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ОБЪЕДИНЕННЫЙ ИНСТИТУТ ЯДЕРНЫХ ИССЛЕДОВАНИЙ

ЛАБОРАТОРИЯ ТЕОРЕТИЧЕСКОЙ ФИЗИКИ

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INTEGRAL EQUATIONS FOR THE $\pi - \pi$ -SCATTERING AT LOW ENERGIES

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Abstract

The accuracy of the equations of Chew and Mandelstam on $\pi-\pi$ -scattering in the unphysical region is examined. A new set of equations is derived from the dispersion relation of fixed momentum transfer and the unitarity condition. The equations are quite different from the equations of Chew and Mandelstam with regards to the contributions from the unphysical region.

I. Introduction

Recently, Chew and Mandelstam/1/ derived a set of integral equations for the π-π-scattering. The singularities of the partial wave scattering amplitudes are located by means of the two-dimensional dispersion relation proposed by Mandelstam/2,3,4/ . The unitary condition in the unphysical region is obtained by analytic continuation from the physical region with the help of the Legendre expansion. The same methcd has been applied by others/5,6/ to the problems of $\pi-N$ - scattering and N-$\tilde{N}$ - annihilation. It was pointed out by Efremov, Meshcherjakov, Tzu and Shirkov/7/, that the method of the analytic continuation by means of the Legendre expansion has serious limitations. Large errors are introduced by neglecting higher waves in the unphysical region. The validity of the Legendre expansion is limited by the bounderies of the spectral functions. Actually, if the contribution from the high energy unphysical region is not cut off, divergent expressions appear in the integral equations. The degree of divergence of the coefficient of the legendre function increases with the degree of the Legendre function. These divergences cannot be removed by a finite number of subtractions. Thus it is unavoidable to cut off the contribution from the unphysical region beyond a certain limit.

Even if the contribution of the high energy region is cut off at $\nu = L = 10$*), there still remains a substantial region where the distance from the physical region is comparable with the distance from the boundary of the spectral functions. The scattering function in this region cannot be approximated by taking only the first two terms from the Legendre expansion.

In this paper the problem of the $\pi-\pi$ - scattering is reexamined. A set of integral equations is derived by using the unitarity condition together with dispersion relation for constant momentum transfer only, which has been rigorously proved first by Bogoljubov et al./8/). The subtractions are made at points different from those proposed by Chew and Mandelstam to avoid further approximations due to the neglection of the higher partial waves.

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* ν is the square of the momentum of the π -meson in the centre of mass system; the mass of the π -meson is taken as unity.

In the next section, the limitation of the analytic continuation by Legendre expansion is investigated. In the third section the integral equations are derived and the problem of subtraction is discussed. In the last section the error due to the neglection of the higher partial waves is estimated. In the region where the unitarity condition holds rigorously, the error caused by dropping the higher partial waves is estimated to be less than 10%.

II. The Limitations due to the Analytic Continuation by Legendre Expansion

The notations used in this paper are almost identical with those used in /1/. For convenience's sake, they are explained in this paragraph. The following three reactions

$$\begin{aligned}
&(\mathrm{I}) \quad (p_1,\alpha)+(p_2,\beta)\rightarrow(-p_3,\gamma)+(-p_4,\delta),\\
&(\mathrm{II}) \quad (p_1,\alpha)+(p_4,\delta)\rightarrow(-p_2,\beta)+(-p_3,\gamma),\\
&(\mathrm{III}) \quad (p_1,\alpha)+(p_3,\gamma)\rightarrow(-p_2,\beta)+(-p_4,\delta),
\end{aligned} \tag{1}$$

are described by a single Green-function

$$T=A(s,t,\bar{t})\delta_{\alpha\beta}\delta_{\gamma\delta}+B(s,t,\bar{t})\delta_{\alpha\gamma}\delta_{\beta\delta}+C(s,t,\bar{t})\delta_{\alpha\delta}\delta_{\beta\gamma}. \tag{2}$$

The p's denote the four-momenta in the inward direction. α, β, γ, δ are the isotopic spin indices. The invariant variables s, t and $\bar{t}$ are defined as

$$\begin{aligned}
s&=-(p_1+p_2)^2=-(p_3+p_4)^2,\\
t&=-(p_1+p_4)^2=-(p_2+p_3)^2,\\
\bar{t}&=-(p_1+p_3)^2=-(p_2+p_4)^2.
\end{aligned} \tag{3}$$

In particular, if ν and θ denote the square of the π - meson momentum and the scattering angle in the centre of mass system of reaction (1) respectively, we have:

$$\begin{aligned}
s&=4(\nu+1),\\
\bar{t}&=-2\nu(1+\cos\theta),\\
t&=-2\nu(1-\cos\theta).
\end{aligned} \tag{4}$$

