'-s} - \frac{1}{\pi}\int_{-\infty}^{-m^2+1} ds' \left\{ \frac{1}{s'-s} \mp \frac{\bar{s}_-}{s'} \cdot \frac{1}{s'-\bar{s}_-} \right\} B_3^{\pm}(s',-1)$$

If D- and higher waves can be neglected in comparison with the S-waves, F- and higher waves can be neglected in comparison with the P -waves, the S- and P- wave scattering amplitudes $f^{\pm}_{S_{\frac{1}{2}}}(s)$, $f^{\pm}_{P_{\frac{1}{2}}}(s)$, $f^{\pm}_{P_{3/2}}(s)$ can be easily expressed in terms of the forward and the backward scattering amplitudes:

$$f^{\pm}_{S_{\frac{1}{2}}}(s) \cong \frac{1}{2}\left\{ f_1^{\pm}(s,1) + f_1^{\pm}(s,-1) \right\}$$
$$f^{\pm}_{P_{\frac{1}{2}}}(s) - f^{\pm}_{P_{3/2}}(s) \cong \frac{1}{2}\left\{ f_2^{\pm}(s,1) + f_2^{\pm}(s,-1) \right\} \tag{27}$$
$$f^{\pm}_{P_{3/2}}(s) \cong \frac{1}{6}\left\{ f_1^{\pm}(s,1) - f_1^{\pm}(s,-1) \right\}$$

The expression $f_1^{\pm}$ and $f_2^{\pm}$ are defined by Chew, Goldberger, Low and Nambu[9] as:

$$f_1^{\pm}(s, x \equiv \cos\phi) = \frac{E+m}{8\pi\sqrt{s}} \left\{ A^{\pm}(s,x) + (\sqrt{s}-m)\, B^{\pm}(s,x) \right\}$$
$$f_2^{\pm}(s, x) = \frac{E-m}{8\pi\sqrt{s}} \left\{ -A^{\pm}(s,x) + (\sqrt{s}+m)\, B^{\pm}(s,x) \right\} \tag{28}$$

E is here the energy of the nucleon in the center of mass system.

IV. The Integral Equations

The next step is to express $A_1^{\pm}$, $B_1^{\pm}$ in terms of the imaginary parts of the πN partial wave scattering amplitudes and $A_3^{\pm}$, $B_3^{\pm}$ in terms of the imaginary parts of the $N\text{-}\tilde{N}$ annihilation amplitudes. The expressions for $A_1^{\pm}$ and $B_1^{\pm}$ can be obtained directly from (27) and (28).

$$\frac{1}{4\pi} A_1^{\pm}(s,1) \cong \frac{\sqrt{s}+m}{E+m}\, \mathrm{Im}\left\{ f_{s_{\frac{1}{2}}}^{\pm}(s) + 3 f_{p_{\frac{3}{2}}}^{\pm}(s) \right\} - \frac{\sqrt{s}-m}{E-m}\, \mathrm{Im}\left\{ f_{p_{\frac{1}{2}}}^{\pm}(s) - f_{p_{3/2}}^{\pm}(s) \right\}$$

$$\frac{1}{4\pi} A_1^{\pm}(s,-1) \cong \frac{\sqrt{s}+m}{E+m}\, \mathrm{Im}\left\{ f_{s_{\frac{1}{2}}}^{\pm}(s) - 3 f_{p_{3/2}}^{\pm}(s) \right\} - \frac{\sqrt{s}-m}{E-m}\, \mathrm{Im}\left\{ f_{p_{\frac{1}{2}}}^{\pm}(s) - f_{p_{\frac{3}{2}}}^{\pm}(s) \right\}$$

$$\frac{1}{4\pi} B_1^{\pm}(s,1) \cong \frac{1}{E+m}\, \mathrm{Im}\left\{ f_{s_{\frac{1}{2}}}^{\pm}(s) + 3 f_{p_{3/2}}^{\pm}(s) \right\} + \frac{1}{E-m}\, \mathrm{Im}\left\{ f_{p_{\frac{1}{2}}}^{\pm}(s) - f_{p_{\frac{3}{2}}}^{\pm}(s) \right\} \tag{29}$$

$$\frac{1}{4\pi} B_1^{\pm}(s,-1) \cong \frac{1}{E+m}\, \mathrm{Im}\left\{ f_{s_{\frac{1}{2}}}^{\pm}(s) - 3 f_{p_{\frac{3}{2}}}^{\pm}(s) \right\} + \frac{1}{E-m}\, \mathrm{Im}\left\{ f_{p_{\frac{1}{2}}}^{\pm}(s) - f_{p_{\frac{3}{2}}}^{\pm}(s) \right\}$$

