ОБЪЕДИНЕННЫЙ
ИНСТИТУТ
ЯДЕРНЫХ
ИССЛЕДОВАНИЙ

Дубна

D- 1929

V.G.Kadyshevsky, R.M.Muradyan
A.N.Tavkhelidze, I.T.Todorov

SU(6)- Symmetry and Its Possible Generalizations

ЛАБОРАТОРИЯ ТЕОРЕТИЧЕСКОЙ ФИЗИКИ

1965

D- 1929

V.G.Kadyshevsky, R.M.Muradyan
A.N.Tavkhelidze, I.T.Todorov

SU(6)- Symmetry and Its Possible Generalizations

1.In a number of recent papers[1-7], the fruitfulness of combining nonrelativistic spin group $SU(2)$ and unitary spin group $SU(3)$ into a larger group $SU(6)$ was exhibited.

In this note we study the algebra of generators of $SU(6)$ and show that it allows a natural relativistic generalization.

2. The algebra of generators of $SU(6)$ may be constructed of those of $SU(2)$ and $SU(3)$ by a certain standard procedure to be described below.

As is well-known, the generators of the two-row representation of $SU(2)$, the Pauli spin matrices σ_i, do not form an algebra with respect to the usual matrix multiplication, because the matrices σ_i^2, equal to the unit matrix

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

cannot be written down as a linear combination of σ_i $(i = 1,2,3)$. If, however, the matrix σ_0 is added to the generators σ_i, then the linear combinations with real coefficients of this larger set of generators will form an associative algebra with respect to the usual product of matrices; at the same time they will generate the Lie algebra of $U(2)$ with respect to the commutation operation. A similar statement holds true for the lowest (n-dimensional) representation of $SU(n)$ for any $n \geq 2$. The corresponding associative algebra will be denoted by $\mathfrak{A}_n$. In particular, when $n = 3$, the algebra of $\mathfrak{A}_3$ is generated by the nine matrices of Gell-Mann[8] $\lambda_1, \ldots, \lambda_8$ and

$$\lambda_0 = \sqrt{\frac{2}{3}} E = \sqrt{\frac{2}{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which generate $U(3)$.

Consider the tensor product of the algebras which may be defined as an algebra of 6x6 matrices, the latter ones being the Kronecker products of the matrices from $\mathfrak{A}_2$ by the matrices from $\mathfrak{A}_3$. As a basis in the tensor product we choose the elements

$$\Lambda_i^{(a)} = \sigma_a \times \lambda_i \quad (a = 0,1,2,3;\ i = 0,1,\ldots,8) \tag{1}$$

The matrices $\Lambda_i^{(a)}$ are normalized so that

$$Sp\, \Lambda_i^{(a)} \Lambda_j^{(\beta)} = 4\,\delta_{ij}\,\delta_{a\beta}\,. \tag{2}$$

It is easily seen that the algebra generated by the elements $\Lambda_i^{(a)}$ coincides with $\mathfrak{A}_6$, so that the matrices (1) can be treated as generators of U (6). If we eliminate the unit matrix $\Lambda_0^{(0)}$ from the set of generators, we get the simple Lie algebra of generators of SU (6).

Let us emphasize that the passage made above to the associative algebras $\mathfrak{A}_2$ and $\mathfrak{A}_3$ is necessary since the "tensor product" of Lie algebras is, in general, not a Lie algebra since the expression for the commutator $[\Lambda_i^{(\alpha)}, \Lambda_j^{(\beta)}]$. incorporates not only the commutators of the matrices σ_α , σ_β and λ_i , λ_j, but also their anticommutators.

3. Since SU (6) is a group of rank 5, among the generators $\Lambda_i^{(\alpha)}$ ($i = \alpha = 0$ being excluded) there are 5 diagonal ones:

$$\Lambda_3^{(0)} = 2I_3 , \quad \Lambda_8^{(0)} = \sqrt{3}\, Y , \quad \Lambda_0^{(3)} = \sqrt{6}\, S_3 , \qquad (3)$$

$$\Lambda_3^{(3)} = 4\mu_3 - 2\nu_3 , \quad \Lambda_8^{(3)} = 2\sqrt{3}\, \nu_3 ,$$

where

$$Y = \sigma_0 \times \frac{\lambda_8}{\sqrt{3}}, \quad I_3 = \sigma_0 \times \tfrac{1}{2}\lambda_3 , \quad S_3 = \tfrac{1}{2}\sigma \times E \qquad (4)$$

are the operators of the hypercharge, of the third components of the isotopic and usual spins, while the operators μ_3 and ν_3 in the lowest representation considered are expressed quadratically in terms of the operators (4):

$$\mu_3 = \tfrac{1}{2}\sigma_3 \times \tfrac{1}{2} (\lambda_3 + \frac{1}{\sqrt{3}}\lambda_8) = (I_3 + \tfrac{1}{2}Y)S_3 \equiv QS_3 \qquad (5)$$

$$\nu_3 = \tfrac{1}{2}\sigma_3 \times \frac{1}{\sqrt{3}} \lambda_8 = YS_3 . \qquad (6)$$

