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ОБЪЕДИНЕННЫЙ ИНСТИТУТ ЯДЕРНЫХ ИССЛЕДОВАНИЙ

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APPROXIMATE EQUATIONS FOR PARTIAL $\pi - K$ SCATTERING AMPLITUDES

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Abstract

Approximate integral equations for partial π-K-scattering amplitudes are obtained with the help of the double Mandelstam representation in the low energy range.

1. Introduction

Investigation of π-K-interaction is of interest not only from the purely theoretical point of view, but also from the "applied" point of view. In fact, $\pi - K$ interaction is inseparable from the study of elastic and inelastic processes of K-meson-hyperon interaction, and in particular, of K-meson-nucleon scattering.

There are two types of interaction considered in perturbation theory. First,

$$\bar{K}\pi K \quad \text{- interaction} \tag{1}$$

which is plotted in fig.1. It was introduced by Schwinger[1] and discussed in detail by Minami[2] and Pais[3]. And, secondly,

$$\bar{K}K\pi\pi \quad \text{- interaction} \tag{2}$$

plotted in fig. 2 and discussed in detail by Barshay[4].

It is not clear at present which of the above types of interaction is more preferable. One may adduce the following consideration in favour of the second type: if the interaction of K-mesons and baryons is assumed to have the form $\bar{Y}KY$, then to make the theory renormalizable it is necessary to introduce the term $g_{K\pi}\bar{K}K\pi^2$ into the Hamiltonian of the interaction (in a way similar to that of introducing the term $\lambda\varphi^4$ in mesodynamics or the term $\varepsilon A_\mu A_\mu \pi^2$ in mesoelectrodynamics).

The analysis of the experimental data on K-N-scattering[3-5] has shown that there is good agreement between experiment and theory in the low energy range (up to 200MeV) if one uses interaction 2 and assumes that the coupling constant of π-K-interaction approximately equals 1-5.

Mention should also be made of Chou-Kuang-chao results[6] about the possible symmetry of the π-K-system. Proceeding from the Hamiltonian in the form

$$H = H_\pi + H_K + g_{K\pi}\pi_\alpha\pi_\alpha\bar{K}_\lambda K_\lambda \tag{3}$$

where H_π is the Hamiltonian describing π-π-interaction, H_K is the Hamiltonian describing K-meson interaction, and assuming Hamiltonian (3) to be invariant with respect to rotations in isotopic π-meson vector and K-meson isospinor spaces he obtained that all the amplitudes of π-K -scattering without charge exchange are equal to each other that all the amplitudes of charge-exchange scattering equal zero, and that the annihilation process $K+\bar{K}\to n\pi$ is going only through the isoscalar state.

The present paper attempts to study π-K -interaction by the method of Mandelstam's double representations. At the first stage of the investigation we have derived approximated integral equations for partial π-K -scattering amplitudes in the low energy range. Investigation of π-K -interaction by the method of double dispersion relation was stimulated, first, by the importance of this interaction for the study of K-N - scattering, and, secondly, by the work of Efremov, Meshcheryakov and Shirkov[7] who successfully avoided kinematics singularities by considering the corresponding symmetrical and antisymmetrical expressions.

The logical sequence of processes for strong interacting particles in studying π-K -scattering is plotted in fig.3. A detailed study of π-π -scattering was made by Chew and Mandelstam[8] and by Chew, Mandelstam and Noyes[9]. In our equations for π-K scattering the π-π -scattering phase-shifts are assumed to be given.

The π-K -scattering amplitude is considered in the complex plane of variable $\vec{q}^{\,2}$ ($\vec{q}^{\,2}$ is the momentum in the centre-of-mass system of $\pi+K\to\pi'+K'$ reaction). A kinematic cut along $\vec{q}^{\,2}$ in the interval $-M^2\leq\vec{q}^{\,2}\leq-\mu^2$, where M and μ are K-meson and π-meson masses respectively, is eliminated by the method proposed in ref.[7] The analyticity region of the imaginary part of π-K -scattering amplitude is determined from the Mandelstam representation and the perturbation theory under the assumption of interaction (2).

A non-physical cut from $\pi+\pi\to K+\bar{K}$ reaction is eliminated by Muskhelishvili-Omnes method[10]. Employing further the unitarity condition we obtained a closed set of approximate integral equations for π-K -scattering amplitudes which involved π-π - scattering phase-shifts. The approximation consisted in taking into account only nearest singularities while deriving the set of integral equations. This, naturally, led to a reasonable restriction on π-K -scattering by the lowest partial waves s - and p . Their set of integral equations was obtained by two methods: 1) for partial amplitudes averaged over all scattering angles and 2) for partial amplitudes taken at the backward scattering angle.

2. π - K Scattering Amplitude and its Isotopic Structure

Matrix elements of the scattering process

$$\text{I} \quad \pi + K \to \pi' + K'$$

$$\text{II} \quad \pi' + K \to \pi + K'$$

and pair production of K-mesons

$$\text{III} \quad \pi + \pi' \to K' + \bar{K}$$

can be represented in the form

$$\langle f|S|i\rangle = \langle f|1 + iT|i\rangle = \delta_{if} + \delta(\Sigma p_i)\frac{i}{(2\pi)^2}\left(\frac{1}{16 p_{10} p_{20} q_{10} q_{20}}\right)^{1/2} T \tag{4}$$

where Σp_i is the sum of four momenta of all particles, p_{10} and p_{20} is the K-meson energy , q_{10} and q_{20} is the π -meson energy, T is the Green's function.

For the scattering process I the differential cross-section is expressed by the formula

$$\frac{d\sigma}{d\Omega} = \frac{1}{(2\pi)^2}\frac{|\vec{q}_2|}{|\vec{q}_1|}\frac{1}{16 p_{10} p_{20}}|T|^2 \tag{5}$$

In the centre-of-mass system $|\vec{q}_2| = |\vec{q}_1|$, $p_{20} = p_{10} = E_K$ and relation (5) assumes the form of $\frac{d\sigma}{d\Omega} = \left|\frac{1}{8\pi E_K} T\right|^2$. Isotopic structure of the π-K -scattering amplitude has the form:

$$T_{\alpha\beta} = A\,\delta_{\alpha\beta} + \frac{1}{2}[\tau_\alpha, \tau_\beta]\,B \tag{6}$$

where A and B are scalar functions dependent on the scalar product of the four momenta p_1 , p_2 , q_1 , q_2 .

The connection of A and B with isotopic states of the π-K -system is given by:

$$A = \frac{2T^{3/2} + T^{1/2}}{3}$$

$$B = \frac{T^{1/2} - T^{3/2}}{3}$$

where $T^{3/2}$ and $T^{1/2}$ are states of isotopic spin 3/2 and 1/2 respectively.