To express $A_3^{\pm}$ and $B_3^{\pm}$ in terms of the reaction amplitudes of the reaction (111), it is neccessary first of all to obtain the relation between S, $x \equiv \cos\phi$ of the reaction (1) and t, $z \equiv \cos\theta$ of the reaction (111). It follows from (6) and (11), that for the backward scattering

$$t = 2m^2 + 2 - s - \frac{1}{s}(m^2-1)^2 \tag{30}$$

It can be shown, that on the contours of integrations occuring in (26)

$$z = \frac{s-\bar{s}}{4pq} = -1 \tag{31}$$

The $N\text{-}\tilde{N}$ annihilation reaction has been studied by Fraser and Fulco[2]. It is straightforward to express $A_3^{\pm}$ and $B_3^{\pm}$ in terms of the N-$\tilde{N}$ annihilation partial wave helicity amplitudes $f_{\pm}^{I,J}$ introduced by them. I and J denote here the isotopic spin and the total angular momentum of the N-$\tilde{N}$ system respectively. "+" and "−" refer here to the two helicity states. Taking into account the unitary condition for the reaction (111), we obtain after simple calculation:

$$A_3^{+}(s,-1) = \frac{8\pi}{\sqrt{6}} \sum_{J}^{\text{Even}} (J+\tfrac{1}{2}) \frac{(pq)^J}{p^2} \left\{ \frac{m}{2}\sqrt{J(J+1)}\, \mathrm{Im} f_-^{0,J}(t) - \mathrm{Im} f_+^{0,J}(t) \right\}$$

$$A_3^{-}(s,-1) = 4\pi \sum_{J}^{\text{Odd}} (J+\tfrac{1}{2}) \frac{(pq)^J}{p^2} \left\{ -\frac{m}{2}\sqrt{J(J+\tfrac{1}{2})}\, \mathrm{Im} f_-^{1,J}(t) + \mathrm{Im} f_+^{1,J}(t) \right\} \qquad (32)$$

$$B_3^{+}(s,-1) = -\frac{4\pi}{\sqrt{6}} \sum_{J}^{\text{Even}} (J+\tfrac{1}{2}) \sqrt{J(J+1)}\, (pq)^{J-1} \mathrm{Im} f_-^{0,J}(t)$$

$$B_3^{-}(s,-1) = 2\pi \sum_{J}^{\text{Odd}} (J+\tfrac{1}{2}) \sqrt{J(J+1)}\, (pq)^{J-1} \mathrm{Im} f_-^{1,J}(t)$$

where on the contour g_-

$$s = (m^2-1)\, e^{-i\varphi}, \qquad 0 \le \varphi \le \pi$$

$$t = 2m^2 + 2 - 2(m^2-1)\cos\varphi \qquad (33)$$

$$pq = \frac{i}{2}(m^2-1)\sin\varphi$$

For $-\infty < s \le -m^2+1$

$$t = 2m^2 + 2 - s - \frac{1}{s}(m^2-1)^2$$

$$pq = \frac{1}{4}\left\{ \frac{1}{s}(m^2-1)^2 - s \right\} \qquad (34).$$

It is interesting to note, that on the contour g_- A_3^{+} and B_3^{-} are real, but A_3^{-} and B_3^{+} are pure imaginary. The first few terms of the expansion (32) might be sufficient to represent $A_3^{\pm}$ and $B_3^{\pm}$ for values of t not far larger than 4, which is the region giving important contribution to the dispersion relations. However, this also belongs to the unphysical region of the reaction 111, where no experiment is available to us. The actual number of terms to be kept has to be determined in the course of solving the integral equation.

