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This paper treats the fluctuations of the metric tensor for two cases:

a/ in the macroworld where these fluctuations are due to the turbulent motion of matter and

b/ in the microworld where these fluctuations are caused by zero vacuum oscillation: It has been shown in the latter case that these fluctuations are essential for the distances of the order of

$$L_0 = \left(\frac{1\pi\gamma\hbar}{c}\right)^{1/2} \cong 10^{-32} cm.$$

Here γ *is the Newtonian gravitational constant.*

1. INTRODUCTION

In those regions of space where there are powerful turbulent motions of matter accompanied by considerable changes in the density of matter or having large irregular velocities of motion ($\frac{v}{c}$ is not small!), the metric tensor $g^{\mu\nu}$ is a random quantity. This implies that the interval of time t_{AB} and the distance x_{AB} separating two physical world-points A and B also become random quantities. Therefore, one may speak only of the probability that $t_{AB} = t$, $x_{AB} = \ell$.

In the microworld such a statistical character of metric may be due to the statistical features of the vacuum, or in other words, to the 'zero' oscillations of the quantized fields. However, we face here a very intricate problem and it is probable that the statistical features of metric which reflect zero vacuum oscillations are essential only in the extremely small volumes which are likely to be beyond the limits of the quantum theory.

Nevertheless, it seems interesting to make a theoretical attempt to enter this region. It is sufficient for the time being to restrict oneself to the simplest problem.

2. FLUCTUATIONS OF METRIC

We will assume that the energy tensor of matter $T^{\mu\nu}$ may be expanded into two terms

$$T^{\mu\nu} = T^{\mu\nu} + \delta T^{\mu\nu} \tag{1}$$

so that $T_0^{\mu\nu}$ describes the global motion of matter characterized by large scales L and long periods of time T , whereas the term $\delta T^{\mu\nu}$ is due to the turbulent motion of matter characterized by small scales and short periods τ ($\lambda \ll L$, $\tau \ll T$). The mean value of $\delta T^{\mu\nu}$ by the time intervals comparable with T or by the scales comparable with L is assumed to be zero. Therefore,

$$< T^{\mu\nu} > = T_0^{\mu\nu} , < \delta T^{\mu\nu} > = 0 \tag{2}$$

where $<\cdots>$ means the averaging over the turbulent motion. Correspondingly the metric tensor (1) may be decomposed into two parts:

$$g^{\mu\nu} = g_0^{\mu\nu} + g^{\mu\nu} + \dots \qquad (3)$$

The magnitude of the turbulent fluctuations of matter $\delta T^{\mu\nu}$ is also assumed to be small, thus the quantities $g^{\mu\nu}$ are also small compared with $g_0^{\mu\nu}$ determining the global space-time metric. Under these assumptions the Einstein gravitational equation may be put as

$$-\frac{1}{2}\Box^2 g^{\mu\nu} = \varkappa t^{\mu\nu} \qquad (4)$$

Here: $\varkappa = \frac{8\pi\gamma}{c^2}$, $\gamma = 6{,}7\cdot 10^{-8}\,\frac{cm^3}{г\cdot сек^2}$ is the Newtonian gravitational constant; $\Box^2 = g_0^{\alpha\beta}\frac{\partial}{\partial x_\alpha}\frac{\partial}{\partial x_\beta}$ and the tensor

$$t^{\mu\nu} = \delta T^{\mu\nu} - \frac{1}{2} g_0^{\mu\nu}\delta T \qquad (5)$$

where $\delta T = g_0^{\alpha\beta}\delta T_{\alpha\beta}$ is an invariant.