Isotopic structure of process III coincides with that of processes I and II, while the connection of T^0 and T^1 (states of isotopic spins 0 and 1, respectively) with A and B is quite simple:

$$T^0 = \sqrt{6}\,A \; ; \; T^1 = 2B \tag{7}$$

3. Kinematics of Reactions

We shall consider A and B factors as functions of invariant variables s_1, s_2 and s_3 determined by the usual method:

$$s_1 = -(p_1+q_1)^2 \; ; \quad s_2 = -(p_1+q_2)^2 \; ; \quad s_3 = -(p_1+p_2)^2$$

In the centre-of-mass system of reaction I these invariant variables take the form:

$$\begin{aligned} s_1 &= M^2+\mu^2+2\vec{q}_1^{\,2}+2\sqrt{(\vec{q}_1^{\,2}+\mu^2)(\vec{q}_1^{\,2}+M^2)} \\ s_2 &= M^2+\mu^2-2\vec{q}_1^{\,2}\cos\vartheta_1-2\sqrt{(\vec{q}_1^{\,2}+\mu^2)(\vec{q}_1^{\,2}+M^2)} \\ s_3 &= -2\vec{q}_1^{\,2}(1-\cos\vartheta_1) \end{aligned} \tag{8}$$

where ϑ_1 is the scattering angle of the incident meson. It follows from relations (6) and (7) that the substitution of $\alpha \rightleftarrows \beta$ corresponds to the substitution of $s_1 \rightleftarrows s_2$ and in this case

$$\begin{aligned} A(s_1, s_2, s_3) &= A(s_2, s_1, s_3) \\ B(s_1, s_2, s_3) &= -B(s_2, s_1, s_3) \end{aligned} \tag{9}$$

Hence, it appears that in expanding A and B factors into partial waves of the third reaction only even waves will appear in the expansion of A and only odd waves in the expansion of B.

4. Analytic Properties of π-K -Scattering Amplitude in the Centre-of-Mass System of the First Reaction

In what follows we assume that the Mandelstam representation holds for the functions $A(s_1,s_2,s_3)$ and $B(s_1,s_2,s_3)$. If we consider functions A and B in the complex variable plane $\vec{q}_1^{\,2}$, at a fixed scattering angle $\cos\vartheta_1 = z_1 = \mathrm{const}$ it is possible to obtain from the Mandelstam representation all the cuts over $\vec{q}_1^{\,2}$; from reaction I the cut is in the interval $[0, +\infty]$, from reaction II in the interval $[-M^2, -\infty]$ for $z_1 > \frac{\mu}{M}$ angles and in the interval $[-\tau_{z_1}, -M^2]$ for $z_1 \le \frac{\mu}{M}$ angles, from reaction III it is in the interval $\left[-\frac{2\mu^2}{1-z_1}, -\infty\right]$.

Apart from these cuts there is one more, a kinematic one, which lies in the interval $-M^2 \le \vec{q}_1^{\,2} \le -\mu^2$. The latter cut is eliminated by the method proposed in ref.[7] For this purpose, alongside with the variables s_1 and s_2 (see 8) we shall introduce new variables

$$\overset{*}{s}_1 = M^2+\mu^2+2\vec{q}_1^{\,2} - 2\sqrt{(\vec{q}_1^{\,2}+\mu^2)(\vec{q}_1^{\,2}+M^2)}$$
$$\overset{*}{s}_2 = M^2+\mu^2-2\vec{q}_1^{\,2}z_1 + 2\sqrt{(\vec{q}_1^{\,2}+\mu^2)(\vec{q}_1^{\,2}+M^2)} \tag{10}$$

If we now require that

$\overset{*}{s}_1 = s_1$ (in variables of reaction II)

and

$\overset{*}{s}_2 = s_2$ (in variables of reaction II)

we shall obtain that $A(\overset{*}{s}_1, \overset{*}{s}_2, s_3)$ is an amplitude of reaction II in its physical region. Fig.4 represents the connection between the variables of the first and second reactions as expresses by relations (IO) and (11).

Thus, side by side with functions $A(s_1, s_2, s_3)$ and $B(s_1, s_2, s_3)$ we shall consider functions $A(\overset{*}{s}_1, \overset{*}{s}_2, s_3)$ and $B(\overset{*}{s}_1, \overset{*}{s}_2, s_3)$. Symmetrical and asymmetrical combinations of these functions will not depend irrationally on the root $K(\vec{q}_1^{\,2}) = \sqrt{(\vec{q}_1^{\,2}+\mu^2)(\vec{q}_1^{\,2}+M^2)}$. Thus, functions

$$\bar{\Phi}_s = \frac{\Phi(\vec{q}_1^{\,2}, z_1, +K(\vec{q}_1^{\,2})) + \Phi(\vec{q}_1^{\,2}, z_1, -K)}{2}$$

and

$$\Phi_a = \frac{\Phi(\vec{q}_1^{\,2}, z_1, +K) - \Phi(\vec{q}_1^{\,2}, z_1, -K)}{2K}$$

where Φ denotes either function (A or B), will not contain a kinematic cut.

The cuts of $\Phi(\vec{q}_1^{\,2}, z_1, -K)$ function from reaction II are in the $[0,+\infty]$ and $[-M^2, -\tau_2]$ intervals for $z_1 \geq \frac{\mu}{M}$; from reaction III they are in the interval $\left[-\frac{2\mu^2}{1-z_1}, -\infty\right]$. There is no cut from reaction I. If we now write the usual Cauchy theorem in the complex plane $\vec{q}_1^{\,2}$ for functions Φ_s and Φ_a and return again to functions Φ we obtain

$$\begin{aligned}\Phi(\vec{q}_1^{\,2}, \pm K) = \frac{1}{\pi}\Big\{ &\int_0^\infty d\vec{q}_1^{\,\prime 2} \frac{\operatorname{Im}\Phi(\vec{q}_1^{\,\prime 2}, +K)\,\gamma(\pm) + \operatorname{Im}\Phi(\vec{q}_1^{\,\prime 2}, -K)\,\gamma(\mp)}{\vec{q}_1^{\,\prime 2} - \vec{q}_1^{\,2}} + \\ &+ \int_{-\infty}^{q_1^2(z_1)} d\vec{q}_1^{\,\prime 2} \frac{\operatorname{Im}\Phi(\vec{q}_1^{\,\prime 2}, +K)}{\vec{q}_1^{\,\prime 2} - \vec{q}_1^{\,2}}\gamma(\pm) + \vartheta\left(z_1 - \frac{\mu}{M}\right)\int_{-\tau_2}^{-M^2} d\vec{q}_1^{\,\prime 2} \frac{\operatorname{Im}\Phi(\vec{q}_1^{\,\prime 2}, -K)}{\vec{q}_1^{\,\prime 2} - \vec{q}_1^{\,2}}\gamma(\mp) + \\ &+ \int_{-\infty}^{-\frac{2\mu^2}{1-z_1}} d\vec{q}_1^{\,\prime 2} \frac{\operatorname{Im}\Phi(\vec{q}_1^{\,\prime 2}, +K)\,\gamma(\pm) + \operatorname{Im}\Phi(\vec{q}_1^{\,\prime 2}, -K)\,\gamma(\mp)}{\vec{q}_1^{\,\prime 2} - \vec{q}_1^{\,2}}\end{aligned} \tag{12}$$

where

$$\vec{q}_1^{\,2}(z_1) = \begin{cases} -M^2 , & \text{for } z_1 \geq \frac{\mu}{M} \\ -\tau_{z_1} , & \text{for } z_1 \leq \frac{\mu}{M} \end{cases}$$

$$\tau_{z_1} = \frac{M^2+\mu^2-2M\mu z_1}{1-z_1^2} \quad ; \quad \gamma(\pm) = 1 \pm \frac{K(\vec{q}_1^{\,2})}{K(\vec{q}_1^{\,\prime 2})}$$

Fig.5 shows the domains of integration over $\vec{q}_1^{\,2}$ in eq.(12) depending on z_1.