Let A^{I} represents the scattering amplitude of the isotopic spin I of the reaction (1). Between A^{I} and A, B, C exist the following relations

$$A^0 = 3A + B + C,$$
$$A^1 = \quad B - C,$$
$$A^2 = \quad B + C. \qquad (5)$$

The scattering amplitude can be expanded in terms of partial waves

$$A^I(\nu, \cos\theta) = \sum_{\substack{l=\text{even},\, I=0,2 \\ l=\text{odd},\, I=1}} (2l+1)\, A_l^I(\nu)\, P_l(\cos\theta), \qquad (6)$$

$$A_l^I(\nu) = \sqrt{\frac{1+\nu}{\nu}}\; e^{i\delta_l^I} \sin\delta_l^I .$$

l is here the quantum number of the orbital angular momentum; δ_l^I are the phase shifts. In the two meson approximation, the unitarity condition can be written as

$$\operatorname{Im} A_l^I(\nu) = \sqrt{\frac{\nu}{1+\nu}}\; |A_l^I(\nu)|^2. \qquad (7)$$

Chew and Mandelstam used the following double-dispersion representation for A

$$\begin{aligned} A(s,t,\bar t) = {} & \frac{1}{\pi^2}\int ds' \int dt' \frac{a_{13}(s',t')}{(s'-s)(t'-t)} \\ & + \frac{1}{\pi^2}\int ds' \int d\bar t' \frac{a_{12}(\bar t', s')}{(s'-s)(\bar t'-\bar t)} \\ & + \frac{1}{\pi^2}\int dt' \int d\bar t' \frac{a_{23}(t',\bar t')}{(\bar t'-\bar t)(t'-t)} . \end{aligned} \qquad (8)$$

a_{12}, a_{13} and a_{23} are the spectral functions. B and C have similar representations.

The limitations of the Legendre expansion (6) in the unphysical region can be seen at once in the following way. Chew and Mandelstam derived their integral equations from the partial wave dispersion relations

$$A_l^I(\nu) = \frac{1}{\pi}\int_0^\infty d\nu' \frac{\operatorname{Im} A_l^I(\nu')}{\nu'-\nu} + \frac{1}{\pi}\int_{-\infty}^{-1} d\nu' \frac{\operatorname{Im} A_l^I(\nu')}{\nu'-\nu} . \qquad (9)$$

(Equation (V.I) in [1]). According to (IV.5) of [1]

$$\mathrm{Im}\, A_{\ell}^{I}(\nu') = \sum_{I'} \alpha_{II'} \int_{0}^{-\nu'-1} \frac{d\nu''}{\nu'} P_{\ell}\left(1+2\frac{\nu''+1}{\nu'}\right) \mathrm{Im}\, A^{I}(\nu'', \cos\theta'') \qquad (10)$$

in the unphysical region $\nu' \lesssim -1$, where

$$\cos\theta'' = 1 + 2\frac{\nu'+1}{\nu''} . \qquad (11)$$

If (6) is put into (10) , and (10) into (9) , we obtain immediately

$$A_{\ell}^{I}(\nu) = \frac{1}{\pi}\int_{0}^{\infty} d\nu' \frac{\mathrm{Im}\, A_{\ell}^{I}(\nu')}{\nu'-\nu} + \frac{1}{\pi}\int_{0}^{\infty} d\nu' \sum_{\ell', I'} \alpha_{II'} (2\ell'+1) \beta_{\ell\ell'}(\nu', \nu)\, \mathrm{Im} A_{\ell'}^{I'}(\nu') \qquad (12)$$

where

$$\beta_{\ell\ell'}(\nu', \nu) = \int_{-\infty}^{-\nu'-1} d\nu'' \frac{1}{\nu''(\nu''-\nu)} P_{\ell}\left(1+2\frac{\nu'+1}{\nu''}\right) P_{\ell'}\left(1+2\frac{\nu''+1}{\nu'}\right). \qquad (13)$$

This expression diverges for $\ell' \geq 1$. With one subtraction it is possible to get rid of the divergence of the coefficient of the p-wave and to introduce a π-π - interaction constant λ into the theory. But the remaining divergence cannot be got rid of by a finite number of subtractions. This difficulty is a result of the unjustified application of the analytic continuation by means of the Legendre expansion, which begins to fail at the boundary of the spectral functions.

Chew and Mandelstam have to cut off the dispersion integral at $|\nu'| = L \simeq 10$, which is roughly the limit of the validity of the Legendre expansion and then discard the contributions from the d- and higher waves. However, even for $|\nu'| \lesssim 10$, there is a substantial region, whose distance from the boundary of the spectral functions is comparable with its distance from the physical region as can be seen from Fig. 1. It is doubtful, that the first two terms of the Legendre expansion are sufficient to represent the scattering function in this region with necessary accuracy.

