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A b s t r a c t

A new method is suggested for solving the problems of the quantum field theory with a fixed source. The formalism is independent of the magnitude of the coupling constant. It is based on the matrix methods for solving the linear differential equations developed by I.A. Lappo-Danilevsky. The solutions are obtained in the form of series for which a concrete form of the n-th order term is known. The S-matrices have been obtained for a scalar charged and scalar symmetrical theory with a fixed source, as well as for the model advanced by Bialynicki-Birula. The renormalization constants have been treated. In passing to a point interaction the renormalized charge in these models does not contain the logarithmic divergencies.

I n t r o d u c t i o n

The assumption about a weak coupling and the application of the perturbation theory to the equations of mesodynamics lead to the results inconsistent with experiment. Therefore, it would be useful to work out a method which would in no way be based upon the coupling constant as a parameter for iteration, and in which the approximations could be assessed on other grounds. As for Tamm-Dankoff method, it turned out, to be unsatisfactory due to the difficulties associated with the renormalizations. Recently a method of dispersion relations has been given considerable attention and proved to be successful. But since this method is based upon the most general principles of covariance, causality, unitarity and spectrality, it may give poorer information than the Hamiltonian of the interacting fields. In view of great mathematical difficulties we encounter in investigating the equations for the quantum field theory, a study of various models of the theory became rather popular.

Special attention is focused on a class of models with a 'fixed source', i.e., when the fermion field is characterized only by spin and isotopic coordinates. Since the experimental data on pion-nucleon interaction at low energies have been accounted for by Chew-Law model /1/, referred to this class, one may think that the given model describes to some extent the real interaction. Therefore, it should be expected that under these simplifying assumptions there remain a number of problems of the exact field theory. In this connection, a knowledge of the exact solutions of such models will enable us to understand the origin of the difficulties in the theory. However, even for a class of models under consideration (with the exception of a trivial case of the interaction between the scalar neutral mesons and the fixed nucleon /2/) there exist no solutions unlike to those mentioned above.

This paper describes a new method for solving the mesodynamics equations for this class of models taking as an example an interacting system of charged scalar mesons with a fixed source. The formalism suggested is independent of the magnitude of the coupling constant, but is based on the matrix *methods for solving the linear differential equations developed by I.A. Lappo-Danilevsky* [3]. *To use the language generally accepted, a new formalixm is equivalent to the perturbation theory when the Hamiltonian of a system of neutral mesons and a fixed nucleon is chosen as an unperturbed Hamiltonian. However, the advantage is that the n-th order term of the approximation is written down in a closed form whereas in the perturbation theory one can only find any concrete term of a series but not the n-th one. This circumstance makes it, in principle, possible to investigate the convergence of series.*

A method for solving the equation for the S-matrix of the scalar charged theory is set forth in Sections 1–3. Section 4 is concerned with the discussion of the renormalization constants in this model. Section 5 is devoted to the description of the extention of the method to the scalar symmetrical theory. In Section 6, the method is applied to the model suggested by Bialynicki-Birula [4]. *All the calculations are given in the Appendix.*

1. Representation of the S-Matrix as a Functional Integral

Consider a system of scalar charged mesons interacting with the fixed extended nucleon. In this model the nucleon has only two isotopic states (proton and neutron). The system is described by a Hamiltonian:

$$H = m_0(\psi^+\psi) + \frac{1}{2}\sum_{i=1}^{2}\int d\vec{x} : [\pi_i^2(\vec{x}) + (\vec{\nabla}\varphi_i(\vec{x}))^2 + \mu^2\varphi_i^2(\vec{x})] : + \\ + g\sum_{i=1}^{2}\int d\vec{x}\,(\psi^+\tau_i\psi)\varphi_i(\vec{x})\rho(\vec{x}) \tag{1.1}$$

where $\psi = v_p C_p + v_n C_n$ *is the operator of the nucleon field,* C_N $(N = p, n)$ *is the operator of the nucleon annihilation,* v_N *is the spinor describing the nucleon* $\left[v_p = \binom{1}{0},\ v_n = \binom{0}{1}\right]$, $\pi_i(\vec{x})$ *and* $\varphi_i(\vec{x})$ *are the operators of the meson field,* $\rho = \sum_{\vec{k}} v(\vec{k}) e^{i\vec{k}\vec{x}}$ *is the nucleon form-factor,* τ_i *are the matrices of isotopic spin ½.*

In the interaction representation the S-matrix satisfies the following equation:

$$i\frac{\partial}{\partial t}S(t,t_0) = H_I(t)S(t,t_0) \\ S(t,t_0)\Big|_{t=t_0} = 1 \tag{1.2}$$

where

$$H_I(t) = g\sum_{i=1}^{2}\left(\psi^+(t),\tau_i\psi(t)\right)\hat{\varphi}_i(t)$$

$$\hat{\varphi}_i(t) = \int d\vec{x}\,\varphi_i(\vec{x},t)\rho(\vec{x}) = \sum_{\vec{k}}\frac{v(k)}{\sqrt{2\omega}}\left[a_{i\vec{k}}e^{-i\omega t} + a^+_{i\vec{k}}e^{i\omega t}\right]$$

$$\psi(t) = \psi\, e^{-im_0 t}$$

In the symbolic form the solution of Eq. (1.2) is

$$S(t,t_0) = T_\psi\, T_{\hat{\varphi}}\, exp\left\{-i\int_{t_0}^{t} d\xi\, H_I(\xi)\right\} \tag{1.3}$$

The main problem of the theory–the representation of the S-matrix as normal products - may be partially solved in a general form[5,6], namely, the expression for the S-matrix may be transformed so as it would be ordered in the meson operators $\hat{\varphi}$. At the same time, however, the nucleon operators ψ and ψ^+ remain entangled (i.e. under the T-product). Such a partial ordering is accomplished by representing the S-matrix as a functional integral.

Following Feynman[5] we suppose that any functional $\mathcal{F}[\Lambda]$ determined over the set of scalar functions $\Lambda(s)$ set in the interval $[t_0, t]$, may be represented as a superposition of the exponential functionals (by analogy with the Fourier integral for usual functions):

$$\mathcal{F}[\Lambda] = \int \delta\Phi(s)\, exp\left\{i\int_{t_0}^{t} ds\,\Lambda(s)\,\Phi(s)\right\}\overline{\mathcal{F}}[\Phi] \tag{1.4}$$

where $\overline{\mathcal{F}}[\Phi]$ is a new functional which is a functional Fourier transform of $\mathcal{F}[\Lambda]$. $\int\delta\Phi ...$ is the functional integration over the space of real scalar functions $\Phi(s)$. Neglecting the mathematical difficulties in the determination of this operation (e.g. the determination of measure in the space of functions $\Phi(s)$) we shall mean by $\int\delta\Phi\, G[\Phi]$ the limit

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} d\Phi_{s_1}\dots\int_{-\infty}^{\infty} d\Phi_{s_n}\, G(\Phi_{s_1},\dots,\Phi_{s_n})$$

where n is the number of points dividing the interval $[t_0, t]$. If $\mathcal{F}[\Lambda]$ is set, then $\overline{\mathcal{F}}[\Phi]$ may be determined from the reverse transformation:

$$\overline{\mathcal{F}}[\Phi] = C\int\delta\Lambda\, exp\left\{-i\int_{t_0}^{t} ds\,\Lambda(s)\Phi(s)\right\}\mathcal{F}[\Lambda] \tag{1.5}$$

where C is the normalization constant.

Then the operator $\mathcal{F}[\hat{\varphi}]$ is determined as

$$\mathcal{F}[\hat{\varphi}] = \int \delta\Phi \; exp\left\{ i\int_{t_0}^{t} ds\, \Phi(s)\hat{\varphi}(s)\right\} \bar{\mathcal{F}}[\Phi] \qquad (1.6)$$

where $\mathcal{F}[\Phi]$ *is set by (1.4) and (1.5), and by the operator*

$$\hat{G}(t,t_0) = exp\left\{ i\int_{t_0}^{t} ds\, \Phi(s)\hat{\varphi}(s)\right\}$$

we mean the solution of the operator differential equation

$$\frac{\partial}{\partial t}\hat{G}(t,t_0) = i\hat{\varphi}(t)\Phi(t)\hat{G}(t,t_0)$$
$$\hat{G}(t,t_0)\big|_{t=t_0} = 1. \qquad (1.7)$$

On the basis of these results, the S-matrix of Eq. (1.2) may be put as

$$S(t,t_0) = \iint \delta\Phi_1\,\delta\Phi_2 \; exp\left\{ i\int_{t_0}^{t} ds\, \hat{\varphi}_j(s)\Phi_j(s)\right\} \times$$
$$\times\, C^2 \iint \delta\Lambda_1\delta\Lambda_2 \; exp\left\{ -i\int_{t_0}^{t} ds\, \Lambda_j(s)\Phi_j(s)\right\} \tilde{S}(t,t_0/\Lambda_1,\Lambda_2). \qquad (1.8)$$

Here $\tilde{S}(t,t_0/\Lambda_1,\Lambda_2)$ *has the meaning of the S-matrix of the system of the classical charged meson field* $\Lambda_1(t)$, $\Lambda_2(t)$ *and the quantized nucleon field* $\psi(t)$, $\psi^+(t)$, *and obeys the equation*

$$i\frac{\partial}{\partial t}\tilde{S}(t,t_0/\Lambda_1,\Lambda_2) = g\sum_{i=1}^{2}(\psi^+\tau_i\psi)\Lambda_i(t)\,\tilde{S}(t,t_0/\Lambda_1,\Lambda_2)$$
$$\tilde{S}(t,t_0/\Lambda_1,\Lambda_2)\big|_{t=t_0} = 1. \qquad (1.9)$$

Since the operator $exp\left\{ i\int_{t_0}^{t} ds\, \hat{\varphi}(s)\,\Phi(s)\right\}$ *satisfies Eq. (1.7) by the definition, it must be considered as time-ordered (a usual T-product). According to Wick's theorem* [7] *the T-product of the meson operators may be expressed in terms of the normal product*

$$T_{\hat{\varphi}}\, exp\left\{ i\int_{t_0}^{t} ds\, \hat{\varphi}_j(s)\Phi_j(s)\right\} = N_{\hat{\varphi}}\left[exp\left\{ \frac{i}{2}\iint_{t_0 t_0}^{t t} d\xi\, d\eta\, \Delta(\xi-\eta)\frac{\delta^2}{\delta\hat{\varphi}_j(\xi)\delta\hat{\varphi}_j(\eta)}\right\} exp\left\{ i\int_{t_0}^{t} ds\, \Phi_j(s)\hat{\varphi}_j(s)\right\} = \right.$$
$$= exp\left\{ -\frac{i}{2}\iint_{t_0 t_0}^{t t} d\xi\, d\eta\, \Delta(\xi-\eta)\, \Phi_j(\xi)\Phi_j(\eta)\right\} :exp\left\{ i\int_{t_0}^{t} ds\, \hat{\varphi}_j(s)\Phi_j(s)\right\}: \qquad (1.10)$$

where the causality function $\Delta(\xi-\eta)$ *is determined by the relation*

$$\langle 0|T\{\hat{\varphi}_i(\xi)\hat{\varphi}_j(\eta)\}|0\rangle = i\delta_{ij}\Delta(\xi-\eta) = i\delta_{ij}\sum_{\vec{k}}\frac{v^2(\vec{k})}{2i\omega}e^{-i\omega|\xi-\eta|}. \qquad (1.11)$$

Finally the S-matrix, disentangled in the meson operators $\hat{\varphi}_1 \hat{\varphi}_2$ may be written as

$$S(t,t_0) = \iint \delta\Phi_1 \delta\Phi_2 \, \exp\Big\{-\frac{i}{2}\iint\limits_{t_0 t_0}^{t\,t} d\xi d\eta \, \Delta(\xi-\eta)\, \Phi_j(\xi)\Phi_j(\eta)\Big\} : \exp\Big\{i\int\limits_{t_0}^{t} ds\, \hat{\varphi}_j(s)\Phi_j(s)\Big\} : \times \qquad (1.12)$$
$$\times\, C^2 \iint \delta\Lambda_1 \delta\Lambda_2 \, \exp\Big\{-i\int\limits_{t_0}^{t} ds\,\Lambda_j(s)\Phi_j(s)\Big\}\, \widetilde{S}(t,t_0/\Lambda_1,\Lambda_2).$$

Thus, the problem of finding the S-matrix of Hamiltonian (1.1) is divided into : 1) the problem of finding the classical $\widetilde{S}$-matrix as a solution of Eq. (1.9) with arbitrary functions $\Lambda_1(t)$, $\Lambda_2(t)$ and 2) the problem of the functional integration of this matrix by (1. 12).

