ОБЪЕДИНЕННЫЙ ИНСТИТУТ ЯДЕРНЫХ ИССЛЕДОВАНИЙ

Лаборатория теоретической физики

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QUASIPOTENTIAL CHARACTER OF THE SCATTERING AMPLITUDE

Дубна 1963.

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Abstract

It is shown that in order to describe the bound states in quantum field theory it is possible to construct a local potential which is a superposition of Yukawa potentials with energy dependent intensities.

It has been shown/1/ that in the framework of quantum field theory a system consisting of two interacting particles can be described by an equation of the Schrödinger type with a generalized complex potential which is dependent on energy and particle momenta. Such an approach allows, on the one hand, to find the scattering amplitude* and, on the other, to study the structure of the bound states.

Apart from such an approach, in/1/ a scheme is developed for constructing a local potential to describe the scattering amplitude. This scheme can be briefly set forth as follows.

For the scattering amplitude in quantum field theory there exists an expansion in the coupling constant. It is quite obvious that considering only the partial sums of this expansion one cannot answer the questions on a possible asymptotic behaviour of the amplitude and on the character of the bound states and the resonances of the system. In order to solve these problems a method of summing up the perturbation theory graphs was treated in/1/. This approach enables us, to construct such a potential for the amplitude of the process, that Schrödinger equation with this potential would yield the ordinary expression for the scattering amplitude in any order of perturbation theory. The knowledge of the perturbation theory potential permits to establish a number of its general features, as we shall see later on.

Since we restricted ouselves to the problem of the accurate description of the scattering amplitude by the very construction of the potential, then the Schrödinger equation thus obtained makes it possible to find the energy spectrum of the bound states and resonances of the system.

It has been established in/1/ that the process of scattering as well as the energy spectrum of the bound states and resonances are described by the equation**

$$[E^2 - q^2 - m^2]\,\psi(q) - \frac{1}{\sqrt{q^2+m^2}} \int V(E,(q-p)^2)\,\psi(p)\,d^3p = 0, \qquad (1)$$

where the potential $V(E,(p-q)^2)$ is a function of the energy E and of the momentum transfer $(p-q)^2$. Hence, for the transition amplitude $T(q,q')$ we have an equation of Lippman-Schwinger type

$$T(q,q') = V(E+i\epsilon,(q-q')^2) + \int \frac{V(E+i\epsilon,(q-p)^2)\,T(p,q')}{[(E+i\epsilon)^2 - p^2 - m^2]\sqrt{p^2+m^2}}\,d^3p. \qquad (2)$$

On the mass shell

$$q^2 = q'^2 = E^2 - m^2$$

the function T coincides with the scattering amplitude M, where

$$S = 1 - i\frac{\pi}{E^2}\,\delta(E-E')\,M. \qquad (3)$$

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* We call the scattering amplitude a quasipotential one if its projections on the even and odd states in the variable $\cos\theta$ can be represented as superpositions of Fourier transforms of Yukawa potentials.

** Here and in what follows p and q denote the three-dimensional momenta.

Presenting the function T in the form

$$T(q,q') = V(E+i\epsilon, (q-q')^2) + \int V(E+i\epsilon, (q-p)^2)\, G(E+i\epsilon; p, k)\, V(E+i\epsilon, (q'-k)^2) \cdot d^3p\, d^3k \qquad (4)$$

we get an equation for the Green function $G(E; q, q')$

$$(E^2 - q^2 - m^2)\sqrt{q^2+m^2}\, G(E; q,q') - \int V(E, (q-p)^2)\, G(E, p, q')\, d^3p = \delta(q-q'). \qquad (5)$$

Before investigating the properties of Eq. (2) we make a remark: The dispersion relation in S with one subtraction without taking the u-channel into account, has the form

$$M(s,t) = V(t) + 1/\pi \int_{4m^2}^{\infty} d\sigma \frac{Im\, M(\sigma, t)}{\sigma - s}. \qquad (6)$$