It can be seen from fig.5 that at $z_1 = 1$ the only non-zero integration domains will be $\vec{q}_1^{\,\prime 2} \geq 0$. It corresponds to the transition to the usual dispersion relations for the forward scattering case.

At $z_1 = -1$ contribution from the negative $\vec{q}_1^{\,\prime 2}$ is made only by the cut from reaction III.

Eq. (12) was derived from the Mandelstam representation without using any approximation. Restricting ourselves only to nearest singularities, namely, taking into account only part of the cut from reaction III we shall further on consider such an equation

$$\Phi(\vec{q}_1^{\,2}, \pm K) = \frac{1}{\pi}\int_0^\infty d\vec{q}_1^{\,\prime 2}\, \frac{\operatorname{Im}\Phi(\vec{q}_1^{\,\prime 2}, +K)\gamma(\pm) + \operatorname{Im}\Phi(\vec{q}_1^{\,\prime 2}, -K)\gamma(\mp)}{\vec{q}_1^{\,\prime 2} - \vec{q}_1^{\,2}} + \frac{1}{\pi}\int_{-\frac{8\mu^2}{1-z_1}}^{-\frac{2\mu^2}{1-z_1}} d\vec{q}_1^{\,\prime 2}\, \frac{\operatorname{Im}\Phi(\vec{q}_1^{\,\prime 2}, +K)\gamma(\pm) + \operatorname{Im}\Phi(\vec{q}_1^{\,\prime 2}, -K)\gamma(\mp)}{\vec{q}_1^{\,\prime 2} - \vec{q}_1^{\,2}} \tag{13}$$

5. Elimination of Nonphysical Cut From Reaction III

In eq. (13) imaginary parts $\operatorname{Im}\Phi(\vec{q}_1^{\,\prime 2}, +K)$ and $\operatorname{Im}\Phi(\vec{q}_1^{\,\prime 2}, -K)$ in the interval $\left[-\frac{2\mu^2}{1-z_1}, -\frac{8\mu^2}{1-z_1}\right]$ will be considered as analytic continuations of the corresponding functions from the region $s_3 \geq 4M^2$.

Let us expand amplitude $\Phi(s_1, s_2, s_3)$ into partial waves of reaction III in the region of physical values of $\vec{q}_3^{\,2}$ and $\cos\vartheta_3$:

$$\Phi(s_1, s_2, s_3) = 2 p_3 q_3 \sum_\ell (2\ell+1)\, \Phi_{\mathrm{III}}^{\ell}(\vec{q}_3^{\,2})\, P_\ell(\cos\vartheta_3) \tag{14}$$

As follows from eq. (9) the summation over even ℓ is taken in relation (14) for $\Phi = A$ and the summation over odd ℓ is taken for $\Phi = B$. Let us also write unitarity relation for reaction III:

$$\operatorname{Im} \Phi(s_1,s_2,s_3) = \frac{1}{32\pi^2} \frac{|\vec{q}_3|}{W(\vec{q}_3^{\,2})} \int d\Omega' \, \Phi_{III}(\Omega') \Pi^*(\Omega'') \tag{15}$$

here Π^* denotes hermitian conjugated amplitude of reaction $\pi+\pi \to \pi+\pi$ [x].

It follows from eqs.(7) and (15) that

$$\operatorname{Im} A = \frac{1}{32\pi^2} \frac{|\vec{q}_3|}{W_3} \int d\Omega' \, A(\Omega') \overset{*}{\Pi}{}^0(\Omega'')$$

$$\operatorname{Im} B = \frac{1}{32\pi^2} \frac{|\vec{q}_3|}{W_3} \int d\Omega' \, B(\Omega') \overset{*}{\Pi}{}^1(\Omega'')$$

If in expanding Π^0 and Π^1 into partial waves we restrict ourselves only to s and p -waves respectively, we obtain

$$\operatorname{Im} A_{III}(\vec{q}_3^{\,2}) = \frac{1}{8\pi} \frac{|\vec{q}_3|}{W_3} A^0_{III}(\vec{q}_3^{\,2}) e^{i\delta_0} \sin\delta_0$$

$$\operatorname{Im} B_{III}(\vec{q}_3^{\,2}) = \frac{1}{8\pi} \frac{|\vec{q}_3|}{W_3} B^1_{III}(\vec{q}_3^{\,2}) e^{i\delta_1} \sin\delta_1 \cos\vartheta_3 \tag{16}$$

It is evident that the expansion of (16) over $z_3 = \cos\vartheta_3$ in the reaction $\pi+\pi' \to K+\bar{K}$ can be analytically continued up to the first singularity at which the imaginary part of reaction III amplitude discontinues. The boundary of the first discontinuity is found in a way similar to that of ref.[11]

The bounding curves for the spectral functions have the form

$$[s_3 - 16\mu^2][s_1 - (M+\mu)^2][s_1 - (M-\mu)^2] - 64 s_1 \mu^4 = 0 \tag{17}$$

(asymptotes $s_1 = (M+\mu)^2$ and $s_3 = 16\mu^2$)

and

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x) Isotopic structure of reaction $\pi+\pi \to \pi+\pi$ has the form (see, for instance, ref.[8]):

$$\Pi^{\alpha\beta,\gamma\varepsilon} = \Pi_1 \delta_{\alpha\beta}\delta_{\gamma\varepsilon} + \Pi_2 \delta_{\alpha\varepsilon}\delta_{\beta\gamma} + \Pi_3 \delta_{\alpha\gamma}\delta_{\varepsilon\beta}$$

while the isotopic states of spin 0(Π^0) and of spin I(Π^1) are related to the coefficients Π_1, Π_2, Π_3 in the following way

$$\Pi^0 = 3\Pi_1 + \Pi_2 + \Pi_3$$

$$\Pi^1 = \Pi_2 - \Pi_3$$

$$[s_3-4\mu^2][s_1-(M+3\mu)^2]-32\mu^3(M+\mu)=0 \tag{18}$$

(asymptotes $s_1=(M+3\mu)^2$ and $s_2=4\mu^2$)

It follows from eqs.(17) and (18) that the imaginary part of the amplitude of the reaction III can be analytically continued over variable $\vec{q}_1^{\,2}$ (at physical z_1) up to $-21.3\,\mu^2$.

The analyticity of the real part of reaction III amplitude is determined from the Mandelstam representation with the help of Heine's theorem[12].

The boundary in variables $\vec{q}_1^{\,2}$ and z_1 of the analytic continuation of the third reaction amplitude is plotted in fig. 6.