2. The Finding of the 'Classical' $\widetilde{S}$-Matrix

Since the nucleon field has only two degrees of freedom and the operators of this field anticommute between each other, then the operator $\widetilde{S}(t,t_0/\Lambda_1,\Lambda_2)$ may be represented as the following expansion over the nucleon operators ψ and ψ^+, which, as can be easily shown, is most general

$$\widetilde{S}(t,t_0/\Lambda_1,\Lambda_2) = 1 + [2(\psi^+\psi) - (\psi^+\psi)^2]\, f(t,t_0/\Lambda_1,\Lambda_2) + \\ + \sum_{j=1}^{3} (\psi^+\tau_j\psi)\, h_j(t,t_0/\Lambda_1,\Lambda_2) \qquad (2.1)$$

where f and h_j are the usual scalar functions. This follows immediately from the relations easily verified.

$$(\psi^+\tau_i\psi)(\psi^+\tau_j\psi) = i\,\varepsilon_{ije}(\psi^+\tau_e\psi) + \delta_{ij}[2(\psi^+\psi) - (\psi^+\psi)^2]$$
$$(\psi^+\tau_i\psi)[2(\psi^+\psi) - (\psi^+\psi)^2] = (\psi^+\tau_i\psi).$$

After substituting (2.1) into Eq. (1.9) and equating the coefficients of identical structures, we obtain the equation system for f and h_j which may be put in the matrix form

$$i\frac{\partial}{\partial t}\mathcal{Y}(t,t_0/\Lambda_1,\Lambda_2) = g\sum_{i=1}^{2} \tau_i\Lambda_i(t)\,\mathcal{Y}(t,t_0/\Lambda_1,\Lambda_2) \qquad (2.2)$$
$$\mathcal{Y}(t,t_0/\Lambda_1,\Lambda_2)\big|_{t=t_0} = I$$

where

$$\mathcal{Y}(t,t_0/\Lambda_1,\Lambda_2) = \begin{pmatrix} 1+f(t,t_0/\Lambda_1,\Lambda_2)+h_3(t,t_0/\Lambda_1,\Lambda_2), & h_1(t,t_0/\Lambda_1,\Lambda_2)-ih_2(t,t_0/\Lambda_1,\Lambda_2) \\ h_1(t,t_0/\Lambda_1,\Lambda_2)+ih_2(t,t_0/\Lambda_1,\Lambda_2), & 1+f(t,t_0/\Lambda_1,\Lambda_2)-h_3(t,t_0/\Lambda_1,\Lambda_2) \end{pmatrix}.$$

The solution of (2.2) is very difficult as it reduces to the solution of the linear differential equation of the second order with two arbitrary functions. As usual such equations are solved by the method of the perturbation theory, i.e., by expanding over the parameter g which is assumed to be small. If the parameter g is large, Eq. (2.2) may be approximately solved using the 'quasi-classical' method. However, in this case the expressions obtained cannot be functionally integrated.

Lappo-Danilevsky developed a method solving the differential equation systems employing the theory of functions of matrices. The method is that the function of matrices may be represented as a finite sum of the main compositions of matrices with the coefficients which may be expanded in series by certain characteristic parameters of matrices. Thus, it is not the constant g but some invariants of the matrices entering the equation turn out to be the expansion parameters . We will not be concerned here with the procedure of obtaining the solution, all the details are given in the monograph by I.A. Lappo-Danilevsky/3/. Omitting very complicated and long transformations of the recurrent relations of Lappo-Danilevsky for Eq. (2.2), we give at once the final expression

$$\begin{aligned} \mathcal{Y}(t,t_0/\Lambda_1,\Lambda_2) = \sum_{q=0}^{\infty} \Big\{ & \frac{(ig)^{2q}}{(2q)!} \int_{t_0}^{t} d\xi_1 \ldots \int_{t_0}^{t} d\xi_{2q}\, \Lambda_1(\xi_1)\ldots \Lambda_1(\xi_{2q}) \times \\ & \times \Big[\mathrm{Ch}\Big(ig\int_{t_0}^{t} ds\, \varepsilon(s-\xi_1)\ldots\varepsilon(s-\xi_{2q})\Lambda_2(s)\Big) - \tau_2\, \mathrm{Sh}\Big(ig\int_{t_0}^{t} ds\, \varepsilon(s-\xi_1)\ldots\varepsilon(s-\xi_{2q})\Lambda_2(s)\Big)\Big] - \\ & - \frac{(ig)^{2q+1}}{(2q+1)!} \int_{t_0}^{t} d\xi_1 \ldots \int_{t_0}^{t} d\xi_{2q+1}\, \Lambda_1(\xi_1)\ldots \Lambda_1(\xi_{2q+1}) \times \\ & \times \Big[\tau_1\, \mathrm{Ch}\Big(ig\int_{t_0}^{t} ds\, \varepsilon(s-\xi_1)\ldots\varepsilon(s-\xi_{2q+1})\Lambda_2(s)\Big) + i\tau_3\, \mathrm{Sh}\Big(ig\int_{t_0}^{t} ds\, \varepsilon(s-\xi_1)\ldots\varepsilon(s-\xi_{2q+1})\Lambda_2(s)\Big)\Big] \Big\} \end{aligned} \tag{2.3}$$

where

$$\varepsilon(x) = \begin{cases} +1, & x>0 \\ -1, & x<0 \,. \end{cases}$$

One may see by a direct substitution that the solution (2.3) satisfies Eq. (2.2) with the required initial condition.

The functions Λ_1 and Λ_2 enter the solution (2.3) quite symmetrically since by expanding the hyperbolic cosine and sine in series and by changing the sequence of summation, one obtains another expression for $\mathcal{Y}(t,t_0)$, where Λ_1 and Λ_2 , τ_1 and τ_2 change their places

$$\mathcal{Y}(t,t_0|\Lambda_1,\Lambda_2) = \sum_{q=0}^{\infty}\Big\{ \frac{(ig)^{2q}}{(2q)!}\int_{t_0}^{t} d\xi_1 \dots \int_{t_0}^{t} d\xi_{2q}\ \Lambda_2(\xi_1)\dots\Lambda_2(\xi_{2q}) \times$$
$$\times\Big[Ch\Big(ig\int_{t_0}^{t} ds\, \varepsilon(s-\xi_1)\dots\varepsilon(s-\xi_{2q})\Lambda_1(s)\Big) - \tau_1\, Sh\Big(ig\int_{t_0}^{t} ds\, \varepsilon(s-\xi_1)\dots\varepsilon(s-\xi_{2q})\Lambda_1(s)\Big)\Big] - \qquad (2.4)$$
$$- \frac{(ig)^{2q+1}}{(2q+1)!}\int_{t_0}^{t} d\xi_1 \dots \int_{t_0}^{t} d\xi_{2q+1}\ \Lambda_2(\xi_1)\dots\Lambda_2(\xi_{2q+1}) \times$$
$$\times\Big[\tau_2\, Ch\Big(ig\int_{t_0}^{t} ds\, \varepsilon(s-\xi_1)\dots\varepsilon(s-\xi_{2q+1})\Lambda_1(s)\Big) - i\tau_3\, Sh\Big(ig\int_{t_0}^{t} ds\, \varepsilon(s-\xi_1)\dots\varepsilon(s-\xi_{2q+1})\Lambda_1(s)\Big)\Big]\Big\}.$$

For series (2.3) and (2.4), a majorating functional may be easily written down because cosine and sine are not greater than unity (Λ_1 and Λ_2 are real) and the remaining series are easy to be summed up.

$$\mathcal{Y}(t,t_0|\Lambda_1,\Lambda_2) \leq (1+\tau_1)\, min\Big\{ exp\Big[g\int_{t_0}^{t} ds|\Lambda_1(s)|\Big],\ exp\Big[g\int_{t_0}^{t} ds|\Lambda_2(s)|\Big]\Big\}. \qquad (2.5)$$

Thus, the solution of Eq. (2.2) is represented as series (2.3) and (2.4) which are convergent uniformly and absolutely for the interval $[t_0, t]$, if, at least, one of the integrals $\int_{t_0}^{t} ds|\Lambda_1(s)|$ and $\int_{t_0}^{t} ds\,|\Lambda_2(s)|$ is limited over $[t_0, t]$.

The relationship between the Lappo-Danilevsky method and the perturbation theory for equations of (2.2) type is shown in Appendix A.

Being aware of $\mathcal{Y}(t,t_0)$, one can easily write an expression for a 'classical' $\widetilde{S}$-matrix expressed by equality (2.1):

$$\widetilde{S}(t,t_0|\Lambda_1,\Lambda_2) = 1 - \big(2(\psi^+\psi) - (\psi^+\psi)^2\big) +$$
$$+\sum_{q=0}^{\infty} \frac{[-ig(\psi^+\tau_1\psi)]^q}{q!}\int_{t_0}^{t} d\xi_1 \dots \int_{t_0}^{t} d\xi_q\ \Lambda_1(\xi_1)\dots\Lambda_1(\xi_q) \times \qquad (2.6)$$
$$\times\Big[\big(2(\psi^+\psi)-(\psi^+\psi)^2\big)\, Ch\Big(ig\int_{t_0}^{t} ds\, \varepsilon(s-\xi_1)\dots\varepsilon(s-\xi_q)\Lambda_2(s)\Big) -$$
$$-(-)^q(\psi^+\tau_2\psi)\, Sh\Big(ig\int_{t_0}^{t} ds\, \varepsilon(s-\xi_1)\dots\varepsilon(s-\xi_q)\Lambda_2(s)\Big)\Big]$$

The formula is symmetrical with respect to the commutation of indices 1 and 2.

Note, that the criterion thus obtained of the uniform and absolute convergence is not sufficient for performing the functional integration since in integrating there may always be found such functions Λ_1 and Λ_2 which do not satisfy the obtained criterion. Neverthesess, we put aside the problem of the correctness of the functional integration procedure, the more as so far the existence of the functional integrals has been proved only for a very narrow class of functionals. Suppose that a series may be integrated by a term. This operation which has not yet been proved may be justified by the circumstance that the S-matrix obtained as a result of integration satisfies the original equation (1.2). This is confirmed by a direct substitution

3. The Finding of the Quantum S-Matrix

The functional integration of the 'classical' $\tilde{S}$-matrix may be performed without any difficulty as the solution of the classical equation has a 'Gaussian' form. A method for calculating similar functional integrals has become known since Wiener's papers[8] and in the applications to the quantum problems it was developed by Feynman[5]. Let us give the final form of the S-matrix. (See Appendix B).

$$S(t,t_0) = 1 - \left(2(\psi^+\psi) - (\psi^+\psi)^2\right) + \tag{3.1}$$

$$\sum_{q=0}^{\infty}\sum_{m=0}^{q}\Bigg\{ \frac{(ig)^{2q}\, i^m}{(2q-2m)!\,2^m m!} \int_{t_0}^{t} d\xi_1 \ldots \int_{t_0}^{t} d\xi_{2q}\, \Delta(\xi_1-\xi_2)\ldots\Delta(\xi_{2m-1}-\xi_{2m}) : \hat{\varphi}_1(\xi_{2m+1})\ldots\hat{\varphi}_1(\xi_{2q}) : \times$$

$$\times\Big[\big(2(\psi^+\psi)-(\psi^+\psi)^2\big) : \mathrm{Ch}\Big(ig\int_{t_0}^{t} ds\, \varepsilon(s-\xi_1)\ldots\varepsilon(s-\xi_{2q})\hat{\varphi}_2(s)\Big) : - (\psi^+\tau_2\psi) : \mathrm{Sh}\Big(ig\int_{t_0}^{t} ds\, \varepsilon(s-\xi_1)\ldots\varepsilon(s-\xi_{2q})\hat{\varphi}_2(s)\Big) : \Big]\times$$

$$\times \exp\Big\{-\frac{i}{2}g^2\int_{t_0}^{t}\int_{t_0}^{t} ds_1\, ds_2\, \varepsilon(s_1-\xi_1)\ldots\varepsilon(s_1-\xi_{2q})\,\Delta(s_1-s_2)\,\varepsilon(s_2-\xi_1)\ldots\varepsilon(s_2-\xi_{2q})\Big\} -$$

$$- \frac{(ig)^{2q+1}\, i^m}{(2q+1-2m)!\,2^m m!} \int_{t_0}^{t} d\xi_1 \ldots \int_{t_0}^{t} d\xi_{2q+1}\, \Delta(\xi_1-\xi_2)\ldots\Delta(\xi_{2m-1}-\xi_{2m}) : \hat{\varphi}_1(\xi_{2m+1})\ldots\hat{\varphi}_1(\xi_{2q+1}) : \times$$

$$\times\Big[(\psi^+\tau_1\psi):Ch\Big(ig\int_{t_0}^{t}ds\,\mathcal{E}(s-\xi_1)\ldots\mathcal{E}(s-\xi_{2q+1})\hat{\varphi}_2(s)\Big):+$$

$$+\,i(\psi^+\tau_3\psi):Sh\Big(ig\int_{t_0}^{t}ds\,\mathcal{E}(s-\xi_1)\ldots\mathcal{E}(s-\xi_{2q+1})\hat{\varphi}_2(s)\Big):\Big]\times$$

$$\times\exp\Big\{-\frac{i}{2}g^2\int_{t_0}^{t}\int_{t_0}^{t}ds_1ds_2\,\mathcal{E}(s_1-\xi_1)\ldots\mathcal{E}(s_1-\xi_{2q+1})\Delta(s_1-s_2)\,\mathcal{E}(s_2-\xi_1)\ldots\mathcal{E}(s_2-\xi_{2q+1})\Big\}\quad\Big\}$$

Expression (3.1) is symmetrical with respect to the commutation of indices one and two , that corresponds to the symmetry in the classical function $\mathcal{Y}(t,t_0/\Lambda_1,\Lambda_2)$ expressed in (2.3) and (2.4).

The obtained expression for the S-matrix of Hamiltonian (1.1) is written down in the normal form both in the nucleon and meson operators. One can see by a direct substitution that the S-matrix satisfies Eq. (1.2) with the initial condition.