The error introduced thereby can be estimated roughly in the following way. From the double-dispersion relation and the crossing relations, it follows

$$\mathrm{Im}\, A^{0}(\nu'', \cos\theta'') = \frac{1}{\pi}\int_{t_0(\nu'')}^{\infty} \frac{dt'}{2\nu''} (2a_{13} + b_{13} + c_{13}) \left\{ \frac{1}{\tau - \cos\theta''} + \frac{1}{\tau + \cos\theta''} \right\}, \qquad (14)$$

$$\mathrm{Im}\,A^1(\nu'',\cos\theta'') = \frac{1}{\pi}\int_{t_0(\nu'')}^{\infty}\frac{dt'}{2\nu''}(b_{13}-c_{13})\left\{\frac{1}{\tau-\cos\theta''}-\frac{1}{\tau+\cos\theta''}\right\},$$

$$\mathrm{Im}\,A^2(\nu'',\cos\theta'') = \frac{1}{\pi}\int_{t_0(\nu'')}^{\infty}\frac{dt'}{2\nu''}(b_{13}+c_{13})\left\{\frac{1}{\tau-\cos\theta''}+\frac{1}{\tau+\cos\theta''}\right\},$$

where

$$\tau = 1+\frac{t'}{2\nu''}. \tag{15}$$

$t_0(\nu'')$ is the boundary of the spectral functions a_{13}, b_{13} and c_{13} which are functions of $s''=4(1+\nu'')$ and t'. They are independent of $\cos\theta''$. The dependence of $A^I(\nu'',\cos\theta'')$ on $\cos\theta''$ is thus explicitly given by the denominators inside the curly brackets. According to the theorem of Heine[9]), we have

$$\frac{1}{\tau\mp\cos\theta''} = \sum_{\ell'}(2\ell'+1)\,Q_{\ell'}(\tau)\,P_{\ell'}(\cos\theta''). \tag{16}$$

$Q_\ell(\tau)$ are the Legendre functions of the second kind. To take only two terms from the Legendre expansion is equivalent to the following approximation:

$$\begin{aligned}\frac{1}{\tau-\cos\theta''}+\frac{1}{\tau+\cos\theta''} &\cong 2\,Q_0(\tau),\\ \frac{1}{\tau-\cos\theta''}-\frac{1}{\tau+\cos\theta''} &\cong 6\cos\theta''\,Q_1(\tau).\end{aligned} \tag{17}$$

Now the error involved can be estimated easily. Let us examine the case of $\nu''=3$, where the unitarity condition begins to break down. The nearest singularities, which give the most important contribution to (14) are in the neighbourhood of the boundary of the spectral function $t_0(\nu''=3)=\frac{64}{3}$. The corresponding value of τ is $\frac{41}{9}$. For this value and the scattering angle $\theta=0,\ \pi/2$ we easily obtain the result, quoted in tab. 1:

	$\cos\theta''$	$\frac{1}{\tau-\cos\theta''}+\frac{1}{\tau+\cos\theta''}$	$2Q_0(t)$	$\frac{1}{\tau-\cos\theta''}-\frac{1}{\tau+\cos\theta''}$	$6\cos\theta''Q_1(t)$
$\cos\theta=1$	1	0.46	0.44	0.101	0.099
$\cos\theta=0$	$\frac{11}{3}$	1.25	0.45	1.00	0.36

Table 1.

At $\theta = 0$ the error of the approximations (17) is only 4% and 2% respectively. But at $\theta = \pi/2$ it errs by a factor 3. It is then evident from Fig. 1, that even if a cut off is made at $|\nu'| = 10$, as proposed in /1/, there is a large region of integration, where the imaginary part of the scattering function cannot be approximated by taking only two terms from the Legendre expansion.

On the otherhand, the result also shows, that the approximation (17) is quite good at $\theta = 0$. In accordance with the idea put forward in /7/ we derive in the following the integral equations for the π-π- scattering from the dispersion relation for fixed momentum transfer, which has been proved rigorously by Bogoljubov et al./8/ The method used is similar to that used by Chew, Goldberger, Low and Nambu/10/ in their paper on π-N - scattering.

