ОБЪЕДИНЕННЫЙ ИНСТИТУТ ЯДЕРНЫХ ИССЛЕДОВАНИЙ

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AXIOMATIC METHOD AND PERTURBATION THEORY

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1. The usual approach to the quantum field theory based on the Hamiltonian formalism was born as an immediate transfer to the field theory of the way, that lead from the classical to the quantum mechanics. To specify the theory one prescribes thereat the form of the Lagrangian. Then, one obtains as a result of variation the equations of motion which after the well-known quantisation procedure turn into the Heisenberg equations for the operator field functions. The theory can be in fact formulated in this way merely in the limits of the perturbation expansion. Indeed, we cannot even write down the equations (the removing-of-infinities-prescriptions!) and not only solve them, otherwise then in terms of successive powers of the coupling constant.

This inherent obstacles of the Hamiltonian method have brought to an other approach – it is often called 'axiomatic' one, though this name doesn't seem to us the best. Namely, one tries to build up the theory on the basis of certain fundamental physical requirements which the solutions of the equations must satisfy without dealing with these equations explicitly. This way becames recently the subject of main interest in connection with the dispersion relations – the only exact result in quantum field theory.

The basic physical principles of the axiomatic method may be formulated in different ways. Thus, we may requite, for instance, the Heisenberg fields commuting on any space-like hypersurface to exist at each point – the pursuits in this direction have been made by Lehmann, Symanzik and Zimmermann (see /1,2/ and numerous further papers). On the other hand, we can follow the programme suggested by Heisenberg /3/ and restrict ourselves to treating the scattering matrix. The latter way was chosen by Bogolubov, Polivanov and the author /4/ * in connection with the theory of dispersion relations **.

In any version of the axiomatic approach there arise natural questions about the compatibility of the system of 'axioms' introduced and its sufficiency in order to define (with what ambiguity?) the theory. The first of this questions can find no definite answer as yet since the existence of a non-contradictory scheme of the quantum field theory is not established at all. The aim of the present paper is the study of the second question. Namely, we shall show that once the perturbation theory is adopted, the formal expansion of the scattering matrix in powers of the coupling constant follows from the basic principles of the axiomatic approach supplemented by assumptions on the transformation properties of the fields considered and about the degrees of growth of the matrix elements with the same ambiguity as in the usual theory.

2. We shall start from the system of basic principles as formulated in PTDR, Sec.2. Specifying the transformation properties of the fields we restrict ourselves for the sake of simplicity to the case of one scalar

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* Hereafter referred to as PTDR.

** The system of basic principles used in PTDR has originated from such a system proposed earlier by Bogolubov [5] within the framework of perturbation theory and the hypothesis of the adiabatic switching on and off of the interaction.

field /out-field φ (x)/. We shall write the (extended out of the energy shell) functional expansion of the scattering matrix in normal products of the φ (x) as

$$S = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int dx_1 \dots dx_n \;\phi^n(x_1, \dots, x_n) \;:\varphi(x_1) \dots \varphi(x_n): \qquad /1/$$

Here $\phi^n(x_1, \dots, x_n)$ are c-functions symmetrical in their arguments. By the n-fold variational differentiation we obtain for these functions the expression

$$\phi^n(x_1, \dots, x_n) = i^n \langle 0 | \frac{\delta^n S}{\delta\varphi(x_1) \dots \delta\varphi(x_n)} | 0 \rangle ,$$

which on making use of the vacuum stability condition (PTDR, I,(6)), may be brought into the form

$$\phi^n(x_1, \dots, x_n) = i^n \langle 0 | \frac{\delta^n S}{\delta\varphi(x_1) \dots \delta\varphi(x_n)} S^+ | 0 \rangle , \qquad /2/$$

more convenient for our further purposes.

By the property PTDR, II(1) the functions $\phi^n(x_1, \dots, x_n)$ with all arguments different are to be generalized functions integrable in one of the classes C (q,r). Should this be true for the coinsident arguments, too, their Fourier transforms $\tilde{\phi}^n(p_1, \dots, p_n)$ defined by the relation

$$\int dx_1 \dots dx_n \, e^{i \sum_1^n xp} \phi^n(x_1, \dots, x_n) = (2\pi)^4 \delta(p_1 + \dots + p_n) \, \tilde{\phi}^n(p_1, \dots, p_n) \qquad /3/$$

(the δ-function comes to take into account the translation invariance) would also be generalized function integrable in some classes C(q',r') and , hence, $\tilde{\phi}(p)$ would be polynomially bounded when any of the momenta tend to infinity. We impose upon the $\tilde{\phi}(p)$ a weaker condition (held in the usual theory) — that of being polynomially bounded at an uniform extension of all the momenta. Namely, we demand to exist for any n a finite growth index – the minimum integer $\Omega(n)$ such that when all the momenta are extended uniformly,

$$p_1 = \xi_1 P \; ; \; \dots \; ; \; p_n = \xi_n P \; ; \qquad P \to \infty , \qquad /4/$$

the function $\tilde{\phi}(\xi, P)$ increases slower than $P^{\Omega(n)+\alpha}$ for any $\alpha > 0$.