Thus, the operation of the functional integration, although not grounded from a mathematical point of view, leads in the given case to a correct result, that is confirmed by a direct substitution. This circumstance points out that the method proposed by Feynman is correct. In expanding by the coupling constant g the series of the usual perturbation theory are obtained with the advantage that here we have an explicit form of the n-th order term of this series while the existing apparatus of the perturbation theory permits to obtain any concfete term of the series but not the n-th one. This shortcoming of the perturbation theory, in our opinion, is the main difficutly in studing the problem of series convergence in the perturbation theory.

To clear up the physical meaning of the iterations in the S-matrix (3. 1) let us return again to Eq. (1.2)

$$i\frac{\partial}{\partial t}S=g\left[(\psi^+\tau_1\psi)\hat{\varphi}_1(t)+(\psi^+\tau_2\psi)\hat{\varphi}_2(t)\right]S\,. \qquad (3.2)$$

The expressions $\hat{\varphi}_1\pm i\,\hat{\varphi}_2$ are the operators of chargedmesons, while the operators $\hat{\varphi}_1$ and $\hat{\varphi}_2$ lead to the creation or annihilation of a definite combination of positive and negative mesons. For instance, the operator $\hat{\varphi}_1$ corresponds to the combination $\frac{1}{2}(\pi^-+\pi^+)$. Now instead of the main nucleon states $\mathcal{V}_p$ and $\mathcal{V}_n$, let us introduce $\mathcal{V}_+=\frac{1}{\sqrt{2}}(\mathcal{V}_p+\mathcal{V}_n)$ and $\mathcal{V}_-=\frac{1}{\sqrt{2}}(\mathcal{V}_p-\mathcal{V}_n)$.

This transformation means the transition to new orts in the isotopic space. In these new orts Eq.(3.2) is written as

$$i\frac{\partial}{\partial t} S = g\left[(\psi'^{+}\tau_3\psi')\hat{\varphi}_1(t) - (\psi'^{+}\tau_2\psi')\hat{\varphi}_2(t)\right] S. \qquad (3.3)$$

Here $\psi' = v_+ C_+ + v_- C_-$. $C_\pm$ *is the particle annihilation operator.*

The operator $\hat{\varphi}_1$ *enters Eq. (3.3) together with the diagonal matrix* τ_3 *and, hence, it is responsible for the emission and absorption of such combination of negative and positive mesons which does not give rise to the transition of a nucleon from the state* v_+ *in* v_- *and conversely. If in the right-hand side of (3.3) the second term had been absent we would have had a neutral theory according to which the emission and absorption of a meson does not change the isotopic coordinates of a nucleon. The solution (3.1) is equivalent to the solution by the perturbation theory when the expression* $(\psi'^{+}\tau_2\psi')\hat{\varphi}_2(t)$ *giving rise to the transitions between the states* v_+ *and* v_- *is assumed to be a perturbation. Note, that it is possible to diagonalize the matrix* τ_2 *enotering into (3.2) with* $\hat{\varphi}_2$ *by another rotation* $v_\pm = \frac{1}{\sqrt{2}}(v_p \pm i v_n)$ *in the isotopic space. The 'perturbing' term will be then* $(\psi'^{+}\tau_2\psi')\hat{\varphi}_1(t)$ *. This situation corresponds to the abovementioned symmetry of the S-matrix with respect to the operators* φ_1 *and* φ_2 *. However, if one restricts oneself to the finite number of terms in series (3.1), then the symmetry will be violated (one operator is in the degree of the exponent in expanding of which any degree of this operator appear, whereas the other will enter this expression in the finite degree. In dealing with the cut off series for the S-matrix there may arise the processes violating the law of charge conservation. This will take place if for the processes involving more than 2n mesons one restricts oneself to the n terms of a series. Therefore, it is necessary to calculate the matrix element of the complete series of the S-matrix and only in the series of the matrix element one may restrict oneself to this or that number of terms. In the language of the perturbation theory a separate term of series (3.1) involves such graphs for which the law of charge conservation does not fulfilled, (for instance* $n \rightarrow p + \pi^+$*). For the complete S-matrix the law of charge conservation is fulfilled exactly and when the matrix elements are calculated correctly, as was pointed out above, there is no violation of this law. Therefore, in the framework of the formalizm developed one may speak that the law of charge conservation is not fulfilled in virtual processes like in the perturbation theory the law of energy conservation is not fulfilled for virtual processes.*

4. Renormalization Constants

To obtain the eigenfunctions and eigenvalues of Hamiltonian (1.1) we make use of the hypothesis of the adiabatic switching on of the interaction*[10] *which may be formulated as follows:*

Let Φ_m *be the eigenfunction of the free Hamiltonian* H_0. *If, further, the solution of the equation for the S-matrix with the adiabatic ally increasing interaction is known*

$$i\frac{\partial}{\partial t}S^{\alpha}(t,t_0)=H_I(t)e^{-\alpha|t|}S^{\alpha}(t,t_0) \qquad (4.1)$$

$$S^{\alpha}(t,t_0)\big|_{t=t_0}=1$$

then the eigenfunctions of the operator $H=H_0+H_I$ *are*

$$C_m\Psi_m^{(\pm)}=\lim_{\alpha\to 0}\frac{S^{\alpha}(0,\pm\infty)\Phi_m}{(\Phi_m,S^{\alpha}(0,\pm\infty)\Phi_m)} \qquad (4.2)$$

where C_m *is the normalization constant, whereas the signs* $(\pm)$ *correspond to the 'outgoing' and 'incoming' waves. The eigenvalue of the energy in the state* $\Psi_m^{(\pm)}$ *is determined by the equality*

$$E_m=\lim_{\alpha\to 0}\frac{(\Phi_m,HS^{\alpha}(0,\pm\infty)\Phi_m)}{(\Phi_m,\,S^{\alpha}(0,\pm\infty)\Phi_m)}\,. \qquad (4.3)$$

The limiting transition allows to determine correctly the quotient, since the numerator and denominator are not determined due to the presence of the infinite phase factor $exp(i\frac{M}{\alpha})$.

The 'adiabatic' S^{α}*-matrix which is the solution of Eq. (4.1) can be easily obtained form (3.1) by substituting there all the differentials* $d\xi_j$, ds_j *for the expression* $d\xi_j e^{-\alpha|\xi_j|}$, $ds_j e^{-\alpha|s_j|}$.

Note, that the introduction of counter terms into the Hamiltonian leads to the automatic switching out of infinite phases. Although such an introduction of counter terms is considered to be a more correct procedure, in calculating the matrix elements it would be more convenient for us to use theorems (4.2)-(4.3) of the adiabatic hypotheses.

* The Green function of Hamiltonian (II) has been obtained in [9].

Since the $\overset{\alpha}{S}$-matrix is set as a series, then the matrix elements will be represented as a limit of the ratio of two series when $\alpha \to 0$. It appears, that if we divide one series into another and collect the terms by the equal degrees of the coupling constant standing before the exponent, then in the terms thus obtained the phase reduces, and, therefore, one may pass to the limit $\alpha \to 0$ in each term separately. In Appendix C this procedure is illustraeted by the calculations of the renormalized coupling constant. For other matrix elements the calculations are being performed analogously.

The expressions obtained proved to be rather complicated. Although the n-th order term could be written down we have not yet succeeded in investigating it to the end. Therefore, we write out only the second and the third approximation. The calculation of the integrals is considerobly simplified in the limiting case of the point interaction when the form-factor $v(\kappa)$ is tending to unity. We choose the form-factor as follows

$$v(\kappa) = \exp\left\{-\frac{\omega-\mu}{2L}\right\}$$

where L has the meaning of the cut-off momentum. The transition to the point interaction will be performed when L is tending to infinity .

Consider first of all the eigenvalues of energy of the one-nucleon state .According to theorem (4.3) we obtain

$$E_N = \lim_{\alpha\to 0} \frac{\langle 0|C_N H S^{\alpha}(0,-\infty)C_N^{+}|0\rangle}{\langle 0|C_N S^{\alpha}(0,-\infty)C_N^{+}|0\rangle} = m_0 + \delta m \qquad (4.4)$$

where

$$\delta m = \lim_{\alpha\to 0} \frac{\langle 0|C_N H_I S^{\alpha}(0,-\infty)C_N^{+}|0\rangle}{\langle 0|C_N S^{\alpha}(0,-\infty)C_N^{+}|0\rangle} =$$

$$= \lim_{\alpha\to 0}\int_{-\infty}^{0} d\sigma\, e^{\alpha\sigma}\, 2g^2\Delta(\sigma)\, \frac{\sum_{q=0}^{\infty}\frac{1}{q!}\left(-\frac{ig^2}{2}\right)^q A_q^{\alpha}}{\sum_{q=0}^{\infty}\frac{1}{q!}\left(-\frac{ig^2}{2}\right)^q a_q^{\alpha}} = -g^2\sum_{\vec{k}}\frac{v^2(\kappa)}{\omega^2} + 2g^2\int_{-\infty}^{0} d\sigma\,\Delta(\sigma)\exp\left\{-g^2\sum_{\vec{k}}\frac{v^2(\kappa)}{\omega^3}\left[1-e^{i\omega\sigma}\right]\right\} + \ldots$$

$$(4.5)$$

$$A_q^{\alpha} = \int_{-\infty}^{0} d\xi_1 \ldots \int_{-\infty}^{0} d\xi_{2q}\, e^{\alpha(\xi_1+\ldots+\xi_{2q})}\Delta(\xi_1-\xi_2)\ldots\Delta(\xi_{2q-1}-\xi_{2q})\,\varepsilon(\sigma-\xi_1)\ldots\varepsilon(\sigma-\xi_{2q}) \times$$

$$\times \exp\left\{-\frac{i}{2}g^2\iint_{-\infty}^{0} ds_1 ds_2\, e^{\alpha(s_1+s_2)}\varepsilon(s_1-\xi_1)\ldots\varepsilon(s_1-\xi_{2q})\Delta(s_1-s_2)\,\varepsilon(s_2-\xi_1)\ldots\varepsilon(s_2-\xi_{2q})\right\}$$

$$a_q^{\alpha} = \int_{-\infty}^{0} d\xi_1 \cdots \int_{-\infty}^{0} d\xi_{2q}\, e^{\alpha(\xi_1+\ldots+\xi_{2q})}\, \Delta(\xi_1-\xi_2)\ldots\Delta(\xi_{2q-1}-\xi_{2q})\times$$

$$\times\, \exp\Big\{-\tfrac{i}{2} g^2 \iint_{-\infty}^{0} ds_1 ds_2\, e^{\alpha(s_1+s_2)} \mathcal{E}(s_1-\xi_1)\ldots\mathcal{E}(s_1-\xi_{2q})\,\Delta(s_1-s_2)\,\mathcal{E}(s_2-\xi_1)\ldots\mathcal{E}(s_2-\xi_{2q})\Big\}.$$

Within the limits of the point interaction the mass renormalization is written down as

$$\delta m \xrightarrow[L\to\infty]{} -g^2 \sum_{k} \frac{1}{\omega^2}\Big[\,1 + \frac{1}{2(1+\frac{g^2}{2\pi^2})} + \ldots\Big], \qquad (4.6)$$

In accordance with its probability meaning the renormalization constant of the fermion field Z_2 is determined by

$$Z_2 = \left|\langle 0| C_N S^{\alpha}(0,-\infty) C_N^{+} |0\rangle\right|^2 = \left|\sum_{q=0}^{\infty}\frac{1}{q!}\left(-\frac{ig^2}{2}\right)^q a_q^{\alpha}\right|^2 =$$

$$= \exp\Big\{-\tfrac{1}{2} g^2 \sum_{k}\frac{v^2(k)}{\omega^3}\Big\}\Big[1 - g^2\, \mathrm{Re}\iint_0^{\infty} d\eta\, d\nu\, \Delta(\nu)\exp\Big\{g^2\sum_{k}\frac{v^2(k)}{\omega^3}\big[-1+e^{-i\omega\eta}+e^{-i\omega\nu}-e^{-i\omega(\eta+\nu)}\big]\Big\}+\ldots\Big]. \qquad (4.7)$$

A series standing inside straight brackets contains an indefinite phase factor $e^{i\frac{M}{\alpha}}$ which disappears in raising this series by a module to the second power. Restricting to the first two terms in the limit, when $L \to \infty$, we have

$$Z_2 \xrightarrow[L\to\infty]{} \left(\frac{1}{L}\right)^{\frac{g^2}{4\pi^2}}\Big[\,1 + \frac{\frac{g^2}{2\pi^2}}{1+\frac{g^2}{2\pi^2}}\ln L + \ldots\Big] \qquad (4.8)$$

The most interesting from the point of view physical is the connection between the renormalized (observed) coupling constant g_r and the unrenormalized constant g. This connection is determined by

$$\frac{g_r}{g} = \left(\Psi_p^{(+)}, (\psi^{+}\tau_{+}\psi)\Psi_n^{(-)}\right) = \lim_{\alpha\to 0}\frac{\langle 0|C_p S^{\alpha}(\infty,0)(\psi^{+}\tau_{+}\psi)S^{\alpha}(0,-\infty)C_n^{+}|0\rangle}{\langle 0|C_p S^{\alpha}(\infty,-\infty)C_p^{+}|0\rangle}. \qquad (4.9)$$

To make a further analysis more convenient, we will assume that the field $\hat{\varphi}_1$ enters the interaction Hamiltonian with the coupling constant g_1, whereas the field $\hat{\varphi}_2$ with the constant g_2.