III. Derivation of the Integral Equations

In our derivation of the equations we will neglect the g - and higher waves in comparison with s-waves, h- and higher waves in comparison with p-wave. Then from (6) we obtain the following expressions

$$\begin{aligned} A_0^0(\nu) &\cong A^0(\nu, t=0) - \frac{2\nu}{3}\left.\frac{\partial A^0(\nu,t)}{\partial t}\right|_{t=0}, \\ A_1^1(\nu) &\cong \frac{2}{5} A^1(\nu, t=0) - \frac{2\nu}{15}\left.\frac{\partial A^1(\nu,t)}{\partial t}\right|_{t=0}, \\ A_0^2(\nu) &\cong A^2(\nu, t=0) - \frac{2\nu}{3}\left.\frac{\partial A^2(\nu,t)}{\partial t}\right|_{t=0}. \end{aligned} \tag{18}$$

With the help of the crossing relations the dispersion relations for fixed momentum transfer can be written as

$$\begin{pmatrix} A(\nu,t) \\ B(\nu,t) \\ C(\nu,t) \end{pmatrix} = \frac{1}{\pi}\int_0^\infty \frac{d\nu'}{\nu'-\nu}\,\mathrm{Im}\begin{pmatrix} A(\nu',t) \\ B(\nu',t) \\ C(\nu',t) \end{pmatrix} + \frac{1}{\pi}\int_0^\infty \frac{d\nu'}{1+\nu+\nu'+\frac{t}{4}}\,\mathrm{Im}\begin{pmatrix} C(\nu',t) \\ B(\nu',t) \\ A(\nu',t) \end{pmatrix}. \tag{19}$$

The dispersion relations for the derivatives of the scattering amplitudes have the following form:

$$\begin{pmatrix} \frac{\partial A(\nu,t)}{\partial t} \\ \frac{\partial B(\nu,t)}{\partial t} \\ \frac{\partial C(\nu,t)}{\partial t} \end{pmatrix} = \frac{1}{\pi}\int_0^\infty \frac{d\nu'}{\nu'-\nu}\,\mathrm{Im}\begin{pmatrix} \frac{\partial A(\nu',t)}{\partial t} \\ \frac{\partial B(\nu',t)}{\partial t} \\ \frac{\partial C(\nu',t)}{\partial t} \end{pmatrix} + \frac{1}{\pi}\int_0^\infty \frac{d\nu'}{1+\nu+\nu'+\frac{t}{4}}\,\mathrm{Im}\begin{pmatrix} \frac{\partial C(\nu',t)}{\partial t} \\ \frac{\partial B(\nu',t)}{\partial t} \\ \frac{\partial A(\nu',t)}{\partial t} \end{pmatrix} -$$

$$- \frac{1}{4\pi} \int_0^\infty \frac{d\nu'}{(1+\nu+\nu'+\frac{t}{4})^2} \, \mathrm{Im} \begin{pmatrix} C(\nu', t) \\ B(\nu', t) \\ A(\nu', t) \end{pmatrix} . \tag{20}$$

Since the Legendre expansion for the imaginary part of the scattering function converges better than that of the real part, we neglect the imaginary parts of the d- and higher waves in comparison with that of the s-waves and neglect the imaginary parts of the f- and higher waves in comparison with that of the p- waves. That this is justified can easily be seen in a similar form as the estimations were made in the last chapter.

From (5), (6), (18), (19) and (20) we then obtain immediately the following equations for the s- and p- partial wave amplitudes in unsubtracted form

$$\begin{aligned} A_0^0(\nu) = {} & \frac{1}{\pi} \int_0^\infty \frac{d\nu'}{\nu'-\nu} \, \mathrm{Im}\, A_0^0(\nu') \\ & + \frac{1}{3\pi} \int_0^\infty \frac{d\nu'}{1+\nu+\nu'} \left[\mathrm{Im}\, A_0^0(\nu') + \left(3\frac{\nu}{\nu'} - 9\right) \mathrm{Im}\, A_1^1(\nu') + 5\, \mathrm{Im}\, A_0^2(\nu') \right] \\ & + \frac{\nu}{18\pi} \int_0^\infty \frac{d\nu'}{(1+\nu+\nu')^2} \left[\mathrm{Im}\, A_0^0(\nu') - 9\, \mathrm{Im}\, A_1^1(\nu') + 5\, \mathrm{Im}\, A_0^2(\nu') \right], \end{aligned} \tag{21}$$

$$\begin{aligned} A_1^1(\nu) = {} & \frac{1}{\pi} \int_0^\infty \frac{d\nu'}{\nu'-\nu} \, \mathrm{Im}\, A_1^1(\nu') + \frac{1}{5\pi} \int_0^\infty \frac{d\nu'}{\nu'} \, \mathrm{Im}\, A_1^1(\nu') \\ & + \frac{1}{15\pi} \int_0^\infty \frac{d\nu'}{1+\nu+\nu'} \left[-2\, \mathrm{Im}\, A_0^0(\nu') + \left(9 - \frac{3}{2}\frac{\nu}{\nu'}\right) \mathrm{Im}\, A_1^1(\nu') + 5\, \mathrm{Im}\, A_0^2(\nu') \right] \\ & + \frac{\nu}{180\pi} \int_0^\infty \frac{d\nu'}{(1+\nu+\nu')^2} \left[-2\, \mathrm{Im}\, A_0^0(\nu') + 9\, \mathrm{Im}\, A_1^1(\nu') + 5\, \mathrm{Im}\, A_0^2(\nu') \right], \end{aligned} \tag{22}$$