Applying now Muskhelishvili-Omnes method[10] to the function

$$\Phi_s e^{-u(\vec{q}_1^{\,2},z_1)}$$

where

$$u(\vec{q}_1^{\,2},z)=\frac{1}{\pi}\int_{-2.79\mu^2}^{-\frac{2\mu^2}{1-z_1}} d\vec{q}_1^{\,\prime 2}\,\frac{\delta_0\left(-\frac{\vec{q}_1^{\,2}(1-z_1)}{2}-\mu^2\right)}{\vec{q}_1^{\,\prime 2}-\vec{q}_1^{\,2}}$$

δ_0 is S π-π-scattering phase-shift, and again returning to function Φ we obtain the following set of equations for A:

$$A(\vec{q}^{\,2},z_1,\pm)=\frac{1}{\pi}\int_0^\infty d\vec{q}^{\,\prime 2}\,\frac{\mathrm{Im}\,A_{I}(\vec{q}_1^{\,\prime 2},z_1)}{\vec{q}_1^{\,\prime 2}-\vec{q}_1^{\,2}}\,\eta_\pm+\frac{1}{\pi}\int_0^\infty d\vec{q}_1^{\,\prime 2}\,\frac{\mathrm{Im}\,A_{II}(\vec{q}_1^{\,\prime 2},z_1)}{\vec{q}_1^{\,\prime 2}-\vec{q}_1^{\,2}}\,\eta_\mp \tag{19}$$

where

$$\eta_\pm(u,\vec{q}_1^{\,\prime 2},\vec{q}_1^{\,2})=\frac{1}{2}\left[e^{u(\vec{q}_1^{\,2},z_1)-u(\vec{q}_1^{\,\prime 2},z_1)}\pm\frac{K(\vec{q}_1^{\,2})}{K(\vec{q}_1^{\,\prime 2})}\right]$$

To eliminate non-physical cut in the equations for B function consider the functions $\tilde{B}=\frac{B}{s_1-s_2}$ and $\frac{\Phi_{sB}e^{-\beta}}{s_1-s_2}$

where

$$\beta=\frac{1}{\pi}\int_{-2.79\mu^2}^{-\frac{2\mu^2}{1-z_1}} d\vec{q}_1^{\,\prime 2}\,\frac{\delta_1\left(-\frac{\vec{q}_1^{\,2}(1-z_1)}{2}-\mu^2\right)}{\vec{q}_1^{\,\prime 2}-\vec{q}_1^{\,2}}$$

δ_1 is P $\pi\pi$-scattering phase-shift.

Function $\frac{\Phi_{SB} e^{-\beta}}{s_1 - s_2}$ does not contain new singularities and the same Mandelstam representation will hold for it as it did for Φ_{SB} .

In a way analogous to that for the function $A(\pm K)$ we obtain the following system for $\tilde{B}$ function:

$$\tilde{B}(\vec{q}_1^{\,2}, z_1, \pm K) = \frac{1}{\pi}\int_0^\infty d\vec{q}_1'^{\,2}\, \frac{\operatorname{Im}\tilde{B}(\vec{q}_1'^{\,2}, z_1, +K)}{\vec{q}_1'^{\,2} - \vec{q}_1^{\,2}}\,\xi_{\pm} + \frac{1}{\pi}\int_0^\infty d\vec{q}_1'^{\,2}\, \frac{\operatorname{Im}\tilde{B}(\vec{q}_1'^{\,2}, z_1, -K)}{\vec{q}_1'^{\,2} - \vec{q}_1^{\,2}}\,\xi_{\mp} \tag{20}$$

where

$$\xi_{\pm} = \frac{1}{2}\left[e^{\beta(q_1^2, z_1) - \beta(\vec{q}_1'^{\,2}, z_1)} \pm \frac{K(\vec{q}_1^{\,2})}{K(\vec{q}_1'^{\,2})} \right]$$

6. Integral Equations for Partial Scattering Amplitudes

In order to pass from eqs. (19) and (20) to partial amplitude equations, we shall use expansion of $T^{1/2}$ and $T^{3/2}$ into Legendre polynomial series and the unitarity condition

$$T^{J}(q_1^2, z_1) = \sum_{\ell}(2\ell+1)\, T_{\ell}^{J}(q_1^2)\, P_{\ell}(z_1)$$

$$T^{J}(\vec{q}_2^{\,2}, z_2) = \sum_{\ell}(2\ell+1)\, T_{\ell}^{J}(\vec{q}_2^{\,2})\, P_{\ell}(z_2) \tag{21}$$

$$\operatorname{Im} T_{\ell}^{J}(\vec{q}_1^{\,2}) = \frac{1}{8\pi}\frac{|\vec{q}_1|}{W(q_1^2)}\,|T_{\ell}^{J}(\vec{q}_1^{\,2})|^2$$

$$\operatorname{Im} T_{\ell}^{J}(\vec{q}_2^{\,2}) = \frac{1}{8\pi}\frac{|\vec{q}_2|}{W(q_2^2)}\,|T_{\ell}^{J}(\vec{q}_2^{\,2})|^2 \tag{22}$$

Restricting ourselves to S- and P -waves we obtain from eqs. (19)-(20) and from the orthogonality condition of Legendre polynomials

$$2\,T_{\ell}^{3/2}(\vec{q}^{\,2}) + T_{\ell}^{1/2}(\vec{q}^{\,2}) = \frac{1}{16\pi^2}\int_0^\infty d\vec{q}'^{\,2}\,\frac{|\vec{q}'|}{W(\vec{q}'^{\,2})}\Big\{ \nu_{\ell}(\vec{q}'^{\,2}, \vec{q}^{\,2})\big[2\,|T_0^{3/2}(\vec{q}'^{\,2})|^2 + |T_0^{1/2}(\vec{q}'^{\,2})|^2\big] + 3\varphi_{\ell}(\vec{q}'^{\,2}, \vec{q}^{\,2})\big[2\,|T_1^{3/2}(\vec{q}'^{\,2})|^2 + |T_1^{1/2}(\vec{q}'^{\,2})|^2\big]\Big\} \tag{23}$$

$$T_{\ell}^{1/2}(\vec{q}^{\,2}) - T_{\ell}^{3/2}(\vec{q}^{\,2}) = \frac{1}{16\pi^2}\int_0^\infty d\vec{q}'^{\,2}\,\frac{|\vec{q}'|}{W(\vec{q}'^{\,2})}\Big\{ \varepsilon_{\ell}(\vec{q}'^{\,2}, q^2)\big[|T_0^{1/2}(q'^2)|^2 - |T_0^{3/2}(\vec{q}'^{\,2})|^2\big] + 3\omega_{\ell}(\vec{q}'^{\,2}, \vec{q}^{\,2})\big[|T_1^{1/2}(\vec{q}'^{\,2})|^2 - |T_1^{3/2}(\vec{q}'^{\,2})|^2\big]\Big\} \tag{24}$$