Using the definition

$$\omega \equiv \sqrt{s} - m$$

$$K_{\pm}(s',s) \equiv \left\{ \frac{1}{s'-s} \pm \frac{\bar{s}_-}{s'} \cdot \frac{1}{s'-\bar{s}_-} \right\} \tag{35}$$

we obtain from (16), (26), (27), (28), (29) the following equations, which in combination with the unitary condition give the integral equations for the πN scattering. The following equations are for the amplitude $f^{\pm}_{s_{\frac{1}{2}}}(s)$.

$$f^{\pm}_{s_{\frac{1}{2}}}(s) = P^{\pm}_{s_{\frac{1}{2}}}(s) + I^{\pm}_{s_{\frac{1}{2}}}(s) + II^{\pm}_{s_{\frac{1}{2}}}(s) + III^{\pm}_{s_{\frac{1}{2}}}(s)$$

$$P^{\pm}_{s_{\frac{1}{2}}}(s) = \frac{\omega(E+m)g^2}{16\pi\sqrt{s}} \left\{ \frac{2}{m^2-s} \mp \left(\frac{1}{m^2-\bar{s}_+} + \frac{1}{m^2-\bar{s}_-} \right) \right\}$$

$$I^{\pm}_{s_{\frac{1}{2}}}(s) = \frac{1}{\pi}\int_{(m+1)^2}^{\infty} ds' \frac{1}{s'-s} \left\{ \frac{E+m}{E'+m} \cdot \frac{\sqrt{s'}+\sqrt{s}}{2\sqrt{s}} \operatorname{Im} f^{\pm}_{s_{\frac{1}{2}}}(s') + \frac{E+m}{2\sqrt{s}} \cdot \frac{\sqrt{s}-\sqrt{s'}}{E'-m} \operatorname{Im}\left[f^{\pm}_{p_{\frac{1}{2}}}(s') - f^{\pm}_{p_{\frac{3}{2}}}(s') \right] \right\}$$

$$\begin{aligned} II^{\pm}_{s_{\frac{1}{2}}}(s) = \pm \frac{E+m}{4\pi\sqrt{s}} \int_{(m+1)^2}^{\infty} ds' \Big\{ & 3K_-(s',s) \frac{2m+\omega'-\omega}{E'+m} \operatorname{Im} f^{\pm}_{p_{\frac{3}{2}}}(s') \\ & + K_+(s',s) \left[\frac{2m+\omega'-\omega}{E'+m} \operatorname{Im} f^{\pm}_{s_{\frac{1}{2}}}(s') - \frac{\omega'+\omega}{E'-m} \operatorname{Im}\left(f^{\pm}_{p_{\frac{1}{2}}}(s') - f^{\pm}_{p_{\frac{3}{2}}}(s') \right) \right] \Big\} \end{aligned} \tag{36}$$

$$\begin{aligned} III^{\pm}_{s_{\frac{1}{2}}}(s) = -\frac{E+m}{16\pi\sqrt{s}} \cdot \frac{1}{\pi} \Big\{ & \int_{-\infty}^{-m^2+1} ds' \left[K_{\pm}(s',s)\, A_3^{\pm}(s',-1) + \omega K_{\mp}(s',s)\, B_3^{\pm}(s',-1) \right] \\ & + 2 \operatorname{Re} \int_{g_-} ds' \frac{1}{s'-s} \left[A_3^{\pm}(s',-1) + \omega B_3^{\pm}(s',-1) \right] \Big\} \end{aligned}$$

Of course, the expressions $A_3^{\pm}$ and $B_3^{\pm}$ in $III^{\pm}_{s_{\frac{1}{2}}}(s)$ are to be substituted by the expressions (32). The corresponding equations for $f^{\pm}_{p_{\frac{1}{2}}}(s) - f^{\pm}_{p_{\frac{3}{2}}}(s)$ are:

$$f^{\pm}_{p_{\frac{1}{2}}}(s) - f^{\pm}_{p_{\frac{3}{2}}}(s) = P^{\pm}_{p_{\frac{1}{2}-\frac{3}{2}}}(s) + I^{\pm}_{p_{\frac{1}{2}-\frac{3}{2}}}(s) + II^{\pm}_{p_{\frac{1}{2}-\frac{3}{2}}}(s) + III^{\pm}_{p_{\frac{1}{2}-\frac{3}{2}}}(s)$$