From the two-particle unitarity

$$i\{ M(p_\alpha, p_\beta) - \overset{+}{M}(p_\alpha, p_\beta) \} + \int M(p_\alpha, p)\, \overset{+}{M}(p, p_\beta)\, d^3p = 0 \qquad (7)$$

and assuming that $V(t)$ and $Im\, M(s,t)$ admit representations of the form

$$V(t) = 1/\pi \int_{\mu^2}^{\infty} \frac{U(\nu)}{\nu - t} d\nu, \quad Im\, M(s,t) = 1/\pi \int_{4\mu^2}^{\infty} \frac{\rho(s,\nu)}{\nu - t} d\nu \qquad (8)$$

it is possible to get /2/ a non-linear equation for the spectral function $\rho(s, \nu)$

$$\rho(\sigma,\nu) = -\frac{2\pi}{q\sqrt{\sigma}} \iint_{\mu^2}^{\infty} r(\sigma+i\epsilon, \nu_1)\, \overset{+}{r}(\sigma+i\epsilon, \nu_2)\, R(\sigma, \nu_1, \nu_2, \nu)\, d\nu_1\, d\nu_2, \qquad (9)$$

where

$$r(s,\nu) = U(\nu) + 1/\pi \int_{4\mu^2}^{\infty} \frac{\rho(\sigma,\nu)}{\sigma - s} d\sigma, \qquad (10)$$

and the function R can be expressed by

$$R(s, t_1, t_2, t) = \frac{1}{2q^2} K\left(1+\frac{t_1}{2q^2}, 1+\frac{t_2}{2q^2}, 1+\frac{t}{2q^2}\right), \qquad (11)$$

$$\frac{1}{2q^2} K\left(1+\frac{t_1}{2q^2}, 1+\frac{t_2}{2q^2}, 1+\frac{t}{2q^2}\right) = \frac{\theta\left(t_1^2+t_2^2+t^2-2(tt_1+tt_2+t_1t_2) - \frac{t_1t_2t}{q^2}\right)}{\left[t_1^2+t_2^2+t^2-2(tt_1+tt_2+t_1t_2) - \frac{t_1t_2t}{q^2}\right]^{1/2}}, \qquad (12)$$

$$s = 4(m^2+q^2).$$

It is easy to see that the solution of the linear equation (2) with the potential $V(t)$ of the form (8) can be reduced to the solution of the non-linear equation for the spectral function.

Indeed, as far as the potential $V(t)$ is real and its Fourier-transform is dependent on the distance only, the amplitude found from Eq. (2) satisfies the unitarity condition (7). It follows from the analysis of Eq. (2) with the potential (8) which is a superposition of Yukawa potentials, that the scattering amplitude has the representation (6), (8), the subtraction constant in (6) coinciding with the potential $V(t)$.

This shows that the investigation of the scattering amplitude with the aid of Eq. (2) is fairly effective since we are dealing here with a linear equation which is a generalization of the well-studied Schrödinger equation, while using the unitarity condition and dispersion relations we have a rather complicated non-linear system of integral equations.

We proceed now to the study of some properties of Eq. (2). In order to obtain the asymptotic behaviour of the scattering amplitude in the s-channel, Eq. (2) employed for discribing the bound states has to be used in the t-channel, where $t = 4E^2$, while s and u are the momentum transfers. To describe the direct and exchange interaction we introduce the potentials for the even and odd states respectively. Then Eq. (2) can be written for the even and odd states separately:

$$T^{\pm}(q, q') = V^{\pm}(E + i\epsilon, (q - q')^2) + \int \frac{V^{\pm}(E + i\epsilon, (q - p)^2)\, T^{\pm}(p, q')}{[(E + i\epsilon)^2 - p^2 - m^2]\sqrt{p^2 + m^2}}\, d^3p . \tag{13}$$

Let the potentials in some energy range have the form

$$V^{\pm}(E, (q - q')^2) = \int_{\mu^2}^{\infty} \frac{U^{\pm}(E, \nu)}{\nu + (q - q')^2}\, d\nu . \tag{14}$$