After making some calculations (see Appendix 3), and restricting to several terms of a series, we get

$$\frac{g_r}{\sqrt{g_1 g_2}} = 1 + g_1^2 \int_0^\infty dx \cdot x \Big(\sum_{\vec{k}} \frac{v^2(k)}{\omega} e^{-i\omega x} \Big) \exp\Big\{ -2g_2^2 \sum_{\vec{k}} \frac{v^2(k)}{\omega^3} \big[1 - e^{-i\omega x} \big] \Big\} -$$

$$- g_1^4 \int_0^\infty dx_1 \int_0^\infty dx_2 \int_0^\infty dx_3 \, (x_1 + x_2) \Big(\sum_{\vec{k}} \frac{v^2(k)}{\omega} e^{-i\omega x_1} \Big) \Big(\sum_{\vec{k}} \frac{v^2(k)}{\omega} e^{-i\omega x_2} \Big) \exp\Big\{ -2g_2^2 \sum_{\vec{k}} \frac{v^2(k)}{\omega^3} \big[2 - e^{-i\omega x_1} - e^{-i\omega x_2} \big] \Big\} \times$$

$$\times \Big[\exp\Big\{ -2g_2^2 \sum_{\vec{k}} \frac{v^2(k)}{\omega^3} \big[e^{-i\omega(x_1+x_3)} + e^{-i\omega(x_2+x_3)} - e^{-i\omega(x_1+x_2+x_3)} - e^{-i\omega x_3} \big] \Big\} - 1 \Big] - \qquad (4.10)$$

$$- g_1^4 \int_0^\infty dx_1 \int_0^\infty dx_2 \int_0^\infty dx_3 \, (x_1 + x_2) \Big[\Big(\sum_{\vec{k}} \frac{v^2(k)}{\omega} e^{-i\omega(x_1+x_3)} \Big) \Big(\sum_{\vec{k}} \frac{v^2(k)}{\omega} e^{-i\omega(x_2+x_3)} \Big) + \Big(\sum_{\vec{k}} \frac{v^2(k)}{\omega} e^{-i\omega(x_1+x_2+x_3)} \Big) \times$$

$$\times \Big(\sum_{\vec{k}} \frac{v^2(k)}{\omega} e^{-i\omega x_3} \Big) \Big] \exp\Big\{ -2g_2^2 \sum_{\vec{k}} \frac{v^2(k)}{\omega^3} \big[2 - e^{-i\omega x_1} - e^{-i\omega x_2} + e^{-i\omega(x_1+x_3)} + e^{-i\omega(x_2+x_3)} - e^{-i\omega(x_1+x_2+x_3)} - e^{-i\omega x_3} \big] \Big\} + \ldots$$

Note, that we may change the places of the constants g_1 *and* g_2 *. This is the consequence of the S-matrix symmetry by the operators* $\hat{\varphi}_1$ *and* $\hat{\varphi}_2$ *as has been already pointed out.*

Formula (4.10) is remarkable because there exists a finite limit when $L \to \infty$ *(see Appendix 3).*

$$\frac{g_r}{\sqrt{g_1 g_2}} = 1 - \frac{g_1^2}{\pi^2} \frac{1}{\frac{g_2^2}{\pi^2}\left(\frac{g_2^2}{\pi^2} + 1\right)} -$$

$$- \frac{g_1^4}{2\pi^4} \int_0^\infty dx_1 \int_0^\infty dx_2 \frac{(x_1 + x_2)}{(1+x_1)^{2+g_2^2/\pi^2}(1+x_2)^{2+g_2^2/\pi^2}} \int_0^\infty dx_3 \left[\left(\frac{(1+x_1+x_3)(1+x_2+x_3)}{(1+x_1+x_2+x_3)(1+x_3)} \right)^{g_2^2/\pi^2} - 1 \right] - \qquad (4.11)$$

$$- \frac{g_1^4}{2\pi^4} \int_0^\infty dx_1 \int_0^\infty dx_2 \int_0^\infty dx_3 \, (x_1 + x_2) \left(\frac{(1+x_1+x_3)(1+x_2+x_3)}{(1+x_1)(1+x_2)(1+x_1+x_2+x_3)(1+x_3)} \right)^{g_2^2/\pi^2} \left[\frac{1}{(1+x_1+x_3)^2(1+x_2+x_3)^2} + \frac{1}{(1+x_1+x_2+x_3)^2(1+x_3)^2} \right]$$

Consider in more detail the first term of (4.10)

$$g_1^2 \int_0^\infty dx \cdot x \sum_k \frac{v^2(k)}{\omega} e^{-i\omega x} \exp\left\{-2g_2^2 \sum_k \frac{v^2(k)}{\omega^3}\left[1-e^{-i\omega x}\right]\right\}. \qquad (4.12)$$

It is easy to notice that in expanding the integrand by g_2 there is obtained a series containing the terms logarithmically divergent by L. The main divergent part of this series is of the form

$$g_1^2 \ln L \cdot \sum_{n=0}^{\infty} \frac{(-g_2^2 \ln L)^n}{n!\,(n+1)} \qquad (4.13)$$

in complete accordance with the result of the perturbation theory. At the same time (4.12) has the limit at $L \to \infty$ equal to

$$-\frac{g_1^2}{\pi^2} \cdot \frac{1}{g_2^2/\pi^2 \cdot (g_2^2/\pi^2 + 1)} .$$

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Therefore, integral (4.12) as a function of g_2^2, has the pole in the point $g_2 = 0$ and, hence, cannot be expanded in a Taylor series in the neighbourhood of $g_2 = 0$. Such a situation also occurs in the further terms of a series, but the restrictions upon g_2^2/π^2 at which the integrals appear to be convergent change in the transition from one order to another. The third integral in (4.10) is convergent already when $g_2^2/\pi^2 > 1$, while in the n-th order the integrals are convergent for $g_2^2/\pi^2 > n-1$. Therefore, in order all the terms of series (4.9) to be finite when the cut-off is taken away ($L \to \infty$), it is necessary to assume g_2 to be a infinitely large quantity. These restrictions on the constant g_2 different for in each term of a series seem to be rather meaningless. To account for this fact, let us recall that the expression for the renormalized constant (4.9) is symmetrical with respect to the substitution $g_1 \rightleftarrows g_2$. Thus, all the conclusions concerning g_2 are also true for g_1 (since (4.9) may be represented as a series in g_2, whereas g_1 will enter only in the index of the exponent), therefore, the assertion that there is a singularity in zero also by g_1 is correct.. I. e., $g_r = f(g_1, g_2)$ cannot be represented by an expansion in the neighbourhood of $g_1 = 0$ or $g_2 = 0$. But series (4.11) is the expansion just in the vicinity of $g_1 = 0$. This is likely to account for the senseless result we mentioned above.

So, the following conclusions may be derived which, however, cannot be yet considered proved: firstly, the exact solution seems to have the singularity at the point $g_1 = 0$ as well as at the point $g_2 = 0$ so that one cannot look for the solution as an expansion in the vicinity of the point $g = g_1 = g_2 = 0$; secondly, although the series of Lappo-Danilevsky is better than that of the perturbation theory, it is not good enough because it represents the solution partially expanded by the coupling constant; thirdly, there are no, as it seems, logarithmic divergencies due to the point interaction in the expression for the renormalized coupling constant.

For a final clearing up of these questions a more detailed study of integrals in the series of Lappo-Danilevsky is necessary.

5. Scalar Symmetrical Theory

The method set forth in previous Sections may be directly applied to the scalar symmetrical theory described by a Hamiltonian:

$$H = m_0(\psi^+\psi) + \frac{1}{2}\sum_{j=1}^{3}\int d\vec{x} :[\pi_j^2(\vec{x}) + (\vec{\nabla}\varphi_j(\vec{x}))^2 + \mu^2\varphi_j^2(\vec{x})]: + \\ + g\sum_{j=1}^{3}\int d\vec{x}\,(\psi^+\tau_j\psi)\varphi_j(\vec{x})\rho(\vec{x}) \tag{5.1}$$

where

$\varphi_j(\vec{x})$ are three real scalar meson fields.

In the interaction representation the equation for the S-matrix is

$$i\frac{\partial}{\partial t}S(t,t_0) = H_I(t)S(t,t_0) \\ S(t,t_0)\big|_{t=t_0} = 1 \tag{5.2}$$

where

$$H_I(t) = g\sum_{j=1}^{3}(\psi^+\tau_j\psi)\hat{\varphi}_j(t)$$

$$\hat{\varphi}_j(t) = \sum_{\vec{k}}\frac{v(k)}{\sqrt{2\omega}}\left[a_{j\vec{k}}e^{-i\omega t} + a_{j\vec{k}}^+ e^{i\omega t}\right].$$

Representing the S-matrix as a functional integral

$$S(t,t_0) = \iiint\delta\Phi_1\delta\Phi_2\delta\Phi_3\, \exp\Big\{-\frac{i}{2}\iint_{t_0 t_0}^{t\,t} d\xi d\eta\,\Delta(\xi-\eta)\Phi_j(\xi)\Phi_j(\eta)\Big\} : \exp\Big\{i\int_{t_0}^{t} ds\,\hat{\varphi}_j(s)\,\Phi_j(s)\Big\}: \times \\ \times C^3\iiint\delta\Lambda_1\delta\Lambda_2\delta\Lambda_3\, \exp\Big\{-i\int_{t_0}^{t} ds\,\Lambda_j(s)\Phi_j(s)\Big\}\,\tilde{S}(t,t_0|\Lambda_1,\Lambda_2,\Lambda_3) \tag{5.3}$$

we get the following equation for the 'classical' $\tilde{S}$-matrix

$$i\frac{\partial}{\partial t}\widetilde{S}(t,t_0|\Lambda_1\Lambda_2\Lambda_3) = g\sum_{j=1}^{3}(\psi^+\tau_j\psi)\Lambda_j(t)\ \widetilde{S}(t,t_0|\Lambda_1\Lambda_2\Lambda_3)$$

$$\widetilde{S}(t,t_0|\Lambda_1\Lambda_2\Lambda_3)\Big|_{t=t_0} = 1. \tag{5.4}$$

Applying Lappo-Danilevsky method to this equation yields the following result

$$\widetilde{S}(t,t_0|\Lambda_1\Lambda_2\Lambda_3) = 1 - \left[2(\psi^+\psi) - (\psi^+\psi)^2\right] +$$

$$+\sum_{q=0}^{\infty}\sum_{p=0}^{\infty}\frac{[-i(\psi^+\tau_1\psi)]^q}{q!}\frac{[-i(\psi^+\tau_2\psi)]^p}{p!}g^{q+p}\times$$

$$\times\int_{t_0}^{t}d\zeta_1\ldots\int_{t_0}^{t}d\zeta_q\int_{t_0}^{t}d\zeta_1\ldots\int_{t_0}^{t}d\zeta_p\prod_{i=1}^{q}\prod_{j=1}^{p}\Lambda_1(\xi_i)\varepsilon(\zeta_i-\zeta_j)\Lambda_2(\zeta_j)\times$$

$$\times\Big[\left(2(\psi^+\psi)-(\psi^+\psi)^2\right)\mathrm{Ch}\Big(ig\int_{t_0}^{t}ds\prod_{i=1}^{q}\prod_{j=1}^{p}\varepsilon(s-\xi_i)\varepsilon(s-\zeta_j)\Lambda_3(s)\Big) -$$

$$-(-)^{q+p}(\psi^+\tau_3\psi)\,\mathrm{Sh}\Big(ig\int_{t_0}^{t}ds\prod_{i=1}^{q}\prod_{j=1}^{p}\varepsilon(s-\xi_i)\varepsilon(s-\zeta_j)\Lambda_3(s)\Big)\Big]. \tag{5.5}$$

One can see by a immediate substitution that the $\widetilde{S}$-matrix obtained satisfies Eq. (5.4). Formula (5.5) is symmetrical with respect to the cyclic commutations of indices 1, 2, 3.

The functional integration of the 'classical' $\widetilde{S}$-matrix is not difficult since the integrals obtained are of a Gaussian type. The result of the integration is

$$S(t,t_0) = 1 - \left[2(\psi^+\psi) - (\psi^+\psi)^2\right] +$$

$$+ \sum_{q=0}^{\infty} \sum_{p=0}^{\infty} \sum_{m=0}^{[\frac{q}{2}]} \sum_{n=0}^{[\frac{p}{2}]} \frac{\left[-i(\psi^+\tau_1\psi)g\right]^q}{(q-2m)!\,2^m m!} \frac{\left[-i(\psi^+\tau_2\psi)g\right]^p}{(p-2n)!\,2^n n!} \times$$

$$\times \int_{t_0}^{t} d\xi_1 \dots \int_{t_0}^{t} d\xi_q \int_{t_0}^{t} d\zeta_1 \dots \int_{t_0}^{t} d\zeta_p \prod_{i=1}^{q} \prod_{j=1}^{p} \varepsilon(\xi_i - \zeta_j) \times$$

$$\times \Delta(\xi_1 - \xi_2) \dots \Delta(\xi_{2m-1} - \xi_{2m}) \Delta(\zeta_1 - \zeta_2) \dots \Delta(\zeta_{2n-1} - \zeta_{2n}) : \hat{\varphi}_1(\xi_{2m}) \dots \hat{\varphi}_1(\xi_q) \hat{\varphi}_2(\zeta_{2n}) \dots \hat{\varphi}_2(\zeta_p) :$$

$$\times \Big[(2(\psi^+\psi) - (\psi^+\psi)^2) : \mathrm{ch}\Big(ig \int_{t_0}^{t} ds \prod_{i=1}^{q} \prod_{j=1}^{p} \varepsilon(s-\xi_i)\varepsilon(s-\zeta_j)\hat{\varphi}_3(s) \Big) : -$$

$$- (-)^{p+q} (\psi^+\tau_3\psi) : \mathrm{sh}\Big(ig \int_{t_0}^{t} ds \prod_{i=1}^{q} \prod_{j=1}^{p} \varepsilon(s-\xi_i)\varepsilon(s-\zeta_j)\hat{\varphi}_3(s) \Big) : \Big] \times \qquad (5.6)$$

$$\times \exp\Big\{ -\frac{i}{2} g^2 \int_{t_0}^{t}\int_{t_0}^{t} ds_1\, ds_2 \prod_{i=1}^{q} \prod_{j=1}^{p} \varepsilon(s_1 - \xi_i)\varepsilon(s_1 - \zeta_j) \Delta(s_1 - s_2) \varepsilon(s_2 - \xi_i)\varepsilon(s_2 - \zeta_j) \Big\}$$

In this expression the symmetry with respect to the commutation of indices 1, 2,3 is conserved.