$$P^{\pm}_{p_{\frac{1}{2}-\frac{3}{2}}}(s) = \frac{(\sqrt{s}+m)(E-m)}{16\pi\sqrt{s}} \left\{ \frac{2}{m^2-s} \mp \left(\frac{1}{m^2-\bar{s}_+} + \frac{1}{m^2-\bar{s}_-} \right) \right\}$$

$$I^{\pm}_{p_{\frac{1}{2}-\frac{3}{2}}}(s) = \frac{1}{\pi}\int_{(m+1)^2}^{\infty} ds' \frac{1}{s'-s} \left\{ \frac{E-m}{E'-m} \cdot \frac{\sqrt{s'}+\sqrt{s}}{2\sqrt{s}} \operatorname{Im}\left[f^{\pm}_{p_{\frac{1}{2}}}(s') - f^{\pm}_{p_{\frac{3}{2}}}(s') \right] + \frac{E-m}{E'+m} \cdot \frac{\sqrt{s}-\sqrt{s'}}{2\sqrt{s}} \operatorname{Im} f^{\pm}_{s_{\frac{1}{2}}}(s') \right\} \tag{37}$$

$$\mathrm{II}^{\pm}_{p_{\frac12}-\frac32}(s) = \mp \frac{E-m}{4\pi\sqrt{s}} \int\limits_{(m+1)^2}^{\infty} ds' \Big\{ 3K_-(s';s)\, \frac{2m+\sqrt{s'}+\sqrt{s}}{E'+m}\, \mathrm{Im}\, f^{\pm}_{p_{\frac32}}(s')$$

$$+ K_+(s';s)\Big[\frac{2m+\sqrt{s'}+\sqrt{s}}{E'+m}\, \mathrm{Im}\, f^{\pm}_{s_{\frac12}}(s') + \frac{2m+\omega-\omega'}{E'-m}\, \mathrm{Im}\big(f^{\pm}_{p_{\frac12}}(s') - f^{\pm}_{p_{\frac32}}(s')\big)\Big]\Big\}$$

$$\mathrm{III}^{\pm}_{p_{\frac12}-\frac32}(s) = -\frac{E-m}{16\pi\sqrt{s}}\cdot\frac{1}{\pi}\Big\{\int\limits_{-\infty}^{-m^2+1} ds'\big[-K_{\pm}(s';s)\, A_3^{\pm}(s',-1) + (\sqrt{s}+m)K_{\mp}(s';s)\, B_3^{\pm}(s',-1)\big]$$

$$+ 2\,\mathrm{Re}\int_{g_-} ds' \frac{1}{s'-s}\big[-A_3^{\pm}(s',-1) + (\sqrt{s}+m)\, B_3^{\pm}(s',-1)\big]\Big\}$$

The equations for the amplitude $f^{\pm}_{p_{\frac32}}(s)$ are

$$f^{\pm}_{p_{\frac32}}(s) = P^{\pm}_{p_{\frac32}}(s) + \mathrm{I}^{\pm}_{p_{\frac32}}(s) + \mathrm{II}^{\pm}_{p_{\frac32}}(s) + \mathrm{III}^{\pm}_{p_{\frac32}}(s)$$

$$P^{\pm}_{p_{\frac32}}(s) = \mp \frac{\omega(E+m)\, g^2}{48\pi\sqrt{s}}\Big\{\frac{1}{m^2-\bar{s}_+} - \frac{1}{m^2-\bar{s}_-}\Big\}$$

$$\mathrm{I}^{\pm}_{p_{\frac32}}(s) = \frac{1}{\pi}\int\limits_{(m+1)^2}^{\infty} ds' \frac{1}{s'-s}\cdot\frac{E+m}{E'+m}\cdot\frac{\sqrt{s'}+\sqrt{s}}{2\sqrt{s}}\, \mathrm{Im}\, f^{\pm}_{p_{\frac32}}(s') \tag{38}$$

$$\mathrm{II}^{\pm}_{p_{\frac32}}(s) = \pm \frac{E+m}{12\pi\sqrt{s}} \int\limits_{(m+1)^2}^{\infty} ds' \Big\{3K_+(s';s)\, \frac{2m+\omega'-\omega}{E'+m}\, \mathrm{Im}\, f^{\pm}_{p_{\frac32}}(s')$$

$$+ K_-(s';s)\Big[\frac{2m+\omega'-\omega}{E'+m}\, \mathrm{Im}\, f^{\pm}_{s_{\frac12}}(s') - \frac{\omega'+\omega}{E'-m}\, \mathrm{Im}\big(f^{\pm}_{p_{\frac12}}(s') - f^{\pm}_{p_{\frac32}}(s')\big)\Big]\Big\}$$

$$\mathrm{III}^{\pm}_{p_{\frac32}}(s) = -\frac{1}{3}\, \mathrm{III}^{\pm}_{s_{\frac12}}(s)$$