The solution of Eq. (2) for the even and odd states in the same energy range is

$$T^{\pm}(q, q') = \int_{\mu^2}^{\infty} \frac{r^{\pm}(q^2, q'^2, \nu; E)}{\nu + (q - q')^2}\, d\nu . \tag{15}$$

Let us check up this statement by resorting to perturbation theory. For this purpose we write down Eq. (2) in a symbolic form

$$T^{\pm} = V^{\pm} + V^{\pm} \times T^{\pm} , \tag{16}$$

and expand T and V in the coupling constant

$$\begin{aligned} T^{\pm} &= T_2^{\pm} + T_4^{\pm} + \dots \\ V^{\pm} &= V_2^{\pm} + V_4^{\pm} + \dots \end{aligned} \tag{17}$$

Substituting into Eq. (16), we get

$$T_{2n}^{\pm} = V_{2n}^{\pm} + \sum_{m=1}^{n-1} V_{2m}^{\pm} \times T_{2n-2m}^{\pm} . \tag{18}$$

For $n = 1$, $T_2 = V_2$; so, the statement is obvious in this case.

We show now that if the functions $T_2, \dots, T_{2n-2}$ can be represented in the form (15), then the function can be put in the same form, as well.

Let us consider the product

$$V_{2m} \times T_{2n-2m} = \int_{\mu^2}^{\infty} d\nu_1 \int_{\mu^2}^{\infty} d\nu_2 \int_{0}^{\infty} du \, \frac{r_{2n-2m}(u, q'^2, E, \nu_2)\, U_{2m}(E, \nu_1)}{[E^2 - m^2 - u]\sqrt{u + m^2}}\, I(\nu_1, \nu_2) , \tag{19}$$

where
$$I(\nu_1,\nu_2)=\int d^3p\,\frac{\delta(p^2-u)}{[\nu_1+(q-p)^2][\nu_2+(p-q')^2]}.$$

Since
$$I(\nu_1,\nu_2)=\frac{\pi}{2}\int\frac{K(q'^2,t',u,\nu_1,\nu_2,q^2)}{t'+(q-q')^2}\,dt',\tag{20}$$

where
$$K=\frac{\theta(\sqrt{t'}-\sqrt{\nu_1}-\sqrt{\nu_2})\,\theta(\Delta)}{\sqrt{\Delta}}$$

Δ is the well-known determinant /4/, then

$$V_{2m}\times T_{2n-2m}=\int_{\mu^2}^{\infty}\frac{d\nu}{\nu+(q-q')^2}\,\chi(q^2,q'^2,E,\nu),$$

with
$$\chi(q^2,q'^2,E,\nu)=\int_0^{\infty}du\int_{\mu^2}^{\infty}d\nu_1\int_{\mu^2}^{\infty}d\nu_2\,\frac{\tau_{2n-2m}(u,q'^2,E,\nu_2)\,U_{2m}(E,\nu_1)\,K(q'^2,\nu,u,\nu_1,\nu_2,q^2)}{[E^2-m^2-u]\sqrt{u+m^2}}.\tag{21}$$

It follows that the function T_{2n} can be represented in the form of Eq. (15). So, the representation for solving Eq.(2) in the form (15) is established.

Let us write the equation for the spectral function. With this aim, we substitute expressions (14) and (15) into (13). Using (20), we get:*

$$\tau^{\pm}(q'^2,q^2,\nu,E)=U^{\pm}(E,\nu)+\iint\frac{Q^{\pm}(q'^2,\nu,u',t',q^2,E)}{(E^2-m^2-u')\sqrt{u'+m^2}}\,\tau(q',u',t',E)\cdot du'\,dt',\tag{22}$$

where
$$Q^{\pm}(q'^2,\nu,u',t',q^2,E)=\tfrac{1}{2}\int K(q'^2,\nu,u',t',\nu,q^2)\,U^{\pm}(E,\nu)\,d\nu_1.\tag{23}$$