$$A_0^2(\nu) = \frac{1}{\pi}\int_0^\infty \frac{d\nu'}{\nu'-\nu}\,\mathrm{Im}\,A_0^2(\nu')$$

$$+\frac{1}{6\pi}\int_0^\infty \frac{d\nu'}{1+\nu+\nu'}\left[2\,\mathrm{Im}\,A_0^0(\nu') + \left(9-3\frac{\nu}{\nu'}\right)\mathrm{Im}\,A_1^1(\nu') + \mathrm{Im}\,A_0^2(\nu')\right]$$

$$+\frac{\nu}{36\pi}\int_0^\infty \frac{d\nu'}{(1+\nu+\nu')^2}\left[2\,\mathrm{Im}\,A_0^0(\nu') + 9\,\mathrm{Im}\,A_1^1(\nu') + \mathrm{Im}\,A_0^2(\nu')\right]. \qquad (23)$$

For the purpose of estimating the order of magnitude of the d- and f- wave amplitudes the following expressions can be used

$$\frac{A_2^0(\nu)}{\nu} = \frac{1}{5\pi}\int_0^\infty d\nu'\,\frac{\mathrm{Im}\,A_1^1(\nu')}{\nu'(1+\nu+\nu')}$$

$$-\frac{1}{30\pi}\int_0^\infty d\nu'\,\frac{\left\{\frac{1}{3}\,\mathrm{Im}\,A_0^0(\nu') - 3\,\mathrm{Im}\,A_1^1(\nu') + \frac{5}{3}\,\mathrm{Im}\,A_0^2(\nu')\right\}}{(1+\nu+\nu')^2},$$

$$\frac{A_3^1(\nu)}{\nu} = \frac{3}{70\pi}\int_0^\infty d\nu'\,\frac{\mathrm{Im}\,A_1^1(\nu')}{\nu'(1+\nu+\nu')}$$

$$+\frac{1}{70\pi}\int_0^\infty d\nu'\left[1+\frac{\nu}{1+\nu+\nu'}\right]\frac{-\frac{1}{3}\,\mathrm{Im}\,A_0^0(\nu') + \frac{3}{2}\,\mathrm{Im}\,A_1^1(\nu') + \frac{5}{6}\,\mathrm{Im}\,A_0^2(\nu')}{(1+\nu')\,(1+\nu+\nu')},$$

$$\frac{A_2^2(\nu)}{\nu} = \frac{1}{10\pi}\int_0^\infty d\nu'\,\frac{\mathrm{Im}\,A_1^1(\nu')}{\nu'(1+\nu+\nu')}$$

$$-\frac{1}{30\pi}\int_0^\infty d\nu'\,\frac{\frac{1}{3}\,\mathrm{Im}\,A_0^0(\nu') + \frac{3}{2}\,\mathrm{Im}\,A_1^1(\nu') + \frac{1}{6}\,\mathrm{Im}\,A_0^2(\nu')}{(1+\nu+\nu')^2}$$

It is to be stressed, that these formulae are not as accurate as the equations (21) - (23), since g- waves are neglected here with respect to d- waves, and h- wave is neglected with respect to f- wave. The substraction is performed by Chew and Mandelstam for $\nu = -\frac{2}{3}$ (or $s = \frac{4}{3}$), where they must integrate along the line $s = \frac{4}{3}$ between $\cos\theta = \pm 1$. They connect the given value λ of the scattering amplitude at the symmetric point $s = t = \bar{t} = \frac{4}{3}$ with the other points lying on this line by means of analytic continuation with help of Legendre expansion. From this expansion they take only the first terms, that means a further approximation is introduced, caused by neglecting non-physical terms.

In our approach we can avoid further approximations, when the subtraction for A and C is made at the point $0''$ ($s = \bar{t} = 2$, $t = 0$), (cf. Fig. 2) where

$$A\left(\nu = -\tfrac{1}{2}, t = 0\right) = C\left(-\tfrac{1}{2}, 0\right) = \Lambda$$

and for A^1 at the point $0'''$ ($s = 4$, $\bar{t} = t = 0$), where

$$A^1(0,0) = B(0,0) - C(0,0) = 0 .$$

Thus only one parameter is introduced. It is to be noticed, that Λ is different from the constant λ introduced in/1/. The connection between Λ and λ is given in the appendix.