These equations are rather lengthy, but their structure is simple. Each of the amplitudes is a superposition of four terms $P^{\pm}$, $\mathrm{I}^{\pm}$, $\mathrm{II}^{\pm}$, $\mathrm{III}^{\pm}$. $P^{\pm}$ is the pole contribution, while $\mathrm{I}^{\pm}$, $\mathrm{II}^{\pm}$, $\mathrm{III}^{\pm}$ are the contributions from the regions of the reactions (I), (II), (III) respectively.

Since the amplitudes of the reaction (III) appear in the above equations, the integral equations have to be solved simultaneously with the integral equations for the N-$\tilde{N}$ annihilation.

It is interesting to compare the equations (36),(37),(38) with the corresponding result of Chew, Goldberger, Low and Nambu[9]. The main difference is of course the terms $III^{\pm}_{s_{\frac{1}{2}}}$, $III^{\pm}_{p_{\frac{1}{2}}-\frac{3}{2}}$, $III^{\pm}_{p_{\frac{3}{2}}}$ appearing in our equations, which are entirely absent in their equations. These terms represent explicitly the effects of the $\pi\pi$ interaction, since IM $f^{I,J}_{\pm}$ vanish within the two mesons approximation of the unitary condition, if the $\pi\pi$ interaction vanishes.

To compare the pole contributions and the contributions from the regions of the reactions (1) and (11), it is convenient to neglect $\mathcal{I}m\, f^{\pm}_{s_{\frac{1}{2}}}$ and $\mathcal{I}m\, f^{\pm}_{p_{\frac{1}{2}}}$ in comparison with $\mathcal{I}m\, f^{\pm}_{p_{\frac{3}{2}}}$ and drop terms smaller than the main terms by a factor of $\left(\frac{\omega}{m}\right)^2$ as they have done. The following approximate expressions for $\mathcal{P}^{\pm}_{s_{\frac{1}{2}}}$, $I^{\pm}_{s_{\frac{1}{2}}}$ and $II^{\pm}_{s_{\frac{1}{2}}}$ are then obtained.

$$\begin{aligned}
\mathcal{P}^{\pm}_{s_{\frac{1}{2}}}(s) &\cong -\frac{g^2}{4\pi}\cdot\frac{1}{2\sqrt{s}}\left\{\left(1-\frac{\omega}{2m}\right)\pm\left(1+\frac{\omega}{2m}\right)\right\} \\
I^{\pm}_{s_{\frac{1}{2}}}(s) &\cong \frac{2m^2}{\pi\sqrt{s}}\int_1^{\infty}\frac{d\omega'}{k'^2}\left\{1+\frac{\omega'}{2m}-\frac{\omega}{2m}\right\}\mathcal{I}m\, f^{\pm}_{p_{\frac{3}{2}}}(s') \\
II^{\pm}_{s_{\frac{1}{2}}}(s) &\cong \pm\frac{2m^2}{\pi\sqrt{s}}\int_1^{\infty}\frac{d\omega'}{k'^2}\left\{1-\frac{\omega'}{2m}-\frac{\omega}{2m}\right\}\mathcal{I}m\, f^{\pm}_{p_{\frac{3}{2}}}(s')
\end{aligned} \tag{39}$$