If we have the S-matrix it is possible to calculate the renormalization constants at $L \to \infty$
The mass renormalization of the one-nucleon state is

$$\delta m = -g^2 \sum_{k} \frac{1}{\omega^2} \cdot \frac{3}{2} \left[1 + \frac{1}{g^2/\pi^2 + 1} + \dots \right] . \qquad (5.7)$$

The renormalization of the nucleon field Z_2 is

$$Z_2 = \left(\frac{1}{L}\right)^{g^2/2\pi^2} \left[1 + \frac{g^2/\pi^2}{g^2/\pi^2 + 1} \ln L + \dots \right]. \qquad (5.8)$$

The renormalization of the coupling constant is determined in a usual manner and is written as

$$
\begin{aligned}
\frac{g_r}{g} = {} & 1 - \frac{2}{g^2/\pi^2+1} - \\
& - \frac{g^2}{2\pi^2}\int_0^\infty dx_1 \int_0^\infty dx_2 \int_0^\infty dx_3 \, (x_1+x_2) \left\{ \frac{1}{(1+x_1)^{2+g^2/\pi^2}(1+x_2)^{2+g^2/\pi^2}} \left[\left(\frac{(1+x_1+x_3)(1+x_2+x_3)}{(1+x_1+x_2+x_3)(1+x_3)} \right)^{\frac{g^2}{\pi^2}} - 1 \right] + \right. \\
& \left. + \frac{1}{(1+x_1)^{g^2/\pi^2}(1+x_2)^{g^2/\pi^2}(1+x_1+x_2+x_3)^2(1+x_3)^2} \left(\frac{(1+x_1+x_3)(1+x_2+x_3)}{(1+x_1+x_2+x_3)(1+x_3)} \right)^{\frac{g^2}{\pi^2}} \right\} - \dots
\end{aligned}
\tag{5.9}
$$

As for the behaviour of a series for g_r one may exactly repeat all what has been said about the renormalization coupling constant of the charged theory (see Sec. 4).

Note, that in the scalar symmetrical theory there is nothing principally new in comparison with the scalar charged theory.

6. On a Model in the Field Theory

In a recent paper of Bialynicki-Birula/4/ a model of the local field theory with a fixed source was treated, in which the nucleon may be in two states different from each other by their mass (we agreed to call these states a proton and a neutron).

The Hamiltonian of the system has the form

$$
\begin{aligned}
H = {} & m_0(\psi^+\psi) + \frac{1}{2}\int d\vec{x} : \left[\pi^2(\vec{x}) + (\vec{\nabla}\varphi(\vec{x}))^2 + \mu^2\varphi^2(\vec{x}) \right] : + \\
& + g\int d\vec{x} \, (\psi^+\tau_1\psi)\varphi(\vec{x})\rho(\vec{x}) + \Delta m_0 (\psi^+\tau_3\psi) \, .
\end{aligned}
\tag{6.1}
$$

Noting that at $\Delta m_0 = 0$ we have an exactly soluable case of scalar mesons with the fixed source it is possible to apply the perturbation theory by the constant Δm_0 without restricting the interaction forces between the nucleon and mesons. In this manner an interesting result has been received in/4/. The charge renormalization proved to be finite which did not contain the logarithmic singularities.

As for the method developed above Hamiltonian (6.1) is of interest because the series of Lappo-Danilevscky coincides here with the series of the perturbation theory by the constant Δm_0. However, as is was mentioned above, the new method enables us to get the n-th order term of a series that the perturbation theory fails to give. In the case in question this advantage allows to find exactly the spec-

trum of eigenvalues of the full Hamiltonian (5.1).

So, let us consider the equation for the S^{α}*-matrix. We shall look at once for the* S^{α} *-matrix in order to make use of formulae (4.2) and (4.3).*

In the interaction representation we have

$$i\frac{\partial}{\partial t}S^{\alpha}(t,t_0) = H_I(t)e^{-\alpha|t|}S^{\alpha}(t,t_0)$$

$$S^{\alpha}(t,t_0)\big|_{t=t_0} = 1$$

where

$$H_I(t) = g(\psi^{+}\tau_1\psi)\hat{\varphi}(t) + \Delta m_0(\psi^{+}\tau_3\psi)$$

$$\hat{\varphi}(t) = \sum_{\vec{k}}\frac{v(k)}{\sqrt{2\omega}}\left(a_{\vec{k}}e^{-i\omega t} + a^{+}_{\vec{k}}e^{i\omega t}\right). \qquad (6.2)$$

Repeating the procedure set forth in sec. 1–3, we obtain the following expression for the S^{α} *-matrix*

$$S^{\alpha}(t,t_0) = 1 - \left[2(\psi^{+}\psi)-(\psi^{+}\psi)^2\right] +$$

$$+\sum_{q=0}^{\infty}\Bigg\{\frac{(i\Delta m_0)^{2q}}{(2q)!}\int_{t_0}^{t}d\xi_1\ldots\int_{t_0}^{t}d\xi_{2q}\,e^{-\alpha(|\xi_1|+\ldots+|\xi_{2q}|)}\times$$

$$\times\Big[\big(2(\psi^{+}\psi)-(\psi^{+}\psi)^2\big):Ch\Big(ig\int_{t_0}^{t}ds\,e^{-\alpha|s|}\varepsilon(s-\xi_1)\ldots\varepsilon(s-\xi_{2q})\hat{\varphi}(s)\Big):-(\psi^{+}\tau_1\psi):Sh\Big(ig\int_{t_0}^{t}ds\,e^{-\alpha|s|}\varepsilon(s-\xi_1)\ldots\varepsilon(s-\xi_{2q})\hat{\varphi}(s)\Big):\Big]\times$$

$$\times\exp\Big\{-\frac{i}{2}g^2\int_{t_0}^{t}\int_{t_0}^{t}ds_1ds_2\,e^{-\alpha(|s_1|+|s_2|)}\varepsilon(s_1-\xi_1)\ldots\varepsilon(s_1-\xi_{2q})\Delta(s_1-s_2)\varepsilon(s_2-\xi_1)\ldots\varepsilon(s_2-\xi_{2q})\Big\} - \qquad (6.3)$$

$$-\frac{(i\Delta m_0)^{2q+1}}{(2q+1)!}\int_{t_0}^{t}d\xi_1\ldots\int_{t_0}^{t}d\xi_{2q+1}\,e^{-\alpha(|\xi_1|+\ldots+|\xi_{2q+1}|)}\times$$

$$\cdot\Big[(\psi^{+}\tau_3\psi):Ch\Big(ig\int_{t_0}^{t}ds\,e^{-\alpha|s|}\varepsilon(s-\xi_1)\ldots\varepsilon(s-\xi_{2q+1})\hat{\varphi}(s)\Big):-i(\psi^{+}\tau_2\psi):Sh\Big(ig\int_{t_0}^{t}ds\,e^{-\alpha|s|}\varepsilon(s-\xi_1)\ldots\varepsilon(s-\xi_{2q+1})\hat{\varphi}(s)\Big):\Big]$$

$$\times\exp\Big\{-\frac{i}{2}g^2\int_{t_0}^{t}\int_{t_0}^{t}ds_1ds_2\,e^{-\alpha(|s_1|+|s_2|)}\varepsilon(s_1-\xi_1)\ldots\varepsilon(s_1-\xi_{2q+1})\Delta(s_1-s_2)\varepsilon(s_2-\xi_1)\ldots\varepsilon(s_2-\xi_{2q+1})\Big\}\Bigg\}$$

Having the S-matrix, it is easy to calculate the renormalization constants. The eigenvalue of the energy of the one-fermion state is (see Appendix D).

$$E_N = \lim_{\alpha \to 0} \frac{\langle 0 | C_N H S^{\alpha}(0,-\infty) C_N^{+} | 0 \rangle}{\langle 0 | C_N S^{\alpha}(0,-\infty) C_N^{+} | 0 \rangle} =$$

$$= m_0 - \frac{1}{2} g^2 \sum_{\vec{\kappa}} \frac{v^2(\kappa)}{\omega^2} + \delta_N \Delta m_0 \exp\left\{-g^2 \sum_{\vec{\kappa}} \frac{v^2(\kappa)}{\omega^3}\right\} \tag{6.4}$$

where

$$\delta_N = \begin{cases} +1 & \text{for the proton} \quad (N=p) \\ -1 & \text{for the neutron} \quad (N=n) \end{cases}$$

Determine the renormalization (physical) quantities

$$m = m_0 - \frac{1}{2} g^2 \sum_{\vec{\kappa}} \frac{v^2(\kappa)}{\omega^2} \tag{6.5a}$$

$$\Delta m = \Delta m_0 \exp\left\{-g^2 \sum_{\vec{\kappa}} \frac{v^2(\kappa)}{\omega^3}\right\} . \tag{6.5b}$$

The renormalization m_0 coincides exactly with the case of scalar mesons in the field with the fixed source. It is interesting to note that in this model the eigenvalue of the energy of the one-fermion state is renormalized by the two renormalization constants instead of one, as usual.

In the case of the transition to the point interaction the requirement of the finiteness of the renormalized constant m and Δm leads to the necessity of considering the unrenormalized quantities m_0 and Δm_0 as infinite, the order of their increasing being different when

$$m_0 \to \frac{g^2}{4\pi^2} L$$

$$\Delta m_0 \to \Delta m \cdot L^{\frac{g^2}{2\pi^2}}$$

where L is the cut-off momentum.

Having expressed Δm_0 in terms of Δm according to (5.5b) and substituting it into (6.3), we obtain the expression for the S^{α} -matrix represented by a series by the observed parameter Δm.

The eigenvalue of the energy of the system consisting of a nucleon and n-mesons with the momenta $\vec{p}_1 \dots \vec{p}_n$ *, is equal (as it should be expected) to*

$$E_{N\pi_1\dots\pi_n} = E_N + \omega_{\vec{p}_1} + \dots + \omega_{\vec{p}_n}$$

where E_N *is given by formula (6.4).*

Such a spectrum of the eigenvalues is natural for the Hamiltonian with the fixed nucleon.

The renormalization constant of the fermion field Z_2 *is determined as follows*

$$Z_2 = \left| \langle 0 | C_N S^{\alpha}(0,-\infty) C_N^{+} | 0 \rangle \right|^2 =$$

$$= \left| \sum_{q=0}^{\infty} \frac{(-i\Delta m_0)^q}{q!} \int_{-\infty}^{0} d\xi_1 \dots \int_{-\infty}^{0} d\xi_q \, e^{\alpha(\xi_1+\dots+\xi_q)} \exp\left\{ -\frac{i}{2} g^2 \iint_{-\infty}^{0} ds_1 ds_2 \, e^{\alpha(s_1+s_2)} \prod_{j=1}^{q} \varepsilon(s_1-\xi_j) \Delta(s_1-s_2) \varepsilon(s_2-\xi_j) \right\} \right|^2 \qquad (6.6)$$

$$= \exp\left\{ -g^2 \sum_{\vec{k}} \frac{v^2(k)}{2\omega^3} \right\} \left[1 + i\Delta m \int_{-\infty}^{0} d\xi \left(e^{g^2 \sum_{\vec{k}} \frac{v^2(k)}{\omega^3} e^{-i\omega\xi}} - e^{g^2 \sum_{\vec{k}} \frac{v^2}{\omega^3} e^{i\omega\xi}} \right) + \dots \right].$$

If $\frac{g^2}{2\pi^2} < 1$ *, all the integrals in square brackets are convergent, when* $v(k) \to 1$ *(i.e.,* $L \to \infty$ *). Thus, when the cut-off is taken away* Z_2 *is tending to zero like* $(1/L)^{g^2/2\pi^2}$*. In accordance with the probability meaning of* Z_2 *the equality of this constant to zero means that the physical nucleon cannot be found in a 'bare' state.*

The renormalization coupling constant is introduced in a usual manner

$$\frac{g_r}{g} = \left(\Psi_p^{(+)}, (\psi^+ \tau_1 \psi) \Psi_n^{(-)} \right) = \lim_{\alpha \to 0} \frac{\langle 0 | C_p S^{\alpha}(\infty,0) (\psi^+ \tau_1 \psi) S^{\alpha}(0,-\infty) C_n^{+} | 0 \rangle}{\sqrt{\langle 0 | C_p S^{\alpha}(\infty,-\infty) C_p^{+} | 0 \rangle \langle 0 | C_n S^{\alpha}(\infty,-\infty) C_n^{+} | 0 \rangle}} \qquad (6.7)$$

Restricting oneself by the two terms in series (6.7) (the switching off of the infinite phase from (6.7) is performed in the same way as in (4.3)) we shall have

$$\frac{g_r}{g} = 1 - 2(i\Delta m)^2 \int_0^{\infty} dx \cdot x \left[\exp\left\{ 2g^2 \sum_{\vec{k}} \frac{v^2(k)}{\omega^3} e^{-i\omega x} \right\} - 1 \right] + \dots \qquad (6.8)$$

In the transition to the point interaction ($v(k) \to 1$, $L \to \infty$ *) all the integrals in series (6.8) are convergent* for* $g^2/2\pi^2 < 1$.