After the subtraction the equations for the s- and p - wave amplitudes (21) – (25) become:

$$\begin{aligned}
A_0^0(\nu) = 5\Lambda &+ \frac{1}{\pi}\int_0^\infty d\nu' \, \mathrm{Im}\, A_0^0(\nu') \left\{ \frac{\nu+\frac{1}{2}}{(\nu'-\nu)(\nu'+\frac{1}{2})} - \frac{\frac{2}{3}(\nu+\frac{1}{2})}{(\nu'+\frac{1}{2})(1+\nu+\nu')} \right. \\
&\qquad \left. + \frac{\frac{1}{3}\nu}{(1+\nu')(1+\nu+\nu')} + \frac{\frac{1}{18}\nu}{(1+\nu+\nu')^2} \right\} \\
&+ \frac{1}{\pi}\int_0^\infty d\nu' \, \mathrm{Im}\, A_1^1(\nu') \left\{ \frac{\frac{9}{2}(\nu+\frac{1}{2})}{(\nu'+\frac{1}{2})(1+\nu+\nu')} - \frac{\frac{3}{2}}{\nu'(\nu'+\frac{1}{2})} \right. \\
&\qquad \left. - \frac{\frac{3}{2}\nu}{(1+\nu')(1+\nu+\nu')} + \frac{\nu}{\nu'(1+\nu+\nu')} - \frac{\frac{1}{2}\nu}{(1+\nu+\nu')^2} \right\} \\
&- \frac{5}{6\pi}\int_0^\infty d\nu' \, \frac{\mathrm{Im}\, A_0^2(\nu')}{1+\nu+\nu'} \left\{ \frac{\nu+\frac{1}{2}}{\nu'+\frac{1}{2}} + \frac{\nu}{1+\nu'} - \frac{\frac{1}{3}\nu}{1+\nu+\nu'} \right\},
\end{aligned} \qquad (25)$$

$$A_1^1(\nu) = \frac{\nu}{90\pi}\int_0^\infty d\nu'\,\mathrm{Im}\,A_0^0(\nu')\,\frac{11+\frac{\nu}{1+\nu+\nu'}}{(1+\nu')(1+\nu+\nu')}$$

$$+\frac{\nu}{\pi}\int_0^\infty d\nu'\,\mathrm{Im}\,A_1^1(\nu')\left\{\frac{1}{\nu'(\nu'-\nu)}-\frac{\frac{1}{10}}{\nu'(1+\nu+\nu')}-\frac{1}{20}\,\frac{11+\frac{\nu}{1+\nu+\nu'}}{(1+\nu')(1+\nu+\nu')}\right\}$$

$$-\frac{\nu}{36\pi}\int_0^\infty d\nu'\,\mathrm{Im}\,A_0^2(\nu')\,\frac{11+\frac{\nu}{1+\nu+\nu'}}{(1+\nu')(1+\nu+\nu')}\;, \tag{26}$$

$$A_0^2(\nu) = 2\Lambda + \frac{1}{\pi}\int_0^\infty d\nu'\,\mathrm{Im}\,A_0^0(\nu')\left\{\frac{-\frac{2}{3}(\nu+\frac{1}{2})}{(1+\nu+\nu')(\nu'+\frac{1}{2})}\right.$$

$$\left.+\frac{\frac{1}{3}\nu}{(1+\nu')(1+\nu+\nu')}+\frac{\frac{1}{18}\nu}{(1+\nu+\nu')^2}\right\}$$

$$+\frac{1}{\pi}\int_0^\infty d\nu'\,\mathrm{Im}\,A_1^1(\nu')\left\{\frac{-\frac{3}{2}}{\nu'(\nu'+\frac{1}{2})}-\frac{\frac{3}{2}\nu}{(1+\nu')(1+\nu+\nu')}\right.$$

$$\left.-\frac{\frac{1}{2}\nu}{\nu'(1+\nu+\nu')}+\frac{\frac{1}{4}\nu}{(1+\nu+\nu')^2}\right\}$$

$$+\frac{1}{\pi}\int_0^\infty d\nu'\,\mathrm{Im}\,A_0^2(\nu')\left\{\frac{\nu+\frac{1}{2}}{(\nu'+\frac{1}{2})(\nu'-\nu)}+\frac{\frac{2}{3}(\nu+\frac{1}{2})}{(\nu'+\frac{1}{2})(1+\nu+\nu')}\right.$$

$$\left.-\frac{\frac{5}{6}\nu}{(1+\nu')(1+\nu+\nu')}+\frac{\frac{1}{36}\nu}{(1+\nu+\nu')^2}\right\}. \tag{27}$$

In these equations the coefficients of the p-wave amplitude on the righthand sides are quite different from the coresponding terms of the Chew-Mandelstam-equations, sometimes they differ even in sign.