The approximate expression for $\mathcal{P}^{\pm}_{p_{\frac{1}{2}}-\frac{3}{2}}$, $I^{\pm}_{p_{\frac{1}{2}}-\frac{3}{2}}$ and $II^{\pm}_{p_{\frac{1}{2}}-\frac{3}{2}}$ are:

$$\begin{aligned}
\mathcal{P}^{\pm}_{p_{\frac{1}{2}}-\frac{3}{2}}(s) &\cong -\frac{f^2k^2}{\omega}\left\{\left(1-\frac{\omega}{2m}\right)\pm\left(1+\frac{\omega}{2m}\right)\right\}\left(1-\frac{\omega}{2m}\right) \\
I^{\pm}_{p_{\frac{1}{2}}-\frac{3}{2}}(s) &\cong -\frac{1}{\pi}\int_1^{\infty}d\omega'\,\frac{k^2}{k'^2}\left\{\frac{1}{\omega'-\omega}+\frac{1}{m}\right\}\mathcal{I}m\, f^{\pm}_{p_{\frac{3}{2}}}(s') \\
II^{\pm}_{p_{\frac{1}{2}}-\frac{3}{2}} &\cong \pm\frac{1}{\pi}\int_1^{\infty}d\omega'\,\frac{k^2}{k'^2}\left\{\frac{1}{\omega'+\omega}-\frac{1}{m}\right\}\mathcal{I}m\, f^{\pm}_{p_{\frac{3}{2}}}(s')
\end{aligned} \tag{40}$$

f is here renormalized but unrationalized pseudo-vector coupling constant defined as:

$$f^2 \equiv \frac{g^2}{16\pi m^2} \tag{41}$$

The approximate expressions for $\mathcal{P}^{\pm}_{p_{\frac{3}{2}}}$, $I^{\pm}_{p_{\frac{3}{2}}}$ and $II^{\pm}_{p_{\frac{3}{2}}}$ are:

$$\mathcal{P}^{\pm}_{p_{\frac{3}{2}}}(s) \cong \pm\frac{2f^2k^2}{3\omega} \tag{42}$$

$$I^{\pm}_{p_{\frac{3}{2}}}(s) \cong \frac{1}{\pi}\int_1^\infty d\omega' \left\{ \frac{1}{\omega'-\omega} + \frac{1}{m} \right\} \mathrm{Im}\, f^{\pm}_{p_{\frac{3}{2}}}(s')$$

$$\mathrm{II}^{\pm}_{p_{\frac{3}{2}}}(s) \cong \pm \frac{1}{\pi}\int_1^\infty \frac{d\omega'}{\omega'+\omega} \left\{ 1 + \frac{2(\omega+\omega')^2}{3k'^2}\left[1 - \frac{k^2}{(\omega+\omega')^2}\right] - \frac{2\omega}{m} - \frac{2(\omega+\omega')^3}{3mk'^2} \right\}$$

Thus the pole contributions and $I^{\pm}_{p_{\frac{1}{2}}-\frac{3}{2}}$ are exactly the same as those obtained by Chew, Goldberger, Low and Nambu within the approximation made. $I^{\pm}_{s_{\frac{1}{2}}}$, $\mathrm{II}^{\pm}_{s_{\frac{1}{2}}}$ and $\mathrm{II}^{\pm}_{p_{\frac{1}{2}}-\frac{3}{2}}$ differ from their corresponding expressions only by terms smaller than the main terms by a factor $\frac{\omega}{m}$. But $I^{\pm}_{p_{\frac{3}{2}}}$ and $\mathrm{II}^{\pm}_{p_{\frac{3}{2}}}$ look quite different. Actually, our equations (38) are of the unsubtracted form. If a subtraction is made at $\omega = 1$, $I^{\pm}_{p_{\frac{3}{2}}}$ becomes similar to their result, but $\mathrm{II}^{\pm}_{p_{\frac{3}{2}}}$ remains different. This would probably cause some changes in the behaviour of the (3.3) resonance.