In eqs.(23),(24) the following notations are introduced

$$W(\vec{q}^{2})=\sqrt{M^2+\mu^2+2\vec{q}^{2}+2\sqrt{(\vec{q}^{2}+\mu^2)(\vec{q}^{2}+M^2)}}$$

$$f(\vec{q}^{2},z_1)=\frac{\tau+2z_1(\vec{q}^{2}+K)-M^2-\mu^2}{2(1-z_1^2)}$$

$$z_2(\vec{q}^{2},z_1)=1-\frac{f(\vec{q}^{2},z_1)}{\vec{q}^{2}}(1-z_1)$$

$$\mathcal{D}(\vec{q}^{2},z)=\frac{2K+2\vec{q}^{2}+\mu^2}{2K}\cdot\frac{\tau z_1-z_1(M^2+\mu^2)+2(\vec{q}^{2}+K)}{\tau(1-z_1^2)}$$

$$\tau=\sqrt{(M^2-\mu^2)^2-4z_1(M^2+\mu^2)(\vec{q}^{2}+K)+4z_1^2M^2\mu^2+4(\vec{q}^{2}+K)^2}$$

$$\tilde{\eta}_-=\eta_-\left[u(f(\vec{q}^{2},z_1),z_1),\ \vec{q}^{2},\ f(\vec{q}^{2},z_1)\right]$$

$$\tilde{\xi}_-=\xi_-\left[\beta(f(\vec{q}^{2},z_1),z_1),\ \vec{q}^{2},\ f(\vec{q}^{\prime 2},z_1)\right]$$

$$\alpha_\pm(\vec{q}^{\prime 2},\vec{q}^{2},z_1)=\xi_\pm\,\frac{s_1(\vec{q}^{2})-s_2(\vec{q}^{2})}{s_1(\vec{q}^{\prime 2})-s_2(q^{\prime 2})}$$

$$\tilde{\alpha}_\pm=\tilde{\xi}_\pm\,\frac{s_1(\vec{q}^{2})-s_2(\vec{q}^{2})}{s_1[f(q^{\prime 2},z_1)]-s_2[f(q^{\prime 2},z_1)]}$$

$$n_\ell(\vec{q}^{\prime 2},\vec{q}^{2})=\int_{-1}^{+1}dz_1\,P_\ell(z_1)\left[\frac{\eta_+}{\vec{q}^{\prime 2}-\vec{q}^{2}}+\frac{\tilde{\eta}_-\mathcal{D}(\vec{q}^{\prime 2},z_1)}{f(\vec{q}^{\prime 2},z_1)-\vec{q}^{2}}\right]$$

$$\varphi_\ell(\vec{q}^{\prime 2},\vec{q}^{2})=\int_{-1}^{+1}dz_1\,P_\ell(z_1)\left[\frac{\eta_+z_1}{\vec{q}^{\prime 2}-\vec{q}^{2}}+\frac{\tilde{\eta}_-z_2(\vec{q}^{\prime 2},z_1)\mathcal{D}(\vec{q}^{\prime 2},z_1)}{f(\vec{q}^{\prime 2},z_1)-\vec{q}^{2}}\right].$$

$$\varepsilon_\ell(\vec{q}^{\prime 2},\vec{q}^{2})=\int_{-1}^{+1}dz_1\,P_\ell(z_1)\left[\frac{\alpha_+}{\vec{q}^{\prime 2}-\vec{q}^{2}}+\frac{\tilde{\alpha}_-\mathcal{D}(\vec{q}^{\prime 2},z_1)}{f(\vec{q}^{\prime 2},z_1)-\vec{q}^{2}}\right]$$

$$\omega_\ell(\vec{q}^{\prime 2},\vec{q}^{2})=\int_{-1}^{+1}dz_1\,P_\ell(z_1)\left[\frac{\alpha_+z_1}{\vec{q}^{\prime 2}-\vec{q}^{2}}+\frac{\tilde{\alpha}_-z_2(\vec{q}^{\prime 2},z_1)\mathcal{D}(\vec{q}^{\prime 2},z_1)}{f(\vec{q}^{\prime 2},z_1)-\vec{q}^{2}}\right]$$

We shall further on proceed from the assumption that in eqs.(23) and (24) it is sufficient to perform one substraction. In the case under consideration there does not exist a convenient symmetrical point for substraction as it happens to be in the case of π-π -scattering, since the K-meson and π -meson masses are different. We shall, therefore, agree upon substracting at the point $\vec{q}_1^{\,2}=0$. In this case, the

S -phase-shifts will be related to scattering lengths $a^{1/2}$ and $a^{3/2}$ by the correspondi[ng] isotopic states $T_0^{1/2}$ and $T_0^{3/2}$ which will appear in equations as parameters. As a result of simple calculations, we obtain the following equations

$$\mathrm{Re}\left[2T_\ell^{3/2}(\vec q^2)+T_\ell^{1/2}(\vec q^2)\right]=\mathrm{Re}\left[2T_\ell^{3/2}(0)+T_\ell^{1/2}(0)\right]+\frac{1}{16\pi^2}P\int_0^\infty d\vec q'^2\,\frac{|\vec q'|}{W(\vec q'^2)}\Big\{\left[n_\ell(\vec q'^2,\vec q^2)-n_\ell(\vec q'^2,0)\right]\left[2|T_0^{3/2}(\vec q'^2)|^2+|T_0^{1/2}(\vec q'^2)|^2\right]+$$
$$+3\left[\varphi_\ell(\vec q'^2,\vec q^2)-\varphi_\ell(\vec q'^2,0)\right]\left[2|T_1^{3/2}(\vec q'^2)|^2+|T_1^{1/2}(\vec q'^2)|^2\right]\Big\} \tag{25}$$

$$\mathrm{Re}\left[T_\ell^{1/2}(\vec q^2)-T_\ell^{3/2}(\vec q^2)\right]=\mathrm{Re}\left[T_\ell^{1/2}(0)-T_\ell^{3/2}(0)\right]+\frac{1}{16\pi^2}P\int_0^\infty d\vec q'^2\,\frac{|\vec q'|}{W(\vec q'^2)}\Big\{\left[\varepsilon_\ell(\vec q'^2,\vec q^2)+\right.$$
$$\left.-\varepsilon_\ell(\vec q'^2,0)\right]\left[|T_0^{1/2}(\vec q'^2)|^2-|T_0^{3/2}(\vec q'^2)|^2\right]+3\left[\omega_\ell(\vec q'^2,\vec q^2)-\omega_\ell(\vec q'^2,0)\right]\left[|T_1^{1/2}(\vec q'^2)|^2-|T_1^{3/2}(\vec q'^2)|^2\right]\Big\} \tag{26}$$

$$\mathrm{Re}\,T_1^{1/2}(0)=\mathrm{Re}\,T_1^{3/2}(0)=0\ .$$

We shall now obtain a set of equations for partial amplitudes in the system of reaction I for backward scattering. In this case $\vec q_1^{\,2}=\vec q_2^{\,2}$ and $z_1=z_2=-1$. To derive the set of equations we shall proceed from eqs. (19) and (20). The validity of the expansion at the point $z_1=-1$ follows from the analyticity properties of the functions A and B at this point.