All the mentioned difficulties, as analytic continuation in unphysical regions, divergent coefficients and the necessity of a cut off, are avoided in these equations.

IV. The Estimation of the Error, Due to the Neglect of Higher Waves in the Physical Region

The error due to the neglecting of d- and higher waves for the imaginary part and g- and higher waves for the real part of the scattering amplitude can be estimated by a similar method, used in section II.

Let us begin with the estimation of the error with respect to the imaginary part. We get the following expressions

$$\left(1-\gamma\frac{\nu}{\nu''}\frac{d}{d\cos\theta''}\right)\left(\frac{1}{\tau-\cos\theta''}+\frac{1}{\tau+\cos\theta''}\right)\Bigg|_{\cos\theta''=1} \cong 2Q_0(\tau), \qquad (28a)$$

$$\left(1-\gamma\frac{\nu}{\nu''}\frac{d}{d\cos\theta''}\right)\left(\frac{1}{\tau-\cos\theta''}-\frac{1}{\tau+\cos\theta''}\right)\Bigg|_{\cos\theta''=1} \cong 6\left(1-\gamma\frac{\nu}{\nu''}\right)Q_1(\tau). \qquad (28b)$$

$\gamma=\frac{1}{3}$ for the s-wave equations, $\gamma=\frac{1}{6}$ for the p-wave equation. τ is put equal to $\frac{41}{9}$ as before. At lower energies τ becomes larger, the error is smaller. At higher energies τ becomes smaller, but the unitarity condition (7) begins to fail. After simple calculations, the following results are obtained:

In the region where the unitarity condition (7) holds, the error of the approximation (28 a) is less than 3% , of the approximation (28 b) less than 8 %.

The estimation of the error with respect to the real part of the scattering amplitude is more difficult. The most severe limitation of the validity of the Legendre expansion is due to a thin slice of the distant singularity, which does not give important contributions to the scattering function. However, a rough estimation shows, the error must be below 10% in the region, where the unitary condition (7) is valid.

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Appendix

The connection between the substraction constant λ , introduced in /1/ and the subtraction constant Λ , used in our equations, is given by the following formula

$$-\lambda = \Lambda - \frac{1}{30\pi}\int_0^\infty d\nu' \, \mathrm{Im}\, A_0^0(\nu') \left\{ \frac{1}{(\nu'+\frac{2}{3})(\nu'+\frac{1}{2})} + \frac{\frac{4}{3}}{(\nu'+\frac{1}{3})(1+\nu')} - \frac{\frac{2}{3}}{(\nu'+\frac{1}{3})(\nu'+\frac{1}{2})} + \frac{\frac{1}{3}}{(\nu'+\frac{1}{3})^2} \right\}$$

$$- \frac{1}{30\pi}\int_0^\infty d\nu' \, \mathrm{Im}\, A_1^1(\nu') \left\{ \frac{9}{\nu'(\nu'+\frac{1}{2})} + \frac{2}{\nu'(\nu'+\frac{1}{3})} - \frac{4(\frac{1}{2}-\frac{1}{\nu'})}{(\nu'+\frac{1}{3})(\nu'+1)} + \frac{\frac{9}{2}}{(\nu'+\frac{1}{3})(\nu'+\frac{1}{2})} - \frac{3}{(\nu'+\frac{1}{3})^2} \right\}$$

$$+ \frac{1}{90\pi}\int_0^\infty d\nu' \, \mathrm{Im}\, A_0^2(\nu') \left\{ \frac{10}{(\nu'+\frac{1}{3})(1+\nu')} + \frac{\frac{5}{2}}{(\nu'+\frac{1}{3})(\nu'+\frac{1}{2})} - \frac{5}{(\nu'+\frac{1}{3})^2} \right\}$$

Due to the neglect of the higher waves, the above relation is not exact. It is estimated, that the error envolved is less than 6%.

Note added in Proof. (31. May 1960)

After this paper was sent to the publishing department, we saw a preprint from Chew and Mandelstam (UCRL-9126,Theory of the low energy $\pi-\pi$ interaction, part II), in which they discovered, that when the p-wave of low energy is large, a cutoff has to be introduced. At least two new parameters appear in their theory. The solutions are unstable. Our conclusion about their equations is thus confirmed.