An equation of the type (22) for the potential scattering was first discussed in /4/, where the asymptotic behaviour of the scattering amplitude was studied. In order to find, with the aid of Eq. (1), the scattering amplitude and, hence, the spectrum of the bound states and resonances, we choose the potential so that the function T on the mass shell would coincide exactly with the scattering amplitude in any order of perturbation theory. This can be achieved if the potential is determined from the equations·

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* The kernal $Q^{\pm}$ is extended to the region of negative values of q^2 and q'^2. Such values appear when the amplitude of the prooess is extended to the region below the threshold.

$$V_2^{\pm} = [T_2^{\pm}], \quad V_4^{\pm} = [T_4^{\pm}] - [V_2^{\pm} \times T_2^{\pm}]$$

$$V_{2n}^{\pm} = [T_{2n}^{\pm}] - \sum_{m=1}^{n-1} [V_{2m}^{\pm} \times T_{2n-2m}^{\pm}] . \tag{24}$$

Here the square brackets designate the transition to the mass shell in the corresponding expressions*.

Now we show that in some energy region if the scattering amplitudes for the even and odd states $M^{\pm}$ can be represented as

$$M^{\pm}(E, (q-q')^2) = \int_{\mu^2}^{\infty} \frac{\sigma^{\pm}(E, \nu)}{\nu + (q-q')^2} d\nu , \tag{25}$$

then the potentials $V^{\pm}$ we have got from eqs. (24) assume in this energy range the form

$$V^{\pm}(E, (q-q')^2) = \frac{1}{\pi} \int_{\mu^2}^{\infty} \frac{U^{\pm}(E, \nu)}{\nu + (q-q')^2} d\nu . \tag{26}$$

Let us check this statement in perturbation theory. Since $[T_2]$ coincides with the scattering amplitude M_2 in the second order, then the above statement for V_2 is evident. In the fourth order

$$V_4 = M_4 - [M_2 \times M_2] .$$

Hence, by Eqs. (20) and (25) V_4 can be also put in the form of (26).

Similar arguments can be easily given in any order of perturbation theory. Thus, it is established in perturbation theory that if the scattering amplitude in a certain energy region can be written down in the form of (25), the potentials $V^{\pm}$ in this energy region are superpositions of Yukawa potentials with energy dependent intensities.

Let us make one more remark. When use is made of the information about the scattering amplitude from perturbation theory we get some subtraction constants which cannot be represented in the form of (25). In reconstructing the potential no account will be taken of these constants. Depending on the power n of the subtraction polynomial the potential $V(E, t)$ will describe correctly only the partial waves with $\ell > n$.

We will show now that in order to study in field theory the bound states in the t-channel it is possible to construct a local potential which is a superposition of Yukawa potentials with the energy dependent intensities

$$V^{\pm}(E, (q-q')^2) = \frac{1}{\pi} \int_{\mu^2}^{\infty} \frac{U^{\pm}(E, \nu)}{\nu + (q-q')^2} d\nu . \tag{27}$$

We state that such a representation follows from the principles of quantum field theory in the energy range

$$-4\mu^2 < t < 4\mu^2 . \tag{28}$$

The validity of the one-dimensional dispersion relations for the scattering amplitude was proved[5] in any order of perturbation theory both in the variable t for fixed value of s in the interval **

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* Upon performing this, the potential will be a fuction of only two independent variables E and t.

