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ON THE MINIMUM NUMBER OF PARTIAL WAVES IN TWO BODY COLLISIONS

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A B S T R A C T

It has been shown how to determine reliably the minimum number of partial waves involved in the interaction if the total elastic cross section and the differential cross section at a given angle are known.

Sec. I. INTRODUCTION

A great number of partial waves are participating in the interaction between two particles at high energies and it is practically impossible to make a complete phase shift analysis. Therefore, it is important to clear up what undoubted information may be obtained from the available experimental data. In particular, it is of interest to determine the least number of the partial waves L_{min}, necessary for the discription of the experimental data (when L_{min} is large enough, it corresponds to the minimum interaction radius).

It has been shown in the paper by Rarita and Schwed/1/ how to determine L_{min} for elastic scattering if the total interaction cross sections are known.

In this paper the inequalities have been proved for the minimum number of partial waves in the two-particle reactions if the total elastic cross section and the differential cross section are known at one or two angles. It has been shown that the account of spins of the interacting particles changes the results unessentially compared with the spinless case if L_{min} is great. Therefore, the inequality (A) may be used practically always.

The inequality (A) is much stronger than that of Rarita – Schwed/1/ if the scattering amplitude has a real part or is spin dependent. It may be also used for describing inelastic reactions of the type $a + b \to c + d$.

Sec. II. INTERACTION BETWEEN SPINLESS PARTICLES

Consider first the case when the particles have no spin. Then

$$\sigma(\vartheta_1) = \frac{1}{4K^2}\left|\sum_{\ell=0}^{\infty}(2\ell+1)A_\ell P_\ell(\cos\vartheta_1)\right|^2. \tag{1}$$

$$\sigma_{el} = \int\sigma(\vartheta)\,d\Omega = \frac{\pi}{K^2}\sum_{\ell=0}^{\infty}(2\ell+1)|A_\ell|^2.$$

Suppose that in these Equations it is possible to restrict oneself to the finite number of partial waves L. Then, making use of Cauchy inequality or defining the conditional minimum σ_{el}, provided $\sigma(\vartheta_1)$ is given, one may prove (see Appendix) that:

$$\Sigma_0 \geqslant \frac{4\pi\,\sigma(\vartheta_1)}{\sigma_{el}}, \tag{A}$$

where

$$\Sigma_0 = \sum_{\ell=0}^{L} (2\ell+1)\left[P_\ell(\cos\vartheta_1)\right]^2 = \frac{(L+1)^2}{\sin^2\vartheta_1}\left\{\left[P_L(\cos\vartheta_1)\right]^2 + \left[P_{L+1}(\cos\vartheta_1)\right]^2 - 2\cos\vartheta_1 P_L(\cos\vartheta_1)P_{L+1}(\cos\vartheta_1)\right\} \quad (2)$$

In the lefthand side of (A) there is a monotonously increasing function L. The inequality will be fulfilled only if L are larger than L_{min}. It is this number L_{min} which is the least one of the partial waves necessary for a simultaneous description of the given σ_{el} and $\sigma(\vartheta_1)$.

For $\vartheta = 0$ the inequality (A) for elastic scattering is stronger than that obtained in /1/. Indeed, in this case (A) is written as

$$(L+1)^2 \geqslant \frac{4\pi\sigma(0)}{\sigma_{el}} = \frac{K^2\sigma_t^2}{4\pi\sigma_{el}} + \frac{4\pi}{\sigma_{el}}\left[Re f(0)\right]^2, \quad (A')$$

while Rarita – Schwed inequality is of the form

$$(L+1)^2 \geqslant \frac{K^2\sigma_t^2}{4\pi\sigma_{el}}. \quad (3)$$

The latter is likely to yield smaller L_{min} than the inequality (A) if the real part of the scattering amplitude is not zero.