From (4) we find

$$g^{\mu\nu}(x) = -2\varkappa\,\Box^{-2} t^{\mu\nu}(x) \qquad (6)$$

where $\Box^{-2}$ is the operator, reverse to $\Box^2$. According to (6) we can write now for the correlations of the metric tensor components at the two space-time points x and x':

$$< g^{\alpha\beta}(x)\, g^{\mu\nu}(x') > = 4\varkappa^2\,\Box_x^{-2}\Box_{x'}^{-2} < t^{\alpha\beta}(x)\, t^{\mu\nu}(x') > \qquad (7)$$

Because of the smallness of $g^{\mu\nu}(x)$, the interval between the two physical world-points A and B

$$\mathcal{S}_{AB} = \int_A^B \sqrt{g^{\mu\nu} dx_\mu dx_\nu} \tag{8}$$

may be represented as follows

$$\mathcal{S}_{AB} = \mathcal{S}^0_{AB} + \frac{1}{2}\int_A^B \frac{g^{\mu\nu} dx_\mu \, dx_\nu}{\sqrt{g^0_{\mu\nu}\, dx_\mu \, dx_\nu}} + \dots \tag{9}$$

The mean value of $< \mathcal{S}_{AB} >$, in the linear approximation, is equal to $\mathcal{S}^0_{AB}$, whereas the root-mean-square deviation $< (\mathcal{S}_{AB} - \mathcal{S}^0_{AB})^2 > = \Delta \mathcal{S}^2_{AB}$, according to (9), may be written as

$$\Delta \mathcal{S}^2_{AB} = \frac{1}{4}\int_A^B dx_\mu \int dx'_\mu < g_{\mu\mu}(x) g_{\mu\mu}(x') > \tag{10}$$

Here the direction of the interval $\mathcal{S}^0_{AB}$ is taken along the Ox_μ axis.

Making use of (7) we express now $\Delta \mathcal{S}^2_{AB}$ in terms of the matter fluctuations

$$\Delta \mathcal{S}^2_{AB} = \frac{\varkappa^2}{2}\int_A^B dx_\mu \int_A^B dx'_\mu \, \Box^{-2}_x \Box^{-2}_{x'} < t^{\mu\mu}(x) t^{\mu\mu}(x') + t^{\mu\mu}(x') t^{\mu\mu}(x) > \tag{11}$$

Thus, in the linear approximation, the problem reduces to the calculation of the double correlations of the tensor $t^{\mu\nu}(x)$.

3. AN ESTIMATION OF METRIC FLUCTUATIONS IN THE MACROWORLD

The motion of matter will be treated as a motion of a perfect compressed fluid.

The tensor of matter for this case reads:

$$T^{\mu\nu} = \left(\rho + \frac{P}{c^2}\right)\frac{u^\mu u^\nu}{c^2} - \frac{P}{c^2}\, g^{\mu\nu} \qquad (12)$$

Here ρ is the rest mass density of the medium, $p = f(\rho)$ is the pressure, u^μ are the velocity components of the medium. From (5) and (12) we obtain

$$t^{\mu\nu}(x) = A^{\mu\nu}(x)\delta\rho + B(x)\frac{\delta(u^\mu u^\nu)}{c^2} \qquad (13)$$

where $A^{\mu\nu}(x) = \left(1 - \frac{V^2}{c^2}\right)\left(\frac{u^\mu u^\nu}{c^2} - \frac{1}{2}\, g_0^{\mu\nu}\right)$ and $B(x) = \left(\rho + \frac{P}{c^2}\right)$, $V^2 = \frac{dP}{d\rho}$ is the square of the velocity of sound. Note, that the density fluctuatiuns $\delta\rho$, by the order of magnitude, are equal to $\frac{\delta u^2}{V^2}\rho$.

Consider now the tensor correlations $g^{\mu\nu}$ outside the volume Ω occupied by the turbulent matter. According to (7), we get

$$< g^{\nu\nu}(x) g^{\nu\nu}(x') > = 4\varkappa^2 \iint_{\Omega\,\Omega} \frac{d^3y\, d^3z}{R(x,y)R(x',z)} < t^{\nu\nu}([t],y)\, t^{\nu\nu}([t'],z) > \qquad (14)$$

where $R(x,y)$ and $R(x',z)$ are the distances between x and y, x' aud z, while $[t] = t - \frac{R(x,y)}{c}$, $[t'] = t' - \frac{R(x',z)}{c}$ are the retarded moments of time.