Thus, we have

$$T^J(\vec q_1^{\,2},z_1)=T^J(\vec q_1^{\,2},-1)+(1+z_1)\partial_{z_1}T^J(\vec q_1^{\,2},z_1)_{z_1=-1}+\dots$$
$$T^J(\vec q_2^{\,2},z_2)=T^J(\vec q_2^{\,2},-1)+(1+z_2)\partial_{z_2}T^J(\vec q_2^{\,2},z_2)_{z_2=-1}+\dots \tag{27}$$

$$\mathrm{Im}\,T^J(q_1^2,z_1)=\frac{|\vec q_1|}{8\pi W(q_1^2)}\left(|T_0^J(\vec q_1^{\,2})|^2+3z_1|T_1^J(\vec q_1^{\,2})|^2+\dots\right)$$
$$\mathrm{Im}\,T^J(\vec q_2^{\,2},z_2)=\frac{|\vec q_2|}{8\pi W(q_2^2)}\left(|T_0^J(\vec q_2^{\,2})|^2+3z_2|T_1^J(\vec q_2^{\,2})|^2+\dots\right) \tag{28}$$

$$T^J(\vec q_1^{\,2},-1)=T_0^J(q_1^2)-3T_1^J(q_1^2)$$
$$\partial_z T^J(\vec q^{\,2},z)\big|_{z=-1}=3T_1^J(q^2) \tag{29}$$

Employing expressions (27)-(29), expanding integrand in eqs. (19),(20) into Taylor series, equating coefficients of identical powers z and performing one substraction at the point $\vec q_1^{\,2}=0$ we shall obtain a different set of equations for partial amplitudes (in all the equations argument -1 denotes that the function is taken at the point $z_1 = z_2 = -1$).

$$\mathrm{Re}\left[2T_0^{3/2}(\vec{q}^2)+T_0^{1/2}(\vec{q}^2)\right] = \mathrm{Re}\left[2T_0^{3/2}(0)+T_0^{1/2}(0)\right] + \frac{P}{8\pi^2}\int_0^\infty d\vec{q}'^2 \frac{|\vec{q}'|}{W(\vec{q}'^2)}\left\{\frac{2|T_0^{3/2}(\vec{q}'^2)|^2+|T_0^{1/2}(\vec{q}'^2)|^2}{\vec{q}'^2(\vec{q}'^2-\vec{q}^2)}\left[\vec{q}'^2\left(V(\vec{q}'^2,\vec{q}^2)-V(\vec{q}'^2,0)\right)+\vec{q}^2V(\vec{q}'^2,0)\right]+\right.$$

$$\left. 3\,\frac{2|T_1^{3/2}(\vec{q}'^2)|^2+|T_1^{1/2}(\vec{q}'^2)|^2}{\vec{q}'^2(\vec{q}'^2-\vec{q}^2)}\left[\vec{q}'^2\left(L(\vec{q}'^2,\vec{q}^2)-L(\vec{q}'^2,0)\right)+\vec{q}^2L(\vec{q}'^2,0)\right]\right\} \tag{30}$$

$$\mathrm{Re}\left[2T_1^{3/2}(\vec{q}^2)+T_1^{1/2}(\vec{q}^2)\right] = \frac{P}{8\pi^2}\int_0^\infty d\vec{q}'^2 \frac{|\vec{q}'|}{W(\vec{q}'^2)}\left\{\frac{2|T_0^{3/2}(\vec{q}'^2)|^2+|T_0^{1/2}(\vec{q}'^2)|^2}{\vec{q}'^2(\vec{q}'^2-\vec{q}^2)}\left[\vec{q}'^2\left(C(\vec{q}'^2,\vec{q}^2)-C(\vec{q}'^2,0)\right)+\vec{q}^2C(\vec{q}'^2,0)\right]+\right.$$

$$\left. 3\,\frac{2|T_1^{3/2}(\vec{q}'^2)|^2+|T_1^{1/2}(\vec{q}'^2)|^2}{\vec{q}'^2(\vec{q}'^2-\vec{q}^2)}\left[\vec{q}'^2\left(Q(\vec{q}'^2,\vec{q}^2)-Q(\vec{q}'^2,0)\right)+\vec{q}^2Q(\vec{q}'^2,0)\right]\right\} \tag{31}$$

$$\mathrm{Re}\left[T_0^{1/2}(\vec{q}^2)-T_0^{3/2}(\vec{q}^2)\right] = \mathrm{Re}\left[T_0^{1/2}(0)-T_0^{3/2}(0)\right] + \frac{P}{8\pi^2}\int_0^\infty d\vec{q}'^2 \frac{|\vec{q}'|}{W(\vec{q}'^2)}\left\{\frac{|T_0^{1/2}(\vec{q}'^2)|^2-|T_0^{3/2}(\vec{q}'^2)|^2}{\vec{q}'^2(\vec{q}'^2-\vec{q}^2)}\left[\vec{q}'^2\left(E(\vec{q}'^2,\vec{q}^2)-E(\vec{q}'^2,0)\right)+\vec{q}^2E(\vec{q}'^2,0)\right]+\right.$$

$$\left. 3\,\frac{|T_1^{1/2}(\vec{q}'^2)|^2-|T_1^{3/2}(\vec{q}'^2)|^2}{\vec{q}'^2(\vec{q}'^2-\vec{q}^2)}\left[\vec{q}'^2\left(F(\vec{q}'^2,\vec{q}^2)-F(\vec{q}'^2,0)\right)+\vec{q}^2F(\vec{q}'^2,0)\right]\right\} \tag{32}$$

$$\mathrm{Re}\left[T_1^{1/2}(\vec{q}^2)-T_1^{3/2}(\vec{q}^2)\right] = \frac{P}{8\pi^2}\int_0^\infty d\vec{q}'^2 \frac{|\vec{q}'|}{W(\vec{q}'^2)}\left\{\frac{|T_0^{1/2}(\vec{q}'^2)|^2-|T_0^{3/2}(\vec{q}'^2)|^2}{\vec{q}'^2(\vec{q}'^2-\vec{q}^2)}\left[\vec{q}'^2\left(G(\vec{q}'^2,\vec{q}^2)-G(\vec{q}'^2,0)\right)+\vec{q}^2G(\vec{q}'^2,0)\right]+\right.$$

$$\left. +\,3\,\frac{|T_1^{1/2}(\vec{q}'^2)|^2-|T_1^{3/2}(\vec{q}'^2)|^2}{\vec{q}'^2(\vec{q}'^2-\vec{q}^2)}\left[\vec{q}'^2\left(H(\vec{q}'^2,\vec{q}^2)-H(\vec{q}'^2,0)\right)+\vec{q}^2H(\vec{q}'^2,0)\right]\right\} \tag{33}$$

where

$$V(\vec{q}'^2,\vec{q}^2) = \eta_+(-1)+\partial_{z_1}\eta_+ + \tilde{\eta}_-(-1)\mathcal{D}(\vec{q}'^2,-1) + (\vec{q}'^2-\vec{q}^2)\partial_{z_1}\left[\tilde{\eta}_-\frac{\mathcal{D}(\vec{q}'^2,z_1)}{f(\vec{q}'^2,z_1)-\vec{q}^2}\right]_{z_1=-1}$$