When σ_{el}, $\sigma(\vartheta_1)$ and $\sigma(\vartheta_2)$ are given, it is possible to obtain in a similar manner a stronger inequality than (A). Its form is rather complicated

$$\rho_{11}\rho_{22} - \rho_{12}^2 \geqslant \frac{1}{\sigma_{el}}\left\{\rho_{11}\sigma(\vartheta_2) - 2|\rho_{12}|\sqrt{\sigma(\vartheta_1)\sigma(\vartheta_2)} + \rho_{22}\sigma(\vartheta_1)\right\}. \quad (4)$$

where

$$\rho_{ik} = \frac{1}{4\pi}\sum_0^L (2\ell+1)P_\ell(\cos\vartheta_i)P_\ell(\cos\vartheta_k) = \frac{L+1}{\cos\vartheta_i - \cos\vartheta_k}\left[P_{L+1}(\cos\vartheta_i)P_L(\cos\vartheta_k) - P_L(\cos\vartheta_i)P_{L+1}(\cos\vartheta_k)\right]. \quad (5)$$

One may also derive the inequalities which make use of a greater number of experimental points. Of course, they will become even more complicated, but will yield a more precise estimate for L_{min} since any additional information may only increase L_{min}. Inequality (4) may be reversed, and it is possible to determine the largest value for the differential cross section at a certain angle if the total elastic cross section and the differential cross section at another angle ϑ_1 are known. So,

$$\sigma(\vartheta_2) \leqslant \frac{\sigma(\vartheta_1)}{\rho_{11}^2}\left\{|\rho_{12}| + \sqrt{\left(\frac{\rho_{11}\sigma_{el}}{\sigma(\vartheta_1)} - 1\right)\left(\rho_{11}\rho_{22} - \rho_{12}^2\right)}\right\}^2 \quad (6)$$

The upper limit for $\sigma(\vartheta_2)$ thus found depends upon the number of partial waves L which are taken into account in sums (5). If $L = L_{min}$, (6) passes into an equality. With increasing L the righthand side of the inequality under consideration becomes larger. For practical estimates, one may obtain definite indications concerning the upper limit $\sigma(\vartheta_2)$ by confining oneself to the reasonable number of L.

Sec. III. INTERACTION BETWEEN SPIN PARTICLES

In this Section we will be concerned with the inequalities employed for determining the minimum number of partial waves- L_{min} if σ_{el} and $\sigma(\vartheta_1)$ are known for particles with spins.

a) It has been proved in the Appendix that for particles with spin ½ and 0, the inequality reads:

$$\mathcal{M}ax\,(\Sigma_0,\Sigma_1) \geqslant \frac{4\pi\,\sigma(\vartheta_1)}{\sigma_{el}} \tag{7}$$

where

$$\Sigma_1 = \sum_{\ell=1}^{L}(2\ell+1)\frac{(\ell-1)!}{(\ell+1)!}\left[P_\ell^{(1)}(\cos\vartheta_1)\right]^2. \tag{8}$$

b) Analogously for nonidentical particles with spins ½ , the inequality may be easily derived

$$\mathcal{M}ap\,(\Sigma_0,\Sigma_1,\Sigma_2) \geqslant \frac{4\pi\,\sigma(\vartheta_1)}{\sigma_{el}} \tag{9}$$

where

$$\Sigma_2 = \sum_{\ell=2}^{L}(2\ell+1)\frac{(\ell-2)!}{(\ell+2)!}\left[P_\ell^{(2)}(\cos\vartheta_1)\right]^2. \tag{10}$$

c) The account of the identity of the Dirac particles leads to a more complicated inequality:

$$\mathcal{M}ax\left\{\Sigma_0',\Sigma_0'',\Sigma_1',\Sigma_1'',\Sigma_2',\Sigma_2''\right\} \geqslant \frac{4\pi\,\sigma(\vartheta_1)}{\sigma_{el}}, \tag{11}$$

where

$$\left.\begin{aligned}\Sigma_m' &= \sum_{\ell=m}^{L}\left[1+(-1)^\ell\right](2\ell+1)\frac{(\ell-m)!}{(\ell+m)!}\left[P_\ell^{(m)}(\cos\vartheta_1)\right]^2,\\ \Sigma_m'' &= \sum_{\ell=m}^{L}\left[1-(-1)^\ell\right](2\ell+1)\frac{(\ell-m)!}{(\ell+m)!}\left[P_\ell^{(m)}(\cos\vartheta_1)\right]^2\end{aligned}\right\} \tag{12}$$

To determine L_{min}, the largest of the sums should be substituted into the lefthand side of inequalities (7), (9), and (11). It is worth while noting that at small angles, Σ_0 is the largest of the sums as Σ_1 , and Σ_2 contains the associated Legendre polynomials. Further, since at high energies L_{min} is sufficiently great, then $\Sigma_m' \approx \Sigma_m'' \approx \Sigma_m$. Therefore, for high energies, the inequality (A) may be used practically for all cases.