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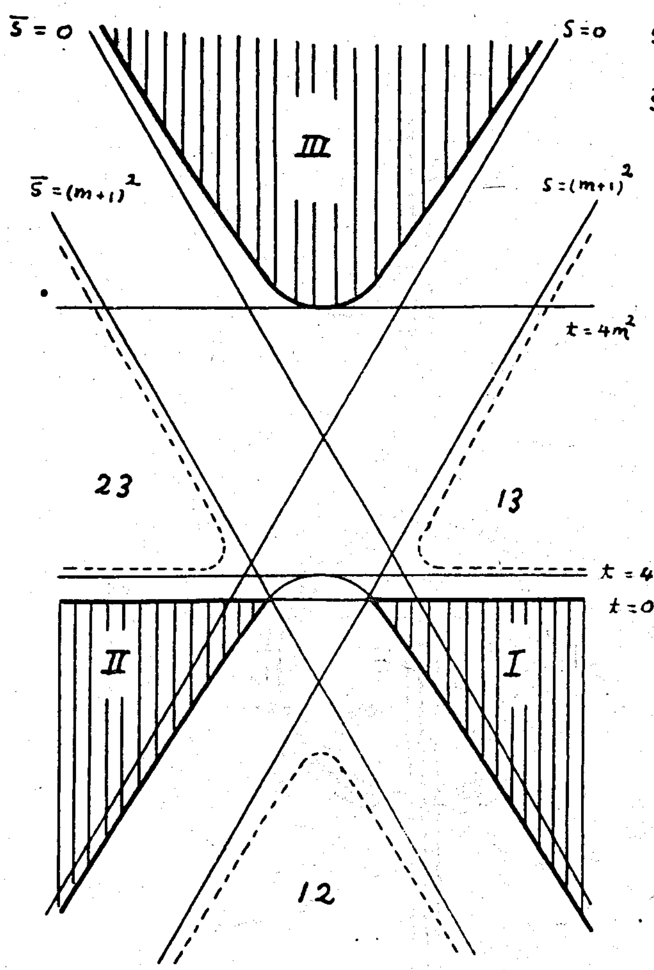