According to (5), in the six-dimensional representation (for the "quark") the operator μ_3 is equal to the product of the electric charge operator by the third component of the spin. Hence, it is proportional to the magnetic moment of the quark. It is reasonable to postulate that in any other representation the operator of the third component of the magnetic moment is the generator μ_3 in this representation. Similarly, if there existed a hyperphoton field (see /9/), then ν_3 could be interpreted as a "hypermagnetic" moment.

The definition of the magnetic moment operator, we have given, corresponds physically to the model of vector addition of magnetic moments of quarks /6,10,11/ x). It should be therefore regarded, from the very beginning, as an approximate definition: the contribution of the orbital angular momentum of the quark system

x) The calculations of the magnetic moments in the framework of this model have been made by B.V. Struminsky.

is not taken into account. This approximation, however, agrees unexpectedly well with experiment. The magnetic moments of all 18 baryons (with spin-parity $1/2^+$ and $3/2^+$) entering the 56-dimensional representation D (3,0,0,0,0)x) of SU (6), as well as the magnetic moments of the transitions between them are expressed, according to /6,7/, in terms of only one constant. In particular, the ratio of the magnetic moments for the neutron and the proton is found to be $-2/3$ (experimentally $\frac{\mu_n}{\mu_p} = -0.68$).

In order to obtain this result from the point of view of our definition of magnetic moment it suffices to construct the wave functions of the proton and the neutron in terms of the components Ψ^{ABC} of a 56-component tensor (here $A = (a,\alpha)$, $B = (b,\beta)$, ..., $a = 1,2,3$ being the index in the unitary space, and $\alpha = 1,2$ - the spin index). Having in view that the wave function of the proton (with a maximum spin projection $s_3 = \tfrac{1}{2}$) is an eigenfunction of the operators Y , Q , S_3 and $S^2 = S_1^2 + S_2^2 + S_3^2$ xx) with eigenvalues 1, 1, 1/2, 3/4, respectively, we obtain

$$\Psi_p = \sqrt{\frac{2}{3}}\, \Psi^{11,21,22} - \frac{1}{\sqrt{3}}\, \Psi^{11,21,12} . \qquad (7)$$

Likewise, the wave function of the neutron is given by

$$\Psi_n = \sqrt{\frac{2}{3}}\, \Psi^{21,21,12} - \frac{1}{\sqrt{3}}\, \Psi^{11,21,22} . \qquad (8)$$

After this, the proton and neutron magnetic moments may be found easily if we note that in the representation D (3, 0, 0, 0, 0) the functions Ψ^{ABC} are eigenfunctions of μ_3 with eigenvalues $\tfrac{1}{2}[1 - p(12) + p(13) - \frac{2}{3}(p(21) + p(31))]$, where $p(a\alpha)$ is the number of indices $(a\alpha)$ among the indices A , B and C .

In a similar manner it is easy to get all the other relations between the magnetic moments of the 56-plet given in /67/ ($\mu_{\Omega^-} = -\mu_p = 3\mu_\Lambda$ etc).

Besides, calculating the matrix elements of the generators μ_3 in the adjoint representation D (1,0,0,0,1) (of dimension 35) we may express by means of μ_3 the magnetic moments of vector-mesons and the magnetic moments of the transitions between the vector and pseudoscalar mesons (e.g., $\mu_{\rho^+} = \mu_{K^{*+}} = \frac{1}{3}\mu_p = \frac{1}{3} < \pi^+ | \mu_3 | \rho^+ >$).

4. The use of SU (6) corresponds to a neglect of the spin-orbital interaction

x) We enumerate the irreducible representations of SU (6) by five non-negative integers $\lambda_1, \dots \lambda_5$, λ_i being the number of columns of i cells in the corresponding Young's pattern.

xx) The operators S_i are determined in the lowest approximation by the matrices $\tfrac{1}{2}\sigma_i \times E$.

and, in general, of the higher orbital angular momenta. We will show that the above scheme can be, in principle, generalized to the case when the quarks possess an arbitrary orbital angular momentum $\ell = 0,1,2,\ldots$.