It should be emphasized that if we restrict ourselves by the minimum number of partial waves determined from the above inequalities, then the spin-flip parts of the corresponding scattering amplitudes are equal to zero. If they are different from zero, then $L > L_{min}$.

If the polarization is taken into account, it is possible to obtain stronger conditions for determining L_{min}. In this case there is a possibility of estimating the contribution of the spin dependent parts of the scattering amplitude to the cross section.

Sec. IV. APPLICATIONS TO P–P SCATTERING AT 8.5 BEV

So, we came to the conclusion that in order to determine the minimum number of partial waves involved in two -particle reactions, at high energies the inequality (A) should be always used.

If the scattering amplitude has a real part or is spin-dependent, the inequality (A) is stronger than that of Rarita–Schwed[1].

Let us consider, for example, the proton-proton scattering at $E_p = 8.5$ BeV. In this case, according to the data of Tzyganov et al[2]

$$\sigma_{el} = (8,6 \pm 0,8)\ mb. , \quad \sigma(2,5^\circ - 5,5^\circ) = (123 \pm 18)\ \frac{mb}{sterad}.$$

From inequality (A) we find

$$L_{min} = 16 \pm 3 . \qquad (13)$$

(The errors in L_{min} are due to experimental ones). Rarita– Schwed inequality leads to a weaker estimate

$$L_{min} = 8 \pm 1 \qquad (14)$$

if $\sigma_{tot} = (30 \pm 3)\ mb.$

The optical model describing these experimental data[2] predicted the effective number of partial waves to be 16 ± 1, and the corresponding interaction radius R equal to

$$(1,6 \pm 0,1) \cdot 10^{-13} \text{ cm.}$$

It follows from our results that any other model will lead to the same or greater interaction radius.

It should be noted in conclusion that all the results obtained are applicable not only to elastic scattering but also to inelastic two-particle collisions, e.g. $\pi^- + p \to \Sigma^- + K^+$ etc. In this case σ_{el} ought to be substituted for the total cross section for the given reaction.

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APPENDIX

Below is given a proof of inequality (7) for the collisions of particles with spins 0 and ½.

It is convenient to write the expressions for $\sigma(\vartheta_1)$ and σ_{el} as

$$\sigma(\vartheta_1) = \frac{1}{4k^2}\left\{\left|\sum_{\ell=0}^{L} x_\ell a_\ell\right|^2 + \left|\sum_{\ell=1}^{L} y_\ell b_\ell\right|^2\right\}. \qquad (16)$$

$$\sigma_{el} = \frac{\pi}{k^2}\left\{\sum_{\ell=0}^{L} |a_\ell|^2 + \sum_{\ell=1}^{L} |b_\ell|^2\right\}.$$

where

$$a_\ell = \frac{1}{\sqrt{2\ell+1}}\left\{(\ell+1)\left(e^{2i\delta_+}-1\right)+\ell\left(e^{2i\delta_-}-1\right)\right\}$$

$$b_\ell = \sqrt{\frac{\ell(\ell+1)}{2\ell+1}}\left\{e^{2i\delta_+} - e^{2i\delta_-}\right\}.$$

$$x_\ell = \sqrt{2\ell+1}\,P_\ell(\cos\vartheta_1), \quad y_\ell = \sqrt{\frac{2\ell+1}{\ell(\ell+1)}}\,P_\ell^{(1)}(\cos\vartheta_1).$$

Now we make use of the well-known Cauchy inequality

$$\left| \sum A_i B_i \right|^2 \leq \left(\sum |A_i|^2 \right) \cdot \left(\sum |B_i|^2 \right) \qquad (18)$$

Then

$$\sigma(\vartheta_1) \leq \frac{1}{4k^2} \left\{ \sum x_\ell^2 \cdot \sum |a_\ell|^2 + \sum y_\ell^2 \sum |b_\ell|^2 \right\} \leq \frac{\sigma_{el}}{4\pi} \, Max \left\{ \sum x_\ell^2 , \sum y_\ell^2 \right\}. \qquad (19)$$

Thus, inequality (7) is proved.

The inequalities (A), (9), and (11) are being derived just in a similar manner. A method described here is much simpler, especially for particles with spin, than that consisting in determining the conditional extremum σ_{el} for the given $\sigma(\vartheta_1)$.

REFERENCES

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