If δu^2 is not small compared with V^2, but still appreciably less than C^2, then among the components $t^{\nu\nu}$ only the term t^{44} is impottant. At the same time $A^{44} \cong \frac{1}{2}$, $B^{44} \ll A^{44}$, therefore : $< t^{44}(y) t^{44}(z) > \cong < \delta\rho(y)\delta\rho(z) >$.

When the medium is sufficiently homogeneous this quantity will depend weakly upon $\frac{1}{2}(y+z)$ and essentially depend upon ($y - z$). Passing now in (14) to the coordinates $\frac{1}{2}(y+z)$ and ($y - z$), it is not difficult to obtain an estimate of (14) for

$$x = (\vec{x}, t) \qquad x' = (\vec{x}, t'):$$

$$< g^{44}(t,\vec{x})\, g^{44}(t',\vec{x}) > \cong \frac{\varkappa^2 \Omega\, \omega(t'-t)\delta\rho^2}{R^2} - \qquad (15)$$

Here $\Omega = \frac{4\pi}{3} R^3$ is the volume of the medium, $\omega(t'-t)$ is the correlation function, which is equal, at $t = t'$, to the fluctuation volume (λ^3), $\delta\rho$ is the amplitude of the medium density fluctuation. As is seen from (15), the fluctuations of the metric tensor are proportional to $\sim R^{1/2}\, \lambda^{3/2}$, where R are the linear dimensions of the medium, λ is the linear scale of turbulentness Correspondingly:

$$\Delta f^2_{AB} \cong \frac{\varkappa^2 \Omega\, \omega \tau\ \delta\rho^2}{R^2} t \cong \varkappa^2 \omega \tau \cdot \delta\rho^2 \cdot R \cdot t \qquad (16)$$

where τ is the time scale of turbulentness.

4. AN ESTIMATION OF METRIC FLUCTUATIONS IN THE MICROWORLD

Let us now evaluate the correlation between the quantities g^{44} at the points x and x' which is due to the oscillations of the scalar field μ, with the nonzero rest mass.

In this case the Lagrange function is

$$\mathcal{L} = \frac{1}{2}\sqrt{g}\left(g^{\alpha\beta} \frac{\partial \Psi}{\partial x_\alpha} \frac{\partial \Psi}{\gamma x_\beta} - \mu^2 \Psi^2 \right) \qquad (17)$$

and the tensor of matter equals

$$T^{\mu\nu}(x) = g_0^{\mu\alpha} g_0^{\nu\beta} \frac{\partial \Psi}{\partial x_\alpha} \frac{\partial \Psi}{\partial x_\beta} - \delta^{\mu\nu} \cdot \mathcal{L} \qquad (18)$$

Due to the nature of the vacuum the quantities $g_0^{\mu\nu}$ have now the Galilean values. It is not the tensor $T^{\mu\nu}$ but only its fluctuations $\delta T^{\mu\nu}$ we are interested in. In order to obtain $\delta T^{\mu\nu}$ from (18) it is sufficient to mean by $\frac{\partial \Psi}{\partial x_\alpha} \frac{\partial \Psi}{\partial x_\beta}$, Ψ^2 etc the normal products of these operators. Therefore, according to (5) and (18), we get

$$t^{44}(x) = \frac{1}{2} g_0^{\alpha\beta} \frac{\partial \Psi}{\partial x_\alpha} \frac{\partial \Psi}{\partial x_\beta} - \mu^2 \Psi^2 \qquad (19)$$

here the products of the operators are considered already normal.