We will assume as before that the quark spin is 1/2. Then the total angular momentum of the quark will take two values $j = \ell \pm ½$. Now we extend the Lie algebra formed by the generators j_1, j_2, j_3 to the least associative algebra which we will denote by $\mathfrak{R}$. The algebra $\mathfrak{R}$ just as the initial Lie algebra is reducible and may be decomposed into a direct sum of irreducible algebras $\mathfrak{A}_{2k}$:

$$\mathfrak{R} = \mathfrak{A}_2 + \mathfrak{A}_2 + \mathfrak{A}_4 + \mathfrak{A}_4 + \ldots + \mathfrak{A}_{2\ell} + \mathfrak{A}_{2\ell} + \ldots \qquad (9)$$

where to each component $\mathfrak{A}_{2k}$ there corresponds the value $j = k - ½$ and a definite value of ℓ equal to $k-1$ or to k. That the irreducible components of $\mathfrak{R}$ are just the algebras $\mathfrak{A}_{2j+1}$ follows from a well-known theorem about finite dimensional factors[x)].

We choose as a basis in $\mathfrak{R}$ a complete set of Hermitian orthonormalized matrices e_a :

$$e_a^+ = e_a , \quad ½ \, Sp \, e_a e_\beta = \delta_{a\beta} \qquad (10)$$

Consider the real Lie algebra generated by the various products $e_a \times \lambda_i$. It can be easily seen, by virtue of (9), that this algebra is reducible and splits into a direct sum of the Lie algebras of groups $U(6k)$, $k = 1,1,2,2\ldots$. If from each of the component in this sum we exclude the unit matrix, we shall obtain the Lie algebra of the group

$$SU(6) \times SU(6) \times SU(12) \times SU(12) \times \ldots , \qquad (11)$$

which allows unified description of spin and orbital angular momentum, on the one hand, and of unitary spin, on the other. In particular, if in the quark model we assume that the main contribution is made by the quark states with total angular momentum $j = \frac{3}{2}$, in the product (11) we may confine ourselves to the factor $SU(12)$. (cf. /13/).

---

x) In our case the factor is defined as an irreducible algebra of matrices invariant under Hermitian conjugation. The theorem of factors may be stated, as follows: any finite dimensional factor is isomorphic with a certain algebra $\mathfrak{A}_r$. (For the theory of factors see, e.g. /12/).

5. At present the idea of unifying the internal symmetry group $SU(3)$ with the Poincare group P(i.e., inhomogeneous Lorentz group) is widely discussed.

Below we suggest a possible solution of this problem which is based on a generalization of our previous reasoning (point 2).

Let $M^{\mu\nu}$ and p^ν stand for the generators of the Poincaré group which satisfy the standard commutation relations

$$[p^\mu, p^\nu] = 0, \quad [M^{\lambda\mu}, p^\nu] = i(g^{\mu\nu} p^\lambda - g^{\lambda\nu} p^\mu)$$
$$[M^{\mu\nu}, M^{\rho\sigma}] = i(g^{\nu\rho} M^{\mu\sigma} - g^{\mu\rho} M^{\nu\sigma} + g^{\nu\sigma} M^{\rho\mu} - g^{\mu\sigma} M^{\rho\nu}) . \qquad (12)$$

If we start from the "physical" (unitary, infinite-dimensional) representation of the Poincaré group, a completion of the corresponding Lie algebra to an associative one with respect to the usual multiplication of generators (cf. p.2.) would require infinitely many new elements. This is so cumbersome that it is hardly of practical interest. Therefore, we will base upon the four-dimensional representation of the Poincaré group in which the generators $M^{\mu\nu}$ and p^ν are described by the matrices[x)]

$$M^{\mu\nu} = ½\sigma^{\mu\nu} = \frac{i}{4}(\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu),$$
$$p^\nu = \kappa \, ½(1 + i\gamma^5)\gamma^\nu , \qquad (13)$$

where $\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}$, and κ is a real constant. It can be easily seen that to extend the set of generators (13) to make it a basis of an associative algebra it is necessary to add two more elements I and $\gamma^5 = \gamma^0 \gamma^1 \gamma^2 \gamma^3$.

Consider now the real Lie algebra, generated by the products

$$\Lambda_i^{(\mu)} = p^\mu \times \lambda_i , \quad \Lambda_i^{(\mu\nu)} = M^{\mu\nu} \times \lambda_i , \quad \Lambda_i^{(5)} = ½\gamma^5 \times \lambda_i ,$$
$$\Lambda_i = ½ I \times \lambda_i . \qquad (14)$$

To this algebra there corresponds a 108-parametrical group G unifying the Poincaré group with that of internal $SU(3)$ symmetry. The group G is seen to have the structure of a semi-direct product

$$G = GL(6,C) \, A_{36} , \qquad (15)$$

where A_{36} is a 36-parametrical invariant Abelian subgroup generated by $\Lambda_i^{(\mu)}$, while the general linear group in six dimensions $GL(6,C)$ provides a

---

x) A realisation of the commutation relations (12) by 4x4 matrices, equivalent to (13) was considered by Yu.M.Shirokov /17/.

synthesis of the spinor (four-row) representation of the Lorentz group and of the three-row representation of $U(3)$.[x)] On the other hand, $GL(6,C)$ splits into a direct product of a two-parametrical gauge group by the special linear group $SL(6,C)$[x)] (of 6x6 matrices with determinant 1)

$$GL(6,C) = e^{i(\alpha+i\beta)} \times SL(6,C). \qquad (16)$$

Eliminating the one-parametrical subgroup of gauge transformations $e^{i\alpha}$ from the group G i.e., omitting in (14) the unit matrix, we are led to a 107-parametrical group G_0 which is the product of the invariant Abelian subgroup A_{36} by a certain non-compact 71 parametrical group.