$$L(\vec{q}'^2,\vec{q}^2) = -\partial_{z_1}\eta_+ - \tilde{\eta}_-(-1)\mathcal{D}(\vec{q}'^2,-1) + (\vec{q}'^2-\vec{q}^2)\partial_{z_1}\left[\tilde{\eta}_-\frac{\mathcal{D}(\vec{q}'^2,z_1)\,z_2(\vec{q}'^2,z_1)}{f(\vec{q}'^2,z_1)-\vec{q}^2}\right]_{z_1=-1}$$

$$C(\vec{q}'^2,\vec{q}^2) = \partial_{z_1}\eta_+ + (\vec{q}'^2-\vec{q}^2)\partial_{z_1}\left[\tilde{\eta}_-\frac{\mathcal{D}(\vec{q}'^2,z_1)}{f(\vec{q}'^2,z_1)-\vec{q}^2}\right]_{z_1=-1}$$

$$Q(\vec{q}'^2,\vec{q}^2) = \eta_+(-1) - \partial_{z_1}\eta_+ + (\vec{q}'^2-\vec{q}^2)\partial_{z_1}\left[\tilde{\eta}_-\frac{\mathcal{D}(\vec{q}'^2,z_1)\,z_2(\vec{q}'^2,z_1)}{f(\vec{q}'^2,z_1)-\vec{q}^2}\right]_{z_1=-1}$$

$$E(\vec{q}'^2,\vec{q}^2) = \alpha_+(-1)+\partial_{z_1}\alpha_+ + \tilde{\alpha}_-(-1)\mathcal{D}(\vec{q}'^2,-1) + (\vec{q}'^2-\vec{q}^2)\partial_{z_1}\left[\tilde{\alpha}_-\frac{\mathcal{D}(\vec{q}'^2,z_1)}{f(\vec{q}'^2,z_1)-\vec{q}^2}\right]_{z_1=-1}$$

$$F(\vec{q}'^2,\vec{q}^2) = -\partial_{z_1}\alpha_+ - \tilde{\alpha}_-(-1)\mathcal{D}(\vec{q}'^2,-1) + (\vec{q}'^2-\vec{q}^2)\partial_{z_1}\left[\tilde{\alpha}_-\frac{\mathcal{D}(\vec{q}'^2,z_1)\,z_2(\vec{q}'^2,z_1)}{f(\vec{q}'^2,z_1)-\vec{q}^2}\right]_{z_1=-1}$$

$$G(\vec{q}'^2,\vec{q}^2) = \partial_2\alpha_+ + (\vec{q}'^2-\vec{q}^2)\,\partial_{z_1}\Big[\tilde{\alpha} - \frac{\mathcal{D}(\vec{q}'^2,z_1)}{f(\vec{q}'^2,z_1)-\vec{q}^2}\Big]_{z_1=-1}$$

$$H(\vec{q}'^2,\vec{q}^2) = -\partial_{z_1}\alpha_+ + \alpha_+(-1) + (\vec{q}'^2-\vec{q}^2)\,\partial_{z_1}\Big[\tilde{\alpha} - \frac{\mathcal{D}(\vec{q}'^2,z_1)\, z_2(\vec{q}'^2,z_1)}{f(\vec{q}'^2,z_1)-\vec{q}^2}\Big]_{z_1=-1}$$

It is observed that in eqs.(25),(26) integration in u and β was performed from $-2.79\mu^2$ to $-\frac{2\mu^2}{1-z_1}$ while in eqs. (30)-(33) from $-4\mu^2$ to $-\mu^2$.

Conclusion

It can be seen from Eqs. (26),(32)-(33) that we have derived systems of approximated non-linear integral equations which may have several solutions.

One of the possible solutions of these equations is the $T^{1/2} = T^{3/2}$ solution, which coincides with the results obtained in refs.[6,13]. However, this solution is to be considered approximate, sinoe the substraction terms $\mathrm{Re}\,T^{1/2}(0)$ and $\mathrm{Re}\,T^{3/2}(0)$ must not, generally speaking, be equal to each other, and, besides, the cross-section of the charge-exchange $K-N$ scattering is known to be non-zero.

Inequality of the substruction terms can be naturelly connected with a different effect exersized by higher-energy baryon states on isotopic spin states $T^{1/2}$ and $T^{3/2}$ of the π-K scattering amplitudes.

It is possible that due to the non-linearity of eqs. (26) and (32)-(33) other solutions than the $T^{1/2}=T^{3/2}$ solution may exist even in the case when $\mathrm{Re}\,T^{1/2}(0)=\mathrm{Re}\,T^{3/2}(0)$.

If one takes into account hyperon states in relations (22) and (28), then eqs. (26) and (32)-(33) will involve additional terms which will even more affect the $T^{1/2}=T^{3/2}$ -solution. However, due to the high energy threshold of hyperon states their role in contributing to eqs. (26), (32)-(33) remains so far unclear.

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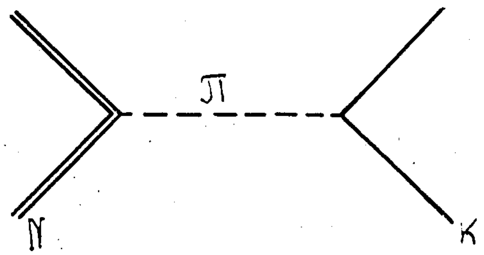