Now, to specify the theory we have (instead of adopting the interaction Lagrangian of the usual approach) to prescribe the growth indices $\Omega(n)$ for all n. Especially, a renormalizable theory does hold but a finite number of functions $\tilde{\Phi}^n(p)$ with positive or zero index. It is also evident that we cannot set the growth indices quite arbitrarily; the problem of what sets of indices are admissible requires a special investigation*.

3. Let us establish now an infinite set of equations connecting the functions $\Phi^n(x)$ with different numbers of arguments. Such a set descend from the causality condition (PTDR, II (2))

$$\frac{\delta}{\delta\varphi(x)}\left(\frac{\delta S}{\delta\varphi(y)} S^+\right) = 0 \quad \text{for} \quad x \lesssim y. \qquad /5/$$

It may be shown by induction** that the more general condition

$$\frac{\delta}{\delta\varphi(x)}\left(\frac{\delta^n S}{\delta\varphi(y_1)\dots\delta\varphi(y_n)} S^+\right) = 0 \qquad \text{if } x \lesssim \text{ all } (y_1,\dots,y_n) \qquad /6/$$

follows from the condition eq./5/. Now, eq./6/ permits to prove again by induction, the operator indentity

$$\frac{\delta^n S}{\delta\varphi(x_1)\dots\delta\varphi(x_n)} S^+ = \frac{\delta^s S}{\delta\varphi(x_{j_1})\dots\delta\varphi(x_{j_s})} S^+ \frac{\delta^{n-s} S}{\delta\varphi(x_{j_{s+1}})\dots\delta\varphi(x_{j_n})} S^+ \qquad /7/$$

which take place, provided (for some $1 \leq s \leq n-1$) the splitting

$$\{x_{j_1},\dots x_{j_s}\} \gtrsim \{x_{j_{s+1}},\dots,x_{j_n}\} \qquad /7a/$$

holds.

In order to come here to the functions $\Phi^n(x)$ let us take the vacuum expectation values of the both sides of eq. /7/. The product of operators in the right-hand-side is to be expanded in the complete set of states*** making use of the formula (PTDR, I, (4)):

$$1 = \sum_{n=0}^{\infty} \frac{1}{n!} \int dk_1 \dots dk_n \; |k_1,\dots,k_n\rangle\langle k_1,\dots,k_n| \qquad /8/$$

Now, the matrix elements so arising may be once again reduced to vacuum expectation values by means of the property PTDR, II, (3). Then we obtain that:

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* Cf. B.V.Medvedev & M.K.Polivanov, to be published in JETP.

** The proofs of the statements /6/–/10/ will be published in JETP (cf. Dubna preprint D-599).

*** We assume here there are no bound states in the theory and, hence, the states entering the sum in eq. /8/ do form a complete system.

$$\Phi^{n}(x_1,\dots,x_n) = \sum_{t=0}^{\infty}\sum_{\nu=0}^{\infty}\sum_{\mu=0}^{\infty}\frac{i^{t+\nu+\mu}}{t!\,\nu!\,\mu!}\int dz_1\dots dz_t\, du_1\dots du_\nu \int du'_1\dots du'_\nu \int dv_1\dots dv_\mu \,\cdot$$
$$\cdot\int dz'_1\dots dz'_t\, dv'_1\dots dv'_\mu \;\Phi^{s+t+\nu}(x_{j_1},\dots,x_{j_s}, z_1,\dots,z_t, u_1,\dots,u_\nu)\, \mathcal{D}^{(-)}(z_1-z'_1)\dots\mathcal{D}^{(-)}(z_t-z'_t)\,\cdot \qquad /9/$$
$$\cdot\,\mathcal{D}^{(-)}(u_1-u'_1)\dots\mathcal{D}^{(-)}(u_\nu-u'_\nu)\,\overset{*}{\Phi}{}^{\nu+\mu}(u'_1,\dots,u'_\nu, v_1,\dots,v_\mu)\,\mathcal{D}^{(-)}(v_1-v'_1)\dots\mathcal{D}^{(-)}(v_\mu-v'_\mu)\,\cdot$$
$$\cdot\,\Phi^{n-s+t+\mu}(z'_1,\dots,z'_t, v'_1,\dots,v'_\mu, x_{j_{s+1}},\dots,x_{j_n}).$$

as soon as the condition /7a/ is fulfilled.