We are able to obtain a secondly quantized realization of the algebra of generators of G if we average (14) over the field operators of the quark $\Psi(x)$ taken at a fixed moment. In particular, to the generators of the commutative subgroup A_{36} there will correspond the operators

$$\tfrac{1}{2}\kappa \int \overset{+}{\Psi}(x)\, \gamma^{\mu} (1 - i\gamma^{5}) \times \lambda_j\, \Psi(x)\, d^3 x . \qquad (17)$$

Let us note that the choice of G (or G_0) as a fundamental group which contains P and $SU(3)$ as subgroups is, to some extent arbitrary. Starting, for instance, from the five dimensional representation of the Poincare group (instead of the four-dimensional one used in the text) we would obtain a larger group containing P and $SU(3)$ as subgroups and having a similar structure.

The right choice should be based on a study of the unitary (infinite-dimensional) representations of the group G and on their physical interpretation. This problem as well as that concerning the role of discrete transformations (spatial and time reflections) will be investigated elsewhere.

The authors take the pleasure to express their deep gratitude to N.N.Bogolubov for stimulating discussions and constant interest in the work, to A.A.Logunov for fruitful discussions, and to A.S.Dynin for useful remarks. The authors are indebted to the participants of the seminar of the Theoretical Physics Laboratory (Joint Institute for Nuclear Research) for the discussion of the paper.

x) This group is introduced for other reasons in /18/, where it is identified with the group $U(6) \times U(6)$. In fact one can state that the complex extensions of the Lie algebras of the groups $U(6) \times U(6)$ and $GL(6,C)$ coincide.

References

1. F.Gürsey and L.A.Radicati. Phys. Rev. Lett., 13, 173 (1964).
2. A.Pais. Phys. Rev. Lett., 13, 175 (1964).
3. F.Gürsey, A.Pais and L.Radicati Phys. Rev. Lett., 13, 299 (1964).
4. T.K.Kuo and T.Yao. Phys. Rev. Lett., 13, 415 (1964).
5. M.A.Bég and V.Singh. Phys. Rev. Lett., 13, 418 (1964); 13, 509 (1964).
6. M.A.Bég, B.W.Lee and A.Pais. Phys. Rev. Lett., 13, 514 (1964).
7. B.Sakita. Phys. Rev. Lett., 13, 643 (1964).
8. M.Gell-Mann. Phys. Rev., 125, 1067 (1962); Physics, 1, 63 (1964).
9. J.J.Sakurai. Ann. of Phys., 11, 1 (1960).
10. M.A.Bég and A.Pais. Preprint, New York, 1964.
11. M.Gell-Mann. Preprint, Pasadena, 1964.
12. М.А. Наймарк, Нормированные кольца. Москва, ГИТТЛ, 1956, гл. 7.
13. P.Freund and B.Lee. Phys. Rev. Lett., 13, 592 (1964).
14. W.D.McGlinn. Phys. Rev. Lett., 12, 467 (1964).
15. M.E.Maer, H.J.Schnitzer, E.C.G.Sudarshan, R.Arharya and M.Y.Han. Phys. Rev., 136B, 888 (1964).
16. U.Ottoson, A.Kiheber and J.Nilsson. Preprint, Gothenburg, 1964.
17. Ю.М. Широков. Теоретико-групповые методы в релятивистской квантовой теории Диссертация. Физический факультет МГУ, стр. 75. Москва, 1959.
18. R.P.Feynman, M.Gell-Man and G.Zweig. Phys. Rev. Lett., 13, 678 (1964).

Note added in proof. After the appearance of this preprint in Russian, Dr.Suranyi has acquainted the authors with a paper of T.Fulton and J.Wess "Symmetry Group Containing Lorentz Invariance and Unitary Spin". Preprint. Vienna (1964), in which similar results are obtained.

Another approach to the relativistic extension of the group $SU(6)$ is given in the recent preprint of R.Delbourgo, A.Salam, J.Strathdee "U(12) and Broken SU(6) Symmetry," Trieste, 1964, in which the tensor product of the algebra of all γ matrices by the algebra $\mathfrak{A}_3$ is considered.

Received by Publishing Department
on December 30, 1964.