** The one-dimensional relations have been obtained[5] for different physical processes. Here we use the results only for equal masses.

$$-4m^2 < s < 4m^2 \tag{29}$$

and in the variable s for fixed t in the interval

$$-4\mu^2 < t < 4\mu^2 . \tag{30}$$

Therefore, for the scattering amplitude one can write the representation*:

$$M(s, t) = \int_{\mu^2}^{\infty} \frac{\sigma_1(s', t)}{s' - s} ds' + \int_{\mu^2}^{\infty} \frac{\sigma_2(u', t)}{u' - u} du' . \tag{31}$$

Here the variable t is in the region (30). Using (31) one can construct the functions $M^{\pm}$:

$$M^{\pm}(s, t) = \int_{\mu^2}^{\infty} \frac{\sigma^{\pm}(s', t)}{s' - s} ds' , \tag{32}$$

where

$$\sigma^{\pm}(s, t) = \sigma_1(s, t) \pm \sigma_2(s, t) .$$

The amplitude $M(s,t)$ is expressed in terms of these functions as

$$M(s, t) = ½ [M^+(s, t) + M^-(s, t) + M^+(u, t) - M^-(u, t)] . \tag{33}$$

Since the momentum transfer $s \leq 0$, then using the representation (32) and the previous statement (see (25) and (26)), the potentials $V^{\pm}$ can be written down in the form of (26). Therefore, their Fourier Ttransforms in the energy range (30) are

$$V^{\pm}(E, r) = 1/\pi \int_{\mu^2}^{\infty} \frac{e^{-\nu r}}{r} U^{\pm}(E, \nu)\, d\nu . \tag{34}$$

Thus, it is established that in quantum field theory in the framework of perturbation theory the amplitudes $M^{\pm}$ can be represented in the form of (32) in some energy region, i.e. they have a quasipotential character**.

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* We do not take into account the subtraction constants since, as was already pointed out, we paid no attention to them in constructing the potential. In perturbation theory the amplitude $M(s,t)$ has no singularities at $t < 4\mu^2$ except for the pole at $t = \mu^2$. The analytical structure of the amplitude obtained on the basis of Eq. (2) will contain poles corresponding to possible bound states in the t -channel. These poles are, in fact, the subtraction constants (in the general case a polynomial) in the s -channel which may affect the asymptotic behaviour of the scattering amplitude. In this case once more the close relation between the bound states and the asymptotic behaviour of the scattering amplitude is manifested.

** Note, that if the amplitude satisfies, for any real s the hypothetical dispersion relation in the momentum transfer, it will be quasipotential. The potentials corresponding to such an amplitude will be represented in the form of (34) for each value of s .

We see therefore that the hypothesis about the validity of the dispersion relations in the momentum transfer leads, for any value of the energie s , to a potential which is a superposition of Yukawa potentials with energy dependent intensities.

As we have seen above, the potentials $V^{\pm}$ reconstructed from this quasipotential scattering amplitude are a superposition of Yukawa potentials with energy dependent intensities.

It is worthwhile to note that in studying the scattering amplitude in the framework of perturbation theory we are not able to draw straighforward conclusions about the existence of resonances and bound states of the system. This does not mean that the system has no such states, but simply that perturbation theory method fails to solve these problem. So, the analytical structure of the scattering amplitude which is obtained from perturbation theory will have no singularities associated with the bound states of the system. The reconstruction of the potential according to perturbation theory and the subsequent investigation of the system by making use of the Schrödinger equation enable us, however, to study the energy spectrum of the bound states and resonances and the asymptotic behaviour of the scattering amplitude. Eq. (2) with the potential (34) was applied in /6/ to investigate the asymptotic behaviour of the scattering amplitude in the s -channel. Following the procedure developed by Fubini and Strofollini for Eq. (22) it has been shown there that the scattering amplitude has the Regge asymptotic behaviour for large s .

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References

1. A.A.Logunov, A.N.Tavkhelidze, preprint of ОИЯИ E-1145 (1962).
2. S.Mandelstam, preprint (1962), Regge Poles as Consequences of Analyticity and Unitarity.
3. R.Blankenbenler et al, Annals. of Phys., 10, 62-93 (1960).
4. S.Fubini, R.Stroffolini, Lectures, August 1962- Trieste.
5. A.A.Logunov, Lu-I-chen, I.T.Todorov, and N.A.Chernikov, preprint of ОИЯИ P-1043 (1962).
6. B.A.Arbuzov, A.A.Logunov et al, preprint of ОИЯИ E-1149 (1962).

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