Thus we arrive at an infinite equation set which the functions $\Phi^{n}(x)$ determining the scattering matrix must satisfy. One may opine that, together with the unitarity condition PTDR, I, (5) which reads in terms of Φ -functions as:

$$\Phi^{n}(x_1,\dots,x_n) + (-1)^n\,\overset{*}{\Phi}{}^{n}(x_1,\dots,x_n) = \delta_{n0} -$$
$$-\sum_{\kappa=1}^{n-1}\sum_{s=0}^{\infty}\frac{(-1)^{\kappa}(-i)^{s}}{s!}\,P\left(\frac{x_1,\dots,x_{n-\kappa}}{x_{n-\kappa+1},\dots,x_n}\right)\int dz_1\dots dz_s\, dz'_1\dots dz'_s\;\Phi^{n-\kappa+s}(x_1,\dots,x_{n-\kappa}, z_1,\dots,z_s)\,\cdot \qquad /10/$$
$$\cdot\,\mathcal{D}^{(-)}(z_1-z'_1)\dots\mathcal{D}^{(-)}(z_s-z'_s)\,\overset{*}{\Phi}{}^{s+\kappa}(z'_1,\dots,z'_s, x_{n-\kappa+1},\dots,x_n)$$

and with some prescription given for the growth indices, this system sufficies to find all the functions $\Phi^{n}(x)$. Let us show that this is really the case at least in the perturbation theory – all the Φ -function can be found up to a finite number of constants.

4. We assume now all the Φ -functions to be expanded in power of some parameter λ :

$$\Phi^{n}(x_1,\dots,x_n) = \delta_{n0} + \sum_{m=1}^{\infty}\lambda^{m}\Phi^{n}_{m}(x_1,\dots,x_n)\,, \qquad /11/$$

the small λ being taking account of the weakness of the interaction. Let us suppose that in this expansion the coefficients of λ^{m} satisfying eqs. /9/,/10/ are already determined for all $m<M$. Then, we shall show we can always find the functions $\Phi^{n}_{M}(x_1,\dots,x_n)$ satisfying (up to λ^{M}) both the eqs. /9/ and the conditions /10/ and shall establish the ambiguity arising.

Let us look for the function $\Phi^{n}_{M}(x_1,\dots,x_n)$. Can its arguments be split into two sets provided by eq. /7a/, the eqs. /9/ hold and we may pick out of them all the terms of the M-th order – then one comes just to the expression of the function considered in terms of the Φ-functions with other numbers of arguments but alwais of the lower orders. Indeed, the expansion /11/ begins for all Φ^n except the Φ^0 with the first order term (if there is no interaction, $S = S^+ = 1$). Hence, the M-th order term in the right-hand-side of eq. /9/ cannot include Φ-functions of the order higher than (M-1). Thus the values of the functions for the arguments allowing any splitting /7a/ may be found from eq. /9/ in terms of the Φ-functions of lower orders, that are already known by supposition. The unitarity condition eq. /10/ may be shown to be satisfied thereat automatically. So, it wants only to determine the functions Φ^n_M for the arguments not allowing the splittings /7a/.

Now, the set of arguments $\{x_1, \dots, x_n\}$ does not allow any splitting /7a/ if and only if all these arguments coincide. In other words, the set of equations /9/ determines the Φ^n_M by given $\Phi^{n'}_m$, $m < M$, up to a function, different from zero only for all the arguments coinciding. Such a function must be built up as a linear combination of δ-function and their derivatives and its Fourier transform has therefore to be a polynomial in $p_1, \dots, p_n$, symmetrical in these variables. By virtue of the assumption on the degrees of growth the power of this polynomial cannot exceed the corresponding growth index $\Omega(n)$ and, hence, a finite number of constants suffices to specify it. For each of these constants either real or imaginary part will be determined by the M-th order unitarity condition, the other remaining arbitrary in our will.

Now, since in a renormalizable theory we can introduce but a finite number of nonnegative growth indices (otherwise no arbitrary polynomial arises), then the whole number of constants needed to get the M-th approximation, the lower being known, has to be finite, too. Finally, taking advantage of the fact the growth index does not depend in our assumptions on the approximation order, we can, just as in the usual renormalization speculations, get together all the constants springing out in each approximation, thus reducing the multitude of constants needed to determine the theory uniquely to a finite number.

The last reasoning referred to the hypothetic convergence of the series /11/ can be avoided without any trouble. Namely, instead of prescribing the values of the constants arising in each approximation, we may fixe the values of the functions $\tilde{\Phi}^n(p)$ in so many points, how many coefficients the corresponding polynomial has, doing this for every n with a nonnegative $\Omega(n)$ and requiring thereafter to preserve this condition in any perturbation theory order. Such a possibility would be in line with the dealing with the renormalized quantities in the usual approach.

For instance, if we demand the growth indices to be equal to 2 for $n = 2$, to be zero for $n = 3$ and $n = 4$, and to be negative otherwise, we come then at the self-interacting scalar field theory with the three-fold and four-fould interaction without derivatives. This theory involves four arbitrary constants. Two of them (corresponding to $n = 2$) will be determined by requiring, there are no mass and wave function renormalizations. To specify the others (corresponding to $n = 3$ and to $n = 4$), we can fix the values of the 3-fold and of the 4-fold "charges" for some fixed momentum sets.

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