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* The condition of convergence $g^2/2\pi^2 < \frac{2}{e}$ given in [4] is not accurate.

The situation in this model is essentially different from that for the charged theory (see (4.10) and further).

For (6.7) and (6.8), $g = 0$ is not a singular point since at this point the integrals are limite in contrast to (4.12). Therefore, here in applying the perturbation theory, i.e. in representing the solution as a series by g^2 , there arise no logarithmic divergences by L characteristic of the field theory. In this connection, the given model, in our opinion, does not reflect certain fundamental difficulties concerning the exact equations of mesodynamics.

In conclusion we give the expression for the matrix element of meson-nucleon-scattering according to this model

$$S_{f \leftarrow i} = \lim_{\alpha \to 0} \frac{\langle 0 | C_N a_{\vec{p}_f} S^{\alpha}(\infty, -\infty) a^+_{\vec{p}_i} C^+_N | 0 \rangle}{\langle 0 | C_N S^{\alpha}(\infty, -\infty) C^+_N | 0 \rangle} = \qquad (6.9)$$

$$= \delta(\vec{p}_i - \vec{p}_f) - 2\pi i \delta(\omega_f - \omega_i) M_{f \leftarrow i}(\omega_f)$$

where

$$M_{f \leftarrow i}(\omega_f) = g^2 \frac{v^2(p_f)}{2\omega_f} \frac{1}{i} \int_{-\infty}^{\infty} d\tau \, e^{-i\omega_f \tau} \lim_{\alpha \to 0} \frac{\sum_{q=0}^{\infty} (-i\delta_N \Delta m_0)^q \frac{1}{q!} B_q^{\alpha}}{\sum_{q=0}^{\infty} (-i\delta_N \Delta m_0)^q \frac{1}{q!} b_q^{\alpha}}$$

$$B_q^{\alpha} = \int_{-\infty}^{\infty} d\xi_1 \ldots \int_{-\infty}^{\infty} d\xi_q \, e^{-\alpha(|\xi_1| + \ldots + |\xi_q|)} \mathcal{E}(\xi_1) \ldots \mathcal{E}(\xi_q) \mathcal{E}(\xi_1 - \tau) \ldots \mathcal{E}(\xi_q - \tau) \times$$

$$\times \exp\left\{ -\frac{i}{2} g^2 \iint_{-\infty}^{\infty} ds_1 ds_2 \, e^{-\varepsilon(|s_1| + |s_2|)} \prod_{j=1}^{q} \mathcal{E}(s_1 - \xi_j) \Delta(s_1 - s_2) \mathcal{E}(s_2 - \xi_j) \right\}$$

$$b_q^{\alpha} = \int_{-\infty}^{\infty} d\xi_1 \ldots \int_{-\infty}^{\infty} d\xi_q \, e^{-\alpha(|\xi_1| + \ldots + |\xi_q|)} \exp\left\{ -\frac{i}{2} g^2 \iint_{-\infty}^{\infty} ds_1 ds_2 \, e^{-\varepsilon(|s_1| + |s_2|)} \prod_{j=1}^{q} \mathcal{E}(s_1 - \xi_j) \Delta(s_1 - s_2) \mathcal{E}(s_2 - \xi_j) \right\}$$

Making use, as usual, of the division to cancel the infinite phase and restricting to the two terms of the expression obtained after this procedure had been performed, we shall have

$$M_{f\leftarrow i}(\omega_f) = -2\delta_N g^2 \frac{v^2(p_f)}{\omega_f^2}\cdot\frac{\Delta m}{\omega_f}\left[1-\delta_N \frac{4i\Delta m}{\omega_f}\int_0^\infty dx \sin^2\frac{x}{2}\left\{e^{2g^2\sum_k \frac{v^2}{\omega^3} e^{-i\frac{\omega}{\omega_f}x}}-1\right\}+\dots\right]. \tag{6.10}$$

Conclusion

The developed method for solving the problems concerning the field theory with the fixed source enables us to find the solutions as series for which the n-th order term is known. At the same time the coupling constant is not a parameter of expansion, and, hence, the assumption about its smallness is not required. One may hope that the knowledge of an explicit form of the n-th order term of a series representing the solution will make it possible to answer the question about the series convergence, at least, for separate models of this class. However, a study of the renormalized coupling constant is likely to lead to the conclusion about the existence of the singular point at the point $g = 0$. This statement cast a serious doubt upon all the methods which make use of the expansion in the constant g. At any rate it follows from formula (4.12) that the logarithmically divergent terms which are absent in our solution appear inevitably in expanding by g. Besides, let us note the following: the application of the little developed method of the functional integration yielding in the given case correct results allows one to hope that in further development of this method it will find more effective application in solving the exact equations of the field theory.

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Appendix A

Taking a simple differential equation of oscillations it is possible to clear up the meaning of iterations in the method of Lappo-Danilevsky. Consider an equation system written in a matrix form

$$i\frac{\partial}{\partial t}\mathcal{Y}(t) = (g_1\tau_1 + g_2\tau_2)\mathcal{Y}(t)$$
$$\mathcal{Y}(0) = I\,. \tag{A.1}$$

Here g_1 and g_2 are constant coefficients. $\mathcal{Y}(t)$ is a two-series matrix. System (A.1) is solved exactly and its solution is written down in the form

$$\mathcal{Y}(t) = \mathrm{Ch}\left(i\sqrt{g_1^2+g_2^2}\,t\right) - \frac{g_1\tau_1 + g_2\tau_2}{\sqrt{g_1^2+g_2^2}}\,\mathrm{Sh}\left(i\sqrt{g_1^2+g_2^2}\,t\right). \tag{A.2}$$

On the other hand, Lappo-Danilevsky method gives the solution as follows

$$\mathcal{Y}(t) = \sum_{q=0}^{\infty}\left\{ \frac{(ig_2)^{2q}}{(2q)!}\int_0^t d\xi_1 \dots \int_0^t d\xi_{2q}\left[\mathrm{Ch}\left(ig_1\int_0^t ds\,\varepsilon(s-\xi_1)\dots\varepsilon(s-\xi_{2q})\right) - \right.\right.$$

$$\left. - \tau_1\,\mathrm{Sh}\left(ig_1\int_0^t ds\,\varepsilon(s-\xi_1)\dots\varepsilon(s-\xi_{2q})\right)\right] -$$

$$- \frac{(ig_2)^{2q+1}}{(2q+1)!}\int_0^t d\xi_1 \dots \int_0^t d\xi_{2q+1}\left[\tau_2\,\mathrm{Ch}\left(ig_1\int_0^t ds\,\varepsilon(s-\xi_1)\dots\varepsilon(s-\xi_{2q+1})\right) + \right. \tag{A.3}$$

$$\left.\left. + \tau_2\tau_1\,\mathrm{Sh}\left(ig_1\int_0^t ds\,\varepsilon(s-\xi_1)\dots\varepsilon(s-\xi_{2q+1})\right)\right]\right\}.$$

Calculating the integrals in (A.3) one can see that the series obtained is a Taylor expansion for function (A.2) in the vicinity of the point $g_2 = 0$. For example

$$\mathrm{Ch}\left(i\sqrt{g_1^2+g_2^2}\,t\right) = \sum_{q=0}^{\infty}\frac{(ig_2)^{2q}}{(2q)!}\int_0^t d\xi_1 \dots \int_0^t d\xi_{2q}\,\mathrm{Ch}\left(ig_1\int_0^t ds\,\varepsilon(s-\xi_1)\dots\varepsilon(s-\xi_{2q})\right) =$$

$$= \mathrm{Ch}(ig_1 t) + \frac{1}{2}(ig_2 t)\frac{g_2}{g_1}\mathrm{Sh}(ig_1 t) + \dots \tag{A.4}$$

The solution in the form of (A.3) may be obtained if the perturbation theory is applied by the constant g_2 to equation (A.1). However, Lappo-Danilevsky presents here the possibility of writing down the n-th order term of the series what is not trivial in the perturbation theory.

Appendix B

The integration of the 'classical' $\tilde{S}$-matrix (2.6) over the classical fields Λ_1 and Λ_2 is based on the following relations

$$C\int \delta\Lambda_2 \exp\Big\{-i\int_{t_0}^{t} ds\,\Lambda_2(s)\Phi_2(s)\Big\} \exp\Big\{ig\int_{t_0}^{t} ds\,\rho(s)\Lambda_2(s)\Big\} = \prod_s \delta(\Phi_2(s) - g\rho(s)) \qquad (B1)$$

where $\rho(s)$ is a certain real function of s.

$$C\int \delta\Lambda_1 \exp\Big\{-i\int_{t_0}^{t} ds\,\Lambda_1(s)\Phi_1(s)\Big\}\ \Lambda_1(\xi_1)\dots\Lambda_1(\xi_n) = i\frac{\delta}{\delta\Phi_1(\xi_1)}\cdots i\frac{\delta}{\delta\Phi_1(\xi_n)}\ \prod_s \delta(\Phi_1(s)). \qquad (B.2)$$

A further integration over the functions Φ_1 and Φ_2 can be also performed without any difficulty

$$\int\delta\Phi_2 \exp\Big\{-\frac{i}{2}\int_{t_0}^{t}\!\!\int_{t_0}^{t} d\xi\, d\eta\, \Delta(\xi-\eta)\Phi_2(\xi)\Phi_2(\eta) + i\int_{t_0}^{t} ds\,\hat{\varphi}_2(s)\Phi_2(s)\Big\} \prod_s \delta(\Phi_2(s) - g\rho(s)) =$$

$$= \exp\Big\{-\frac{i}{2}g^2\int_{t_0}^{t}\!\!\int_{t_0}^{t} d\xi\, d\eta\, \Delta(\xi-\eta)\rho(\xi)\rho(\eta) + ig\int_{t_0}^{t} ds\,\hat{\varphi}_2(s)\rho(s)\Big\} \qquad (B.3)$$

and

$$I_n = I_n(\xi_1,\dots,\xi_n) = \int\delta\Phi_1 \exp\Big\{-\frac{i}{2}\int_{t_0}^{t}\!\!\int_{t_0}^{t} d\xi\, d\eta\, \Delta(\xi-\eta)\Phi_1(\xi)\Phi_1(\eta)\Big\} \exp\Big\{i\int_{t_0}^{t} ds\,\hat{\varphi}_1(s)\Phi_1(s)\Big\}\times$$

$$\times\, i\frac{\delta}{\delta\Phi_1(\xi_1)}\cdots i\frac{\delta}{\delta\Phi_1(\xi_n)}\ \prod_s \delta(\Phi_1(s)) = \qquad (B.4)$$

$$= (-)^n\Big[i\frac{\delta}{\delta\Phi_1(\xi_1)}\cdots i\frac{\delta}{\delta\Phi_1(\xi_n)} \exp\Big\{-\frac{i}{2}\int_{t_0}^{t}\!\!\int_{t_0}^{t} d\xi\, d\eta\, \Delta(\xi-\eta)\Phi_1(\xi)\Phi_1(\eta) + i\int_{t_0}^{t} ds\,\hat{\varphi}_1(s)\Phi_1(s)\Big]_{\Phi(s)=0}.$$

Let us transform the function I_n *so that it would have a more convenient form. In (3.4) the variational derivatives may be substituted by particular derivatives, then*

$$I_n=(-i)^n\left[\frac{\partial^n}{\partial z_1\dots\partial z_n}\exp\left\{-\frac{1}{2}\sum_{i,j=1}^{n}\Delta_{ij}z_iz_j+\sum_{j=1}^{n}z_ja_j\right\}\right]_{z_1=\dots=z_n=0} \quad (3.5)$$

where

$$\Delta_{ij}=i\,\Delta(\xi_i-\xi_j)$$
$$a_j=i\,\hat{\varphi}_1(\xi_j)\;.$$

Differentiating over z_n *and putting* $z_n=0$ *, we get*

$$I_n=(-i)^n\left[\frac{\partial^{n-1}}{\partial z_1\dots\partial z_{n-1}}\left(-\sum_{j=1}^{n-1}\Delta_{jn}z_j\right)\exp\left\{-\frac{1}{2}\sum_{i,j=1}^{n-1}\Delta_{ij}z_iz_j+\sum_{j=1}^{n-1}z_ja_j\right\}\right]_{z_1=\dots=z_{n-1}=0}+(-i)a_nI_{n-1} \quad (3.6)$$