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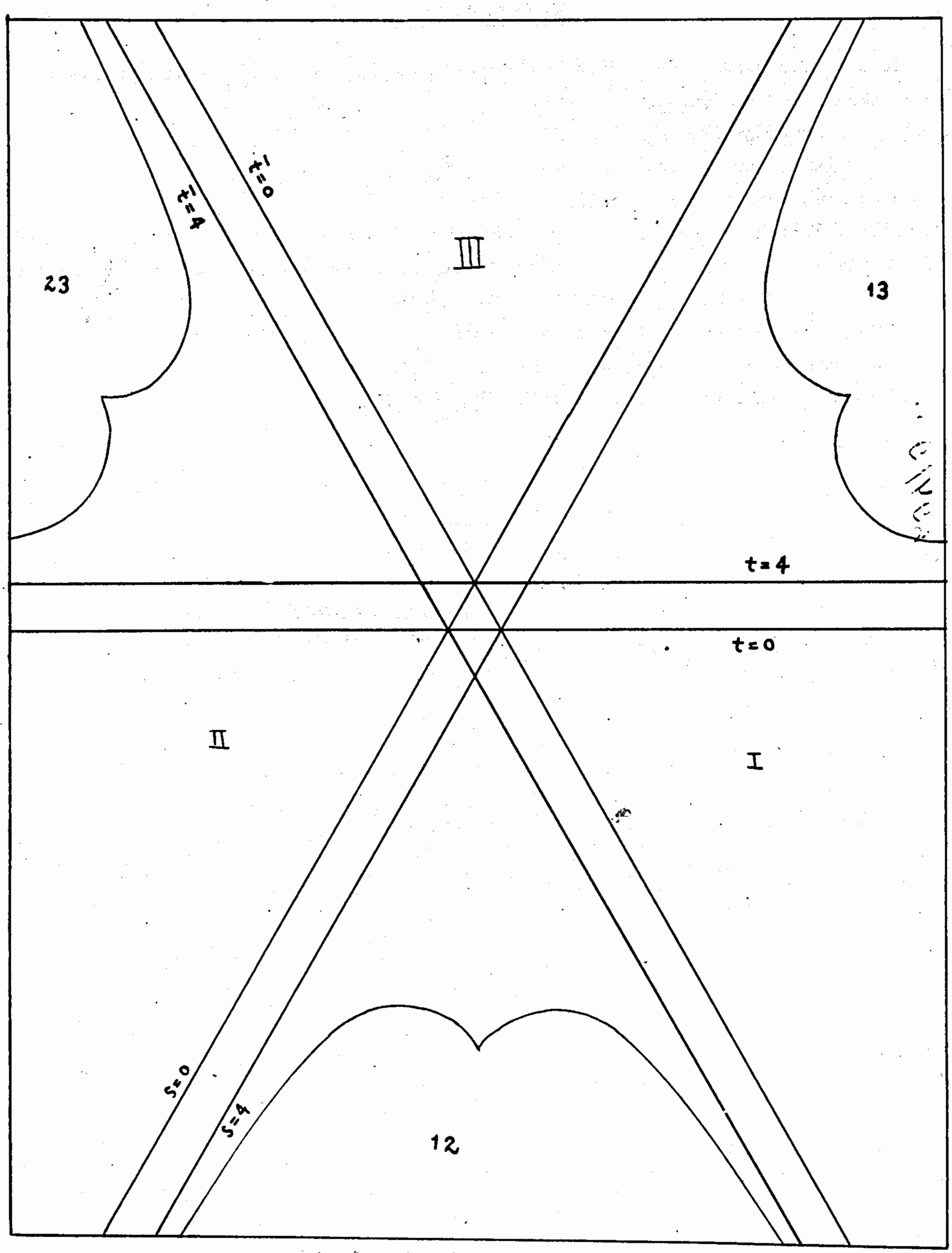

Fig. 1.

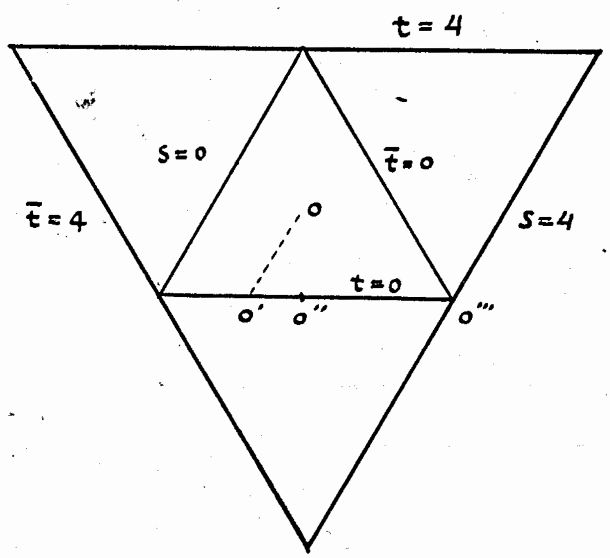

Fig. 2.

694/9 пр.

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