Fig.1a

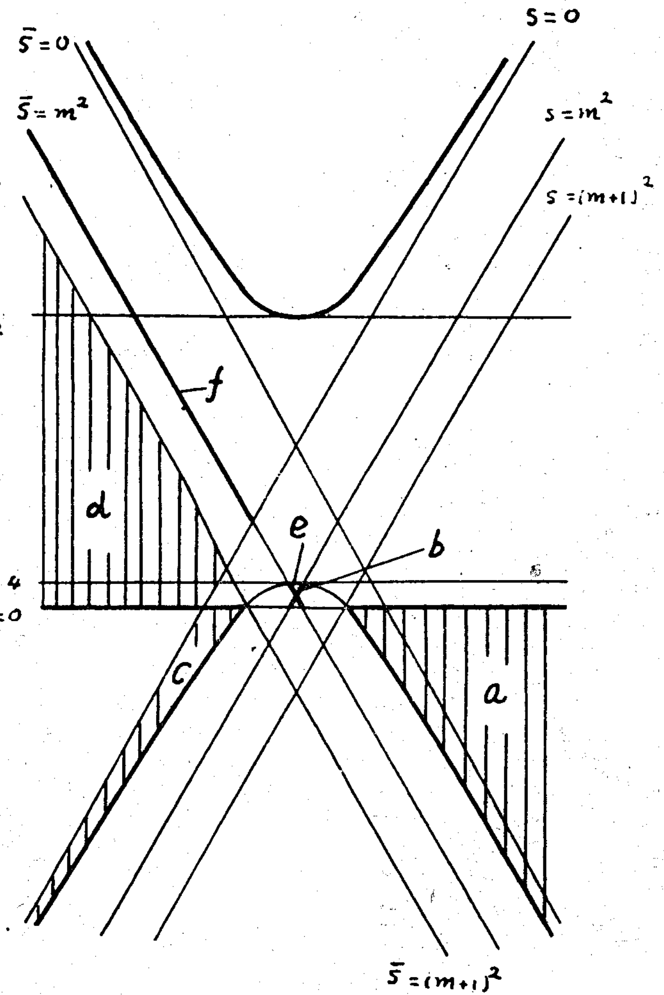

Fig.1b

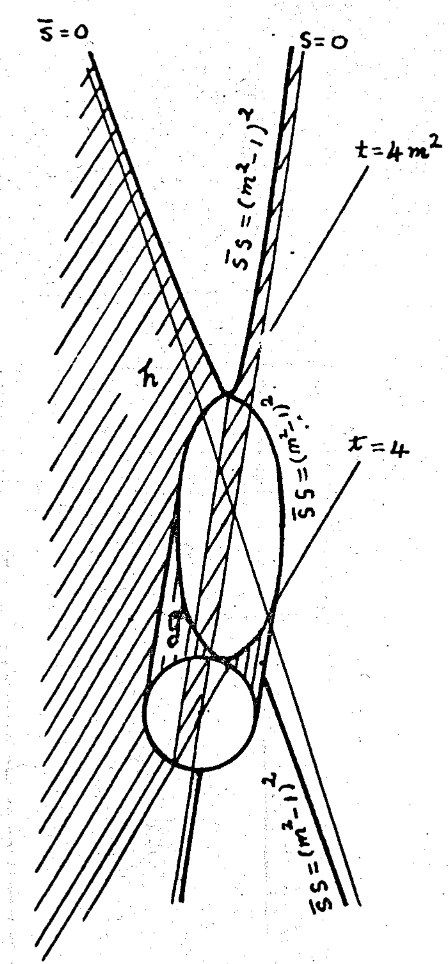

Fig.1c

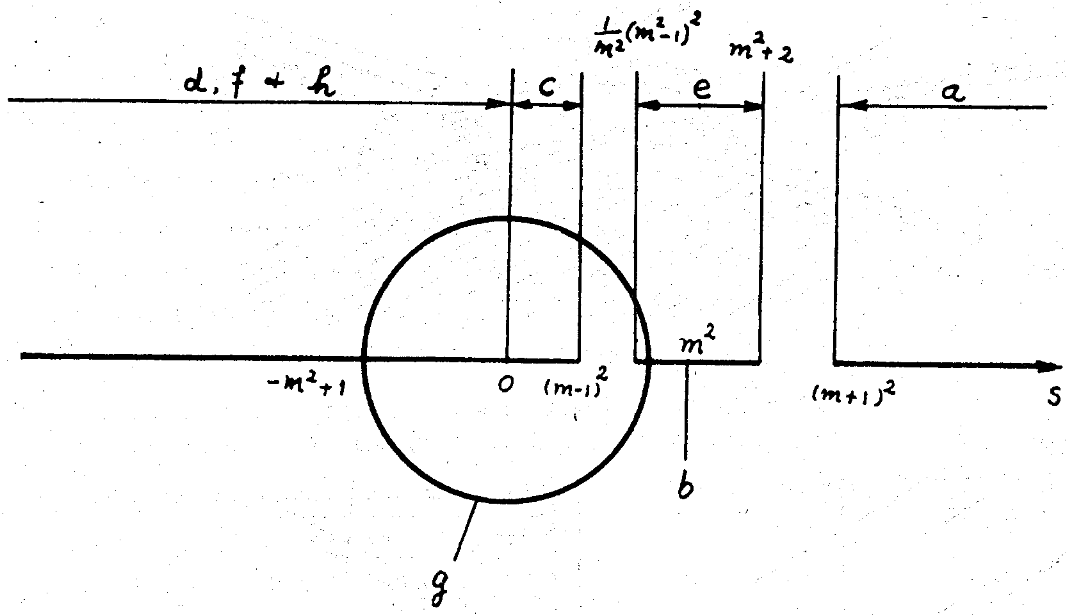

Fig.2

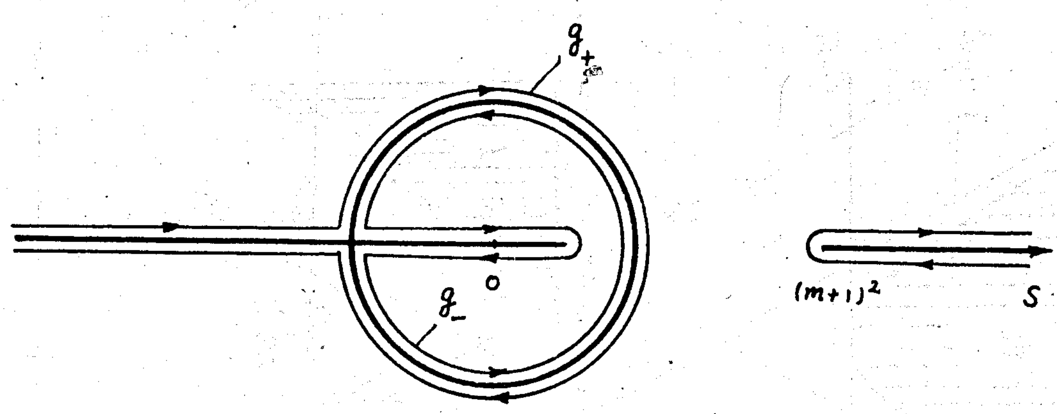

Fig.3