Note, that $I_n(\xi_1,\dots,\xi_n)$ *is a completely symmetrical function with respect to the commutations* $\xi_1,\dots,\xi_n$. *It is integrated over* $\xi_1,\dots,\xi_n$ *within identical limits also with a completely symmetrical function. Therefore, one may consider that it is not* $\sum_{j=1}^{n-1}\Delta_{jn}z_j$ *which stands before the exponent in (3.6) but* $(n-1)\Delta_{n-1,n}z_{n-1}$ *. Thereby we violate the symmetry of function (3.4). However, this does not affect the result of the integration over* $\xi_1,\dots,\xi_n$ *. Thus, the following recurrent relation is obtained*

$$I_n=(n-1)\Delta_{n-1,n}I_{n-2}-i\,a_nI_{n-1}\,. \quad (3.7)$$

Knowing I_1 *and* I_2 *(they can be easily obtained directly from (3.4)) it is not difficult to prove by the method of mathematical induction that*

$$I_n=(-i)^n\sum_{m=0}^{[\frac{n}{2}]}\frac{n!\,i^m}{2^m m!(n-2m)!}\,\Delta_{12}\dots\Delta_{2m-1,2m}\,a_{2m+1}\dots a_n$$

or

$$I_n(\xi_1,\dots,\xi_n)=(-)^n\left[i\frac{\delta}{\delta\Phi_1(\xi_1)}\dots i\frac{\delta}{\delta\Phi_1(\xi_n)}\exp\left\{-\frac{i}{2}\iint_{t_0t_0}^{tt}d\xi d\eta\,\Delta(\xi-\eta)\Phi_1(\xi)\Phi_1(\eta)+i\int_{t_0}^{t}ds\,\hat{\varphi}_1(s)\Phi_1(s)\right\}\right]_{\Phi_1=0}=$$

$$=\sum_{m=0}^{[\frac{n}{2}]}\frac{n!\,i^m}{2^m m!(n-2m)!}\,\Delta(\xi_1-\xi_2)\dots\Delta(\xi_{2m-1}-\xi_{2m})\,\hat{\varphi}_1(\xi_{2m+1})\dots\hat{\varphi}_1(\xi_n) \quad (3.8)$$

$$\left[\frac{n}{2}\right] = \begin{cases} \frac{n}{2}\,, & \text{if} \quad n = 2K \\ \frac{n-1}{2}\,, & \text{if} \quad n = 2K+1\,. \end{cases}$$

Appendix C

The renormalized coupling constant in the charged scalar theory is determined by (4.8). Consider first the matrix element standing in the numerator

$$M_1^\alpha = \langle 0| C_p S^\alpha(\infty,0)(\psi^+ \tau_+ \psi) S^\alpha(0,-\infty) C_n^+ |0\rangle =$$

$$= \langle 0| C_p S^\alpha(\infty,0) \frac{1}{2}\left[(\psi^+\tau_1\psi) + i(\psi^+\tau_2\psi)\right] S^\alpha(0,-\infty) C_n^+ |0\rangle = \qquad (6.1)$$

$$= \langle 0| C_p \frac{i}{2g}\left[\frac{\delta}{\delta\hat{\varphi}_1(0)} + i\frac{\delta}{\delta\hat{\varphi}_2(0)}\right] S^\alpha(\infty,-\infty) C_n^+ |0\rangle\,.$$

Since the S-matrix is symmetrical with respect to the commutation of indices 1 and 2, then

$$M_1^\alpha = \langle 0| C_p \frac{i}{g}\, i \frac{\delta}{\delta\hat{\varphi}_2(0)} S^\alpha(\infty,-\infty) C_n^+ |0\rangle\,. \qquad (C.2)$$

Substituting into (C.2) the expression for the S-matrix (3.1), we obtain

$$M_1^\alpha = \sum_{q=0}^{\infty} \frac{1}{q!}\left(-\frac{g^2}{2}\right)^q A_q^\alpha$$

where

$$A_q^\alpha = \int_{-\infty}^{\infty} d\xi_1 \ldots \int_{-\infty}^{\infty} d\xi_{2q}\, e^{-\alpha(|\xi_1|+\ldots+|\xi_{2q}|)}\, i\Delta(\xi_1-\xi_2)\ldots i\Delta(\xi_{2q-1}-\xi_{2q})\, \varepsilon(\xi_1)\varepsilon(\xi_2)\ldots\varepsilon(\xi_{2q}) \times \qquad (C.3)$$

$$\times \exp\left\{-\frac{i}{2}g^2 \iint_{-\infty}^{\infty} ds_1 ds_2\, e^{-\alpha(|s_1|+|s_2|)}\, \varepsilon(s_1-\xi_1)\ldots\varepsilon(s_1-\xi_{2q})\Delta(s_1-s_2)\,\varepsilon(s_2-\xi_1)\ldots\varepsilon(s_2-\xi_{2q})\right\}.$$

The matrix element in the denominator of formula (4.8) may be written as follows

$$M_2^\alpha = \langle 0| C_p S^\alpha(\infty,-\infty) C_p^+ |0\rangle = \sum_{q=0}^{\infty} \frac{1}{q!}\left(-\frac{g^2}{2}\right)^q a_q^\alpha \qquad (C.4)$$

where

$$a_q^{\alpha} = \int\limits_{-\infty}^{\infty} d\zeta_1 \ldots \int\limits_{-\infty}^{\infty} d\zeta_{2q}\, e^{-\alpha(|\zeta_1|+\ldots+|\zeta_{2q}|)}\, i\Delta(\zeta_1-\zeta_2)\ldots i\Delta(\zeta_{2q-1}-\zeta_{2q}) \times$$

$$\times \exp\left\{-\frac{i}{2}g^2 \iint\limits_{-\infty}^{\infty} ds_1 ds_2\, e^{-\alpha(|s_1|+|s_2|)}\, \mathcal{E}(s_1-\zeta_1)\ldots\mathcal{E}(s_1-\zeta_{2q})\Delta(s_1-s_2)\mathcal{E}(s_2-\zeta_1)\ldots\mathcal{E}(s_2-\zeta_{2q})\right\}.$$

With the accuracy of the first degree of α the integral in the exponent is equal

$$J_n(\zeta_1,\ldots,\zeta_n) = -\frac{i}{2}g^2 \iint\limits_{-\infty}^{\infty} ds_1 ds_2\, e^{-\alpha(|s_1|+|s_2|)}\mathcal{E}(s_1-\zeta_1)\ldots\mathcal{E}(s_1-\zeta_n)\Delta(s_1-s_2)\mathcal{E}(s_2-\zeta_1)\ldots\mathcal{E}(s_2-\zeta_n) =$$

$$= -\frac{1}{i\alpha}\cdot\frac{1}{2}g^2\sum_{\vec{k}}\frac{v^2(k)}{\omega^2} - g^2\sum_{\vec{k}}\frac{v^2(k)}{\omega^3}\left[n + 2\sum_{\ell=2}^{n}\left\{\prod_{j\neq\ell}^{n}\mathcal{E}(\zeta_\ell-\zeta_j)\right\}\sum_{m=1}^{\ell-1}\left\{\prod_{i\neq 1}^{n}\mathcal{E}(\zeta_m-\zeta_i)\right\}e^{-i\omega|\zeta_m-\zeta_\ell|}\right]. \qquad (C.5)$$

The infinite phase $\exp\left\{-\frac{1}{i\alpha}\cdot\frac{1}{2}g^2\sum_{\vec{k}}\frac{v^2(k)}{\omega^2}\right\}$ is identical in all the terms of series M_1^{α} and M_2^{α}. Therefore, it can be canceled out.

If $\zeta_1 > \zeta_2 > \ldots > \zeta_n$, then formula (C.5) becomes simpler

$$J_n(\zeta_1,\ldots,\zeta_n) = -g^2\sum_{\vec{k}}\frac{v^2(k)}{\omega^3}\left[n + \sum_{\ell=2}^{n}\sum_{m=1}^{\ell-1}(-)^{\ell+m}e^{-i\omega(\zeta_m-\zeta_\ell)}\right]. \qquad (C.6)$$

The relation (4.8) with account of (C.3) and (C.4) may be rewritten as follows

$$\frac{g_r}{g} = \lim_{\alpha\to 0}\frac{M_1^{\alpha}}{M_2^{\alpha}} = \lim_{\alpha\to 0}\frac{\sum\limits_{q=0}^{\infty}\frac{1}{q!}\left(-\frac{g^2}{2}\right)^q A_q^{\alpha}}{\sum\limits_{q=0}^{\infty}\frac{1}{q!}\left(-\frac{g^2}{2}\right)^q a_q^{\alpha}} = \lim_{\alpha\to 0}\sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{g^2}{2}\right)^n \bar{A}_n^{\alpha} \qquad (C.7)$$

where $\bar{A}_n^{\alpha}$, A_q^{α}, a_q^{α} are related by

$$A_n^{\alpha} = \sum_{p+q=n}\frac{n!}{p!q!}\, a_p^{\alpha}\,\bar{A}_q^{\alpha}$$

from where

$$\bar{A}_n^{\alpha} = (A_n^{\alpha} - a_n^{\alpha}) - \sum_{q=1}^{n-1}\frac{n!}{q!(n-q)!}\,\bar{A}_q^{\alpha}\, a_{n-q}^{\alpha}.$$

This recurrent relation allows to calculate the n-th order term, if all the previous ones are known.

The presence of the infinite phase in M_1^{α} and M_2^{α} is expressed at $\alpha = 0$ in the divergence of a part of integrals in A_q^{α} and a_q^{α}. However, in $\bar{A}_n^{\alpha}$ at $\alpha \to 0$ all the integrals are convergent. This means that the infinite phase is thereby canceled. Therefore, in the expression for $\bar{A}_n^{\alpha}$ one may put $\alpha = 0$, i.e.,

$$\bar{A}_n = \lim_{\alpha \to 0} \bar{A}_n^{\alpha} = (A_n - a_n) - \sum_{q=1}^{n-1} \frac{n!}{q!(n-q)!} \bar{A}_q a_{n-q} \tag{C.9}$$

where

$$A_q = A_q^{\alpha}\big|_{\alpha=0} \; ; \quad a_p = a_p^{\alpha}\big|_{\alpha=0} .$$

Finally we obtain

$$\frac{g_r}{g} = \sum_{q=0}^{\infty} \frac{1}{q!}\left(-\frac{g^2}{2}\right)^q \bar{A}_q \tag{C.10}$$

where $\bar{A}_q$ is determined from (C.9), whereas A_q and a_q are taken from (C.3) and (C.4) by $\alpha = 0$.

Consider the first term $(\bar{A}_0 = 1)$

$$\bar{A}_1 = (A_1 - a_1) = \int_{-\infty}^{\infty} d\tau_1 \int_{-\infty}^{\infty} d\tau_2 \; i\Delta(\tau_1 - \tau_2)[\varepsilon(\tau_1)\varepsilon(\tau_2) - 1]\, e^{J_2(\tau_1,\tau_2)} =$$

$$= \int_{-\infty}^{\infty} d\tau_1 \int_{-\infty}^{\tau_1} d\tau_2 \; 2i\Delta(\tau_1 - \tau_2)[\varepsilon(\tau_1)\varepsilon(\tau_2) - 1] \exp\left\{-g^2 \sum_{\vec{k}} \frac{v^2(k)}{\omega^3}\left[2 - 2e^{-i\omega(\tau_1 - \tau_2)}\right]\right\} .$$

Making the substitution of the variables $\tau_1 = \nu$, $\tau_1 - \tau_2 = \eta$ we get

$$\bar{A}_1 = \int_{-\infty}^{\infty} d\nu \int_0^{\infty} d\eta \; 2i\Delta(\eta)[\varepsilon(\nu)\varepsilon(\nu+\eta) - 1] \exp\left\{-2g^2 \sum_{\vec{k}} \frac{v^2(k)}{\omega^3}\left[1 - e^{-i\omega\eta}\right]\right\} = \tag{C.11}$$

$$= -2 \int_0^{\infty} d\eta \cdot \eta \sum_{\vec{k}} \frac{v^2(k)}{\omega} e^{-i\omega\eta} \cdot \exp\left\{-2g^2 \sum_{\vec{k}} \frac{v^2(k)}{\omega^3}\left[1 - e^{-i\omega\eta}\right]\right\}$$

since

$$\int_{-\infty}^{\infty} d\nu [\varepsilon(\nu)\varepsilon(\nu+\eta) - 1] = -2 \int_0^{\eta} d\nu = -2\eta .$$

Now let us pass to the limit in $\bar{A}_1$ by $L \to \infty$. As is known the causality functions have the singularities for small values of the argument. Choosing the form-factor in the form

$$v(\kappa) = \exp\left\{-\frac{\omega-\mu}{2L}\right\}$$

and regarding L sufficiently large, we obtain the behaviour of the causality functions for small arguments (by $\left|\frac{\mu}{L}+i\mu\eta\right| \ll 1$)

$$\sum_{\vec{k}} \frac{v^2(\kappa)}{\omega} e^{-i\omega\eta} \sim \frac{1}{\pi^2}\frac{1}{\left(\frac{1}{L}+i\eta\right)^2} \tag{C.12}$$

$$\sum_{\vec{k}} \frac{v^2(\kappa)}{\omega^3} e^{-i\omega\eta} \sim -\frac{1}{2\pi^2}\ln\left(\frac{\mu}{L}+i\mu\eta\right).$$