Expandiug, as usual, the field χ into a Fourier series

$$\chi = \frac{1}{V^{1/2}} \sum_{K} \sum_{K'} \left(\frac{\hbar}{2\omega_K}\right)^{1/2} \left(a_K e^{i(K,x)} + a_K^+ e^{-i(K,x)}\right) \qquad (20)$$

where V is the normalized volume, $\omega_K = c\sqrt{K^2+\mu^2}$, $K=(\vec{K},\omega)$, a_K, a_K^+ are the annihilation and production operators of the field particles. A substitution of (20) into (19) yields

$$t^{44}(x) = \frac{1}{V}\sum_{K}\sum_{K'}\left(\frac{\hbar^2}{2\omega_K\omega_{K'}}\right)^{1/2}\left\{A_{KK'}(x)a_K a_{K'} + B_{KK'}(x)a_K^+ a_{K'} + B^*_{KK'}(x)a^+_{K'}a_K + A^*_{KK'}(x)a^+_K a^+_{K'}\right\} \qquad (21)$$

and

$$A_{KK'}(x) = -\frac{1}{2}\left[2\mu^2 + (K,K')\right]e^{i(K+K',x)} \qquad (22)$$

$$B_{KK'}(x) = -\frac{1}{2}\left[2\mu^2 - (K,K')\right]e^{-i(K-K',x)} \qquad (22')$$

From (21) and (7) and by averaging over the vacuum, we find

$$\frac{1}{2}\langle g_{44}(x)g_{44}(x') + g_{44}(x')g_{44}(x)\rangle =$$

$$= \hbar^2\varkappa^2 \iint \frac{d^3K\, d^3K'}{\omega_K\,\omega_{K'}}\left[\frac{2\mu^2+(K,K')}{(K+K',K+K')}\right]^2 \cos(K+K',x-x') \qquad (23)$$

For $x=(\vec{x},ct)$, $x'=(\vec{x},ct')$, $t'-t=T$

we get

$$\frac{1}{2}\langle g_{44}(x)g_{44}(x') + g_{44}(x')g_{44}(x)\rangle =$$

$$= \frac{2\pi\hbar^2\varkappa^2}{c^2}\int_0^K\!\!\int_0^K \frac{K^2dK\,K'^2dK'}{\omega\,\omega'}\left\{\frac{1}{2} + \frac{\mu^2}{2KK'}\lg\frac{\mu^2+\omega\omega'+KK'}{\mu^2+\omega\omega'-KK'} + \frac{\mu^2}{\mu^2+K^2+K'^2}\right\}\cos(\omega+\omega',T) \qquad (24)$$

This integral is divergent at the upper limit. by $\mathcal{K}\to\infty$

If the rest mass of the field particles is zero ($\mu=0$), then for $\mathcal{K}cT \gg 1$ the integral in (16) is tending to zero like $1/T^2$, whereas for small times $\mathcal{K}cT \ll 1$ it behaves like $\mathcal{K}^4$ viz:

$$\frac{1}{2}\langle g_{44}(t)\,g_{44}(t')+g_{44}(t')\,g_{44}(t)\rangle = \frac{2\pi\hbar^2\varkappa^2}{c^2}\,\frac{K^4}{8}\,, \quad K c T \ll 1$$
$$= \frac{2\pi\hbar^2\varkappa^2}{c^2}\,\frac{K^2}{c^2T^2}\cos 2K_cT, \quad K_cT \gg 1 \qquad (25)$$

It is seen from here, that the metric fluctuations become essential, if $cT \ll \frac{1}{K} = L_0$ and the scale L_0 is determined by the formula

$$L_0 = \left(\frac{\hbar\varkappa}{c}\right)^{1/2} = 0{,}82\cdot 10^{-32}\ \text{cm.} \qquad (26)$$

This scale is much larger than the gravitational radii of particles $L_g = \varkappa\mu$ (μ is the mass of particles),which are usually treated as characteristic dimensions of that region of space in which the gravitational effects in the microworld could be essential. However, it is still considerably smaller even than those small scales which are characteristic of weak interactions ($\sim 10^{-16}$ cm).

Note, that the mass of the field particles is of no importance for the metric fluctuations until the Compton length of the particle $L_c = \frac{\hbar}{\mu c}$ is longer than its gravitational radius L_g since $L_0 = (L_g L_c)^{1/2}$, then the condition $L_c > L_g$ is equivalent to the condition $L_0 > L_g$.

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