Fig.1

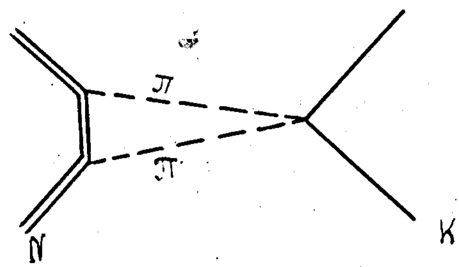

Fig.2

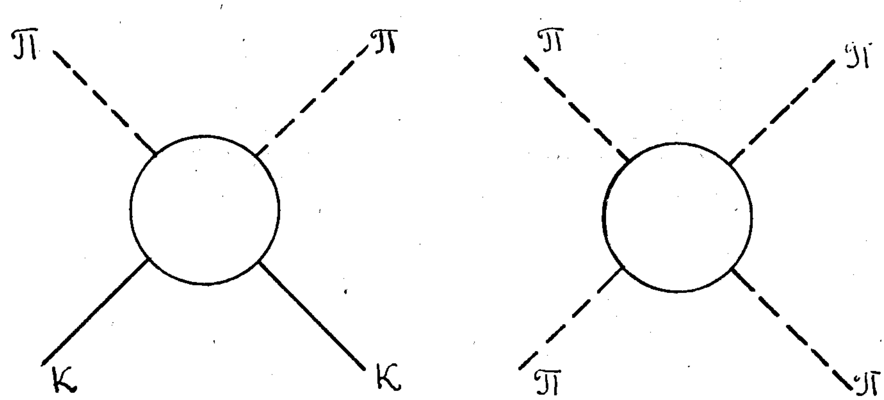

Fig.3

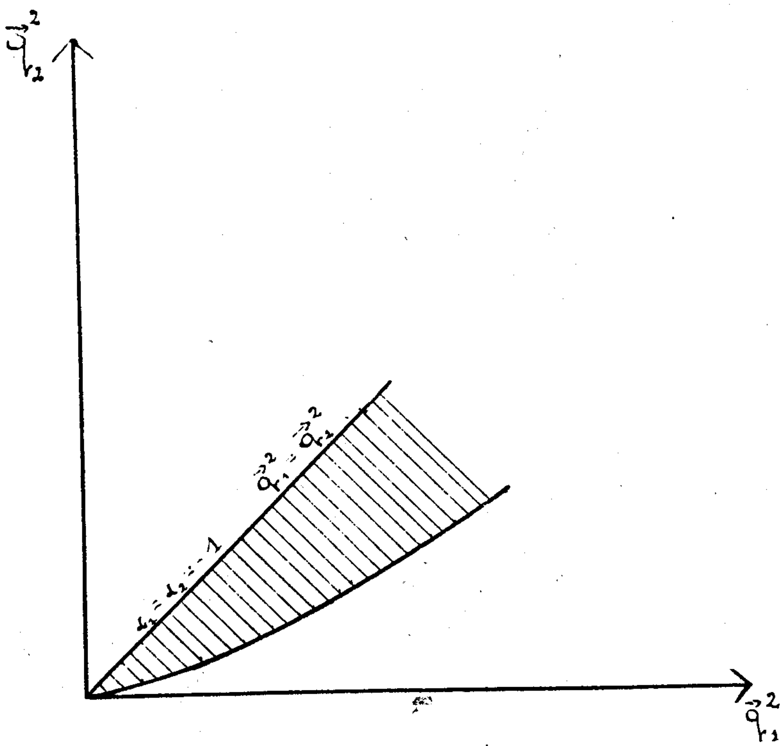

Fig.4

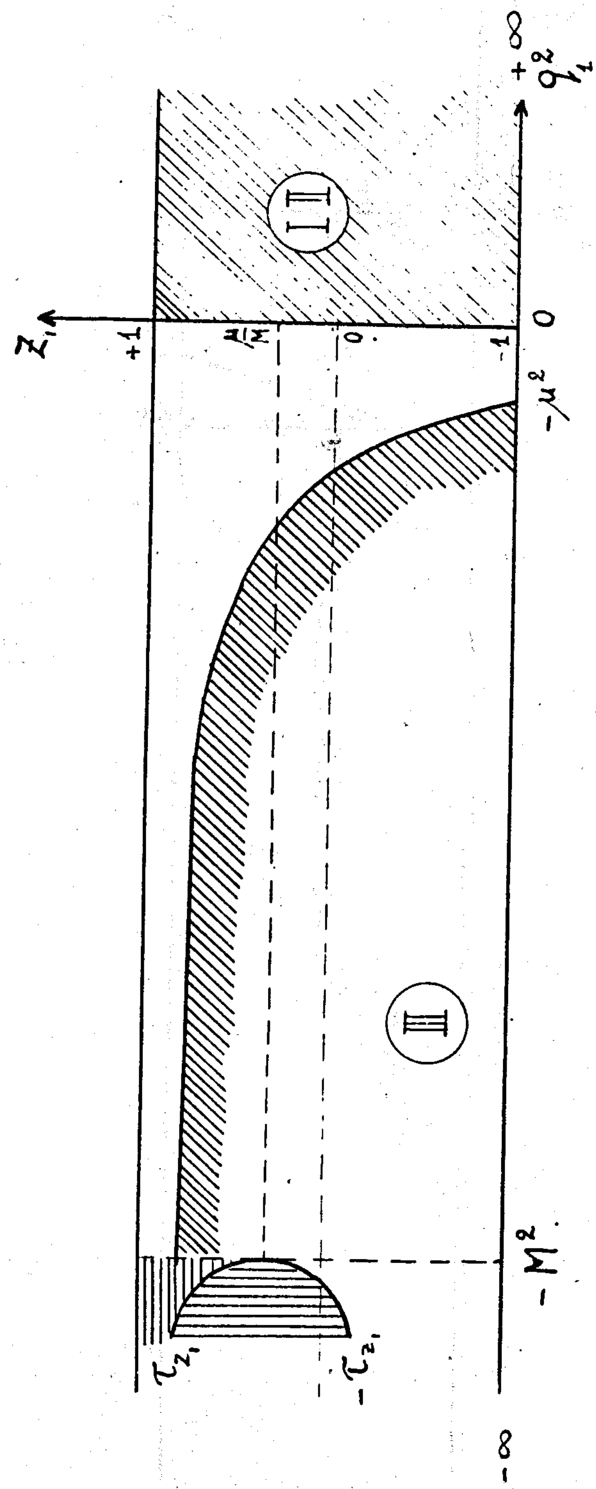

Fig.5

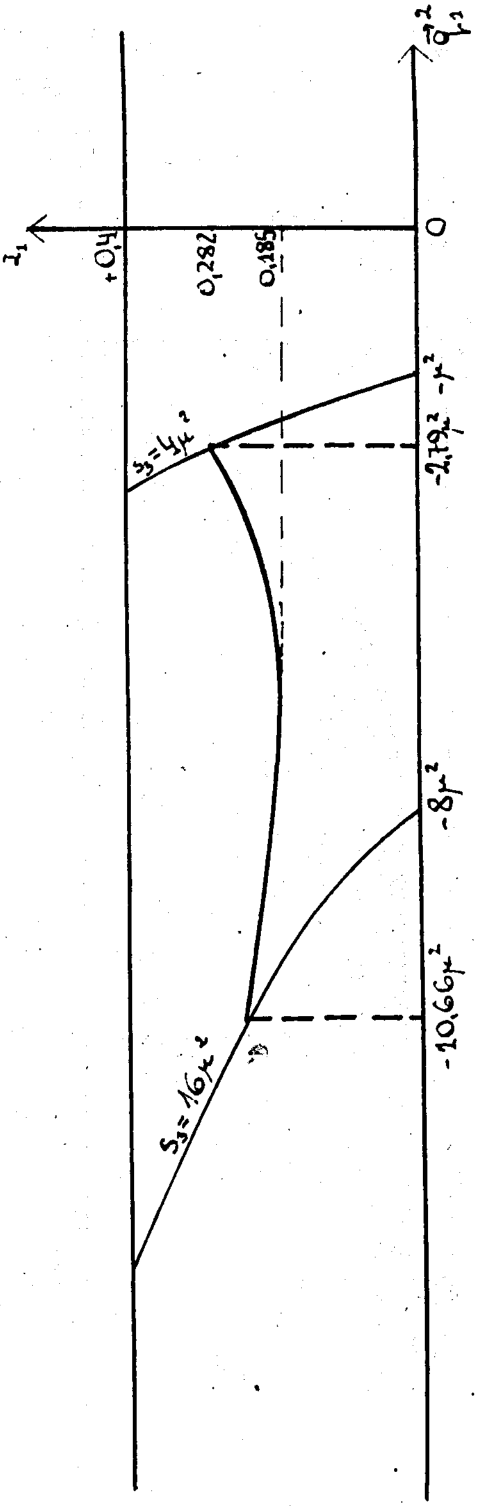

Fig.6