Let us present now the causality functions as follows

$$\sum_{\vec{k}} \frac{v^2(\kappa)}{\omega} e^{-i\omega\eta} = \frac{1}{\pi^2}\cdot\frac{1}{\left(\frac{1}{L}+i\eta\right)^2}\,\mathcal{F}_1(\eta) \tag{C.13}$$

$$\sum_{\vec{k}} \frac{v^2(\kappa)}{\omega^3} e^{-i\omega\eta} = -\frac{1}{2\pi^2}\ln\left(\frac{\mu}{L}+i\mu\eta\right) + \ln \mathcal{F}_2(\eta)$$

where

$$\mathcal{F}_1(0) = \mathcal{F}_2(0) = 1.$$

Then the integral (C.11) with account of (C.13)

is

$$\bar{A}_1 = -\frac{2}{\pi^2}\int_0^\infty d\eta\cdot\eta\,\frac{\left(\frac{1}{L}\right)^{\frac{g^2}{\pi^2}}}{\left(\frac{1}{L}+i\eta\right)^{2+\frac{g^2}{\pi^2}}}\,\mathcal{F}(\eta) \tag{C.14}$$

where

$$\mathcal{F}(\eta) = \mathcal{F}_1(\eta)\left[\mathcal{F}_2(\eta)\right]^{g^2/\pi^2}; \qquad \mathcal{F}(0) = 1$$

The function $\mathcal{F}(\eta)$ ensures the convergence on infinity. As can be easily seen, at $L\to\infty$ the integral is divergent at the lower limit. Let us divide the integral in (C.14) into two

$$\overline{A}_1 = -\frac{2}{\pi^2}\left(\frac{1}{L}\right)^{\frac{g^2}{\pi^2}}\int_0^1 \frac{d\eta\cdot\eta}{\left(\frac{1}{L}+i\eta\right)^{2+g^2/\pi^2}}\,\mathcal{F}(\eta) - \frac{2}{\pi^2}\left(\frac{1}{L}\right)^{\frac{g^2}{\pi^2}}\int_1^\infty \frac{d\eta\cdot\eta}{\left(\frac{1}{L}+i\eta\right)^{2+g^2/\pi^2}}\,\mathcal{F}(\eta). \tag{C.15}$$

At the limit $L\to\infty$ the second term disappears, since the integral is convergent on all the interval $[1, \infty]$. The first term gives the finite contribution . Indeed, making the substitution $i\eta=\frac{1}{L}y$ we get

$$\overline{A}_1 = \frac{2}{\pi^2}\int_0^{iL}\frac{dy\cdot y\;\mathcal{F}(\frac{1}{iL}y)}{(1+y)^{2+g^2/\pi^2}} \xrightarrow[L\to\infty]{} \frac{2}{\pi^2}\int_0^{i\infty}\frac{dy\cdot y}{(1+y)^{2+g^2/\pi^2}} = \frac{2}{\pi^2}\int_0^\infty\frac{dx\cdot x}{(1+x)^{2+g^2/\pi^2}} = \frac{2}{\pi^2}\,\frac{1}{\frac{g^2}{\pi^2}\left(\frac{g^2}{\pi^2}+1\right)}. \tag{C.16}$$

Here we passed from the integration over the ray $[0, i\infty]$ to $[0, \infty]$, since the integrand is analytical in the region $0\le \arg z\le\frac{\pi}{2}$.

In the transition to the limit by $L\to\infty$ in formulae (C.15) and (C.16) the given speculations may be proved with mathematical rigour.

By analogy one may obtain $\overline{A}_2$, $\overline{A}_3$ etc.

Appendix D

According to formula (4.3) the eigenvalue of the energy of the one-fermion state is determined as follows

$$E_N = \lim_{\alpha\to 0}\frac{\langle 0|C_N H S^\alpha(0,-\infty)C_N^+|0\rangle}{\langle 0|C_N S^\alpha(0,-\infty)C_N^+|0\rangle} = m_0 + \delta_N\Delta m_0 + \delta E_N \tag{D.1}$$

where

$$\delta E_N = \lim_{\alpha\to 0}\frac{\langle 0|C_N\, g(\psi^+\tau_1\psi)\hat{\varphi}(0)S^\alpha(0,-\infty)C_N^+|0\rangle}{\langle 0|C_N S^\alpha(0,-\infty)C_N^+|0\rangle}.$$

Consider the matrix element standing in the numerator

$$M_1^\alpha = \langle 0|C_N H_I S^\alpha(0,-\infty)C_N^+|0\rangle = \langle 0|C_N g(\psi^+\tau_1\psi)\hat{\varphi}(0) S^\alpha(0,-\infty)C_N^+|0\rangle$$

Substituting into it the S-matrix from (6.3), we get

$$M_1^\alpha = -\sum_{q=0}^{\infty}(-i\delta_N\Delta m_0)^q\int_{-\infty}^{0}d\xi_1\int_{-\infty}^{\xi_1}d\xi_2\ldots\int_{-\infty}^{\xi_{q-1}}d\xi_q\, e^{\alpha(\xi_1+\ldots+\xi_q)}\, ig^2\int_{-\infty}^{0}ds\, e^{\alpha s}\prod_{j=1}^{q}\varepsilon(s-\xi_j)\, i\Delta(s)\times$$

$$\times\exp\Big\{-\tfrac{i}{2}g^2\iint_{-\infty}^{0}ds_1 ds_2\, e^{\alpha(s_1+s_2)}\prod_{j=1}^{q}\varepsilon(s_1-\xi_j)\,\Delta(s_1-s_2)\,\varepsilon(s_2-\xi_j)\Big\}. \qquad (D.2)$$

The matrix element in the denomerator of formula (D.1) is obtained analogously

$$M_2^\alpha = \langle 0|C_N S^\alpha(0,-\infty)C_N^+|0\rangle = \sum_{q=0}^{\infty}(-i\delta_N\Delta m_0)^q\int_{-\infty}^{0}d\xi_1\int_{-\infty}^{\xi_1}d\xi_2\ldots\int_{-\infty}^{\xi_{q-1}}d\xi_q\, e^{\alpha(\xi_1+\ldots+\xi_q)}\times \qquad (D.3)$$

$$\times\exp\Big\{-\tfrac{i}{2}g^2\iint_{-\infty}^{0}ds_1 ds_2\, e^{\alpha(s_1+s_2)}\prod_{j=1}^{q}\varepsilon(s_1-\xi_j)\,\Delta(s_1-s_2)\,\varepsilon(s_2-\xi_j)\Big\}.$$

The integral standing in the degree of the exponent is equal to (by $\xi_1>\xi_2>\ldots>\xi_q$)

$$I_q(\xi_q,\ldots,\xi_1) = -\tfrac{i}{2}g^2\iint_{-\infty}^{0}ds_1 ds_2\, e^{\alpha(s_1+s_2)}\prod_{j=1}^{q}\varepsilon(s_1-\xi_j)\,\Delta(s_1-s_2)\,\varepsilon(s_2-\xi_j) =$$

$$= -\tfrac{1}{4}g^2\sum_{\vec{k}}\frac{v^2(k)}{\omega^2}\Big(\frac{1}{i\alpha}+\frac{1}{\omega}\Big) - g^2\sum_{\vec{k}}\frac{v^2(k)}{\omega^3}\Big[q+\sum_{\ell=1}^{q}(-)^\ell e^{i\omega\xi_\ell}+2\sum_{\ell=2}^{q}\sum_{m=1}^{\ell-1}(-)^{\ell+m}e^{-i\omega(\xi_m-\xi_\ell)}\Big]. \qquad (D.4)$$

The first component in (D.4) is identical for all the terms of a series both for the numerator and denumerator and, hence, it cancels out. Having calculated the integral

$$ig^2\int_{-\infty}^{0}ds\, e^{\alpha s}\varepsilon(s-\xi_1)\ldots\varepsilon(s-\xi_q)\, i\Delta(s) = \tfrac{1}{2}g^2\sum_{\vec{k}}\frac{v^2(k)}{\omega^2} + g^2\sum_{\vec{k}}\frac{v^2(k)}{\omega^2}\sum_{\ell=1}^{q}(-)^\ell e^{i\omega\xi_\ell}. \qquad (D.5)$$

and substituted it into (D.2), we get

$$M_1^\alpha = -\frac{1}{2} g^2 \sum_{\vec{k}} \frac{v^2(k)}{\omega^2} \cdot M_2^\alpha -$$

$$-\sum_{q=1}^{\infty} (-i\delta_N \Delta m_0)^q \int_{-\infty}^{0} d\zeta_1 \cdots \int_{-\infty}^{\zeta_{q-1}} d\zeta_q \, e^{\alpha(\zeta_1+\ldots+\zeta_q)} g \sum_{\vec{k}} \frac{v^2(k)}{\omega^2} \sum_{\ell=1}^{q} (-)^\ell e^{i\omega\zeta_\ell} \, e^{I_q(\zeta_q,\ldots,\zeta_1)} =$$

$$= -\frac{1}{2} g^2 \sum_{\vec{k}} \frac{v^2(k)}{\omega^2} \cdot M_2^\alpha - \tag{D.6}$$

$$-\sum_{q=1}^{\infty} (-i\delta_N \Delta m_0)^q \int_{-\infty}^{0} d\zeta_1 \cdots \int_{-\infty}^{\zeta_{q-1}} d\zeta_q \, e^{\alpha(\zeta_1+\ldots+\zeta_q)} i\left(\frac{\partial}{\partial\zeta_1} + \ldots + \frac{\partial}{\partial\zeta_q}\right) e^{I_q(\zeta_q,\ldots,\zeta_1)} .$$

Consider now the q- th order term of a series

$$R_q = \int_{-\infty}^{0} d\zeta_1 \cdots \int_{-\infty}^{\zeta_{q-1}} d\zeta_q \, e^{\alpha(\zeta_1+\ldots+\zeta_q)} i\left(\frac{\partial}{\partial\zeta_1} + \ldots + \frac{\partial}{\partial\zeta_q}\right) e^{I_q(\zeta_q,\ldots,\zeta_1)} =$$

$$= \int_{-\infty}^{0} d\zeta_1 \int_{-\infty}^{\zeta_1} d\zeta_2 \cdots \int_{-\infty}^{\zeta_{q-2}} d\zeta_{q-1} \, e^{\alpha(\zeta_1+\ldots+\zeta_{q-1})} \times$$

$$\times \Big\{ \int_{\zeta_1}^{0} d\sigma \, e^{\alpha\sigma} \, i\frac{\partial}{\partial\sigma} \, e^{I_q(\zeta_{q-1},\ldots,\zeta_1,\sigma)} +$$

$$+ \sum_{j=q-2}^{1} \int_{\zeta_{j+1}}^{\zeta_j} d\sigma \, e^{\alpha\sigma} \, i\frac{\partial}{\partial\sigma} \, e^{I_q(\zeta_{q-1},\ldots,\zeta_{j+1},\sigma,\zeta_j,\ldots,\zeta_1)} +$$

$$+ \int_{-\infty}^{\zeta_{q-1}} d\sigma \, e^{\alpha\sigma} \, i\frac{\partial}{\partial\sigma} \, e^{I_q(\sigma,\zeta_{q-1},\ldots,\zeta_1)} \Big\} .$$

Calculating with the accuracy up to α , we obtain

$$R_q = \int\limits_{-\infty}^{0} d\zeta_1 \ldots \int\limits_{-\infty}^{\zeta_{q-2}} d\zeta_{q-1}\, e^{\alpha(\zeta_1+\ldots+\zeta_{q-1})}\, i\left[e^{I_q(\zeta_{q-1},\ldots,\zeta_1,0)} - e^{I_q(-\infty,\zeta_{q-1},\ldots,\zeta_1)} \right] =$$

$$= i\left[1 - \exp\left\{-g^2 \sum_{\vec{k}} \frac{v^2(k)}{\omega^3}\right\}\right] \int\limits_{-\infty}^{0} d\zeta_1 \ldots \int\limits_{-\infty}^{\zeta_{q-2}} d\zeta_{q-1}\, e^{\alpha(\zeta_1+\ldots+\zeta_{q-1})}\, e^{I_{q-1}(\zeta_{q-1},\ldots,\zeta_1)}.$$

Substituting the obtained expression into (D.6) we get

$$M_1^{\alpha} = -\frac{1}{2} g^2 \sum_{\vec{k}} \frac{v^2(k)}{\omega^3} \cdot M_2^{\alpha} - i\left[1 - \exp\left\{-g^2 \sum_{\vec{k}} \frac{v^2(k)}{\omega^3}\right\}\right](-i\delta_N \Delta m_0)\, M_2^{\alpha} . \qquad (D.7)$$

From here formula (Y.4) follows immediately

$$E_N = m_0 + \delta_N \Delta m_0 - \frac{1}{2} g^2 \sum_{\vec{k}} \frac{v^2(k)}{\omega^2} - \delta_N \Delta m_0 \left[1 - \exp\left\{-g^2 \sum_{\vec{k}} \frac{v^2}{\omega^3}\right\}\right] = \qquad (D.8)$$

$$= m_0 - \frac{1}{2} g^2 \sum_{\vec{k}} \frac{v^2(k)}{\omega^2} + \delta_N \Delta m_0 \exp\left\{-g^2 \sum_{\vec{k}} \frac{v^2(k)}{\omega^3}\right\}.$$

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