ОБЪЕДИНЕННЫЙ ИНСТИТУТ ЯДЕРНЫХ ИССЛЕДОВАНИЙ

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ON DISPERSION RELATIONS IN QUANTUM ELECTRODYNAMICS

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Abstract

A method of writing dispersion relations in quantum electrodynamics is considered. The proof is carried out in the lowest orders of perturbation theory improved by means of the renormalization group.

1. Infrared singularities

Dispersion relations for the photon-electron scattering in the forward direction were written as long ago as 1954 /1/. However a further application of dispersion relations to quantum electrodynamics met a difficulty which is due to the infrared singularities. The physical meaning of this difficulty is that the amplitudes of the processes in which the charged particles and a finite number of photons are involved, equals zero, if the charged particles are scattered in the non-forward direction and differ from zero for the forward scattering. Therefore, e.g., the vertex function in electrodynamics is non-analytical one. The dependence of the scattering amplitudes on the momentum transfer is non-analytical as well.

The cross sections of processes with the infinite number of particles (soft photons) are non-zero. As for the dispersion relation method, it was developed up to the present only for the amplitudes of processes with a finite number of particles.

Meanwhile, we may also consider the amplitudes of the processes with a finite number of particles in electrodynamics if we use the formula of the factorization of the infrared divergences /2/ :

$$M_\lambda = e^{F_\lambda} M \quad , \tag{1}$$

where M is the matrix element calculated by introducing the mass $\sqrt{\lambda}$ into the photon propagator, and the function F_λ is of the form :

$$F_\lambda = -\sum_{i<j} z_i a_i z_j a_j F\left((p_i a_i + p_j a_j)^2 \right), \tag{2}$$

where the summation is performed over all the charged particles, z_i is the sign of the charge, and $a_i = 1$ or -1 for the outgoing or the incoming particle with the momentum p_i respectively. The function F is equal to*:

$$F((p'-p)^2) = \frac{i\alpha}{8\pi^3} \int \frac{dk}{k^2-\lambda} \left(\frac{2p'-k}{2p'k-k^2} - \frac{2p-k}{2pk-k^2} \right)^2 . \tag{3}$$

(α is the fine structure constant) and can be represented in the form :

$$F(t) = \frac{t}{\pi} \int_4^\infty \frac{Im\, F(t')\, dt'}{t'(t'-t-i\epsilon)} \quad , \tag{4}$$

$$Im\, F(t) = \frac{\alpha}{4} \sqrt{\frac{t-4}{t}} \left(\frac{2t-4+\lambda}{t-4} \ln \frac{t-4+\lambda}{\lambda} - 1 \right) \tag{5}$$

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* The system of units $\hbar = c =$ (the electron mass) $= 1$. The vector product $ab = a^0 b^0 - \vec{a}\,\vec{b}$.

Considering the coefficient of $\ln\lambda$ in F_λ we can show that its real part is positive in the physical region, it vanishes for the forward scattering (for electron-electron scattering also for backward scattering) and becomes negative in the unphysical region. If in the limit $\lambda \to 0$ the quantity M is finite, this means that the matrix element M_λ at $\lambda = 0$ vanishes in the physical region, is finite for forward scattering and is infinite in a part of the unphysical region. The latter property can be connected with existence of Coulomb bound states.

The assumption that the quantity M in (1) is finite at $\lambda \to 0$ is not strictly proved but it is very plausible [2]. In the following we shall consider the analytical properties of M at $\lambda = 0$ for some processes in the lowest orders of perturbation theory improved with the aid of the renormalization group.

2. Vertex function

For the vertex with three lines corresponding to two real charged particles and a virtual photon with the squared mass t, the quantity M in third order perturbation theory is an analytic function in the t - plane with the cut from 4 to infinity. The diagrams with intermediate photons occur starting with the seventh order. They have the cut from zero to infinity. Thus, for the vertex function the quantity M possesses ordinary normal analytical properties.

3. Compton effect

We consider the quantity M for the scattering of photons on electrons. We denote the squares of the total energies of the direct and crossed processes by s and u respectively, and the square of the momentum transfer by t.

In second order perturbation theory M contains the terms $M_s^{(2)}$ and $M_u^{(2)}$ which have poles at the points $s=1$ and $u=1$ respectively.

The diagrams of fourth order, besides other terms, give pole terms which depend on the additional electron magnetic moment μ'. We take these terms into account in $M_s^{(2)}$ and $M_u^{(2)}$, by replacing the γ matrices by $\gamma + \mu'\sigma^{nm}q_m$ (q is the momentum transfer). The other terms of fourth order give the following contribution:

$$M^{(4)} = M_s^{(2)}[\beta(t)\ln(1-s)+\gamma(t)] + M_{sa}^{(4)} + M_u^{(2)}[\beta(t)\ln(1-u)+\gamma(t)] + M_{ua}^{(4)} \tag{6}$$

$$\beta(t) = \frac{\alpha t}{\pi}\int_4^\infty \frac{t'-2}{\sqrt{t'(t'-4)}}\,\frac{dt'}{t'(t'-t-i\epsilon)} \tag{7}$$

$$\gamma(t) = -\frac{\alpha t}{2\pi}\int_4^\infty \left[\frac{(t'-2)\ln t'}{\sqrt{t'(t'-4)}} - \tfrac{1}{2}\sqrt{\frac{t'-4}{t'}}\,\right]\frac{dt'}{t'(t'-t-i\epsilon)} \,. \tag{8}$$

The quantities $M_{sa}^{(4)}$ and $M_{ua}^{(4)}$ (more exactly the coefficients of the spinor terms) are analytical functions of the variables s,t and u,t and have only branch points as singularities and satisfy a Mandelstam representation [3].

We see that in fourth order there are terms with the anomalous singularities $(1-s)^{-1} \ln(1-s)$ and $(1-u)^{-1} \ln(1-u)$.

By applying the equation of the renormalization group[4] with respect to s and choosing the integration constant from correspondence with perturbation theory (6) we get the whole amplitude near the point $s=1$ has a singularity of the form:

$$\exp\left[\beta(t)\ln(1-s) + \gamma(t)\right]\left(M_s^{(2)} + \ldots\right) \tag{9}$$

and a similar one near $u=1$. In (9) the terms of the perturbation series of $[\alpha \ln(1-s)]^n$ and α^2 order are summed up. It is reasonable to assume that the amplitude M is of the form:

$$M = M_s^{(2)} \exp\left[\beta(t)\ln(1-s) + \gamma(t)\right] + M_u^{(2)} \exp\left[\beta(t)\ln(1-u) + \gamma(t)\right] + M_a \tag{10}$$

where β and γ are the series, whose first terms are given by (7) and (8), in $M_{s,u}^{(2)}$ the additional magnetic moment is taken into account and

$$M_a = M_{sa}^{(4)} + M_{ua}^{(4)} + \ldots \tag{11}$$

is an analytical function with branch points satisfying the Mandelstam representation.

The real part of the coefficient $\beta(t)$ in (10) and (7) is negative in the physical region of the variable t ($t<0$, $t>4$) and positive at $0<t<4$. Therefore the quantity M near $s=1$ (and analogously near $u=1$) has a singularity of the form

$$(1-s)^{-1+\beta(t)}, \tag{12}$$

which, in the physical region of t, is stronger than a pole.

Finally, we consider one consequence of equation (10). Namely, we assume that quantity of the fourth order M_a may be neglected in the whole physical region of the energy. Then for M we have an asymptotic behaviour of the Regge type ($s^{\alpha(t)}$) where the power of s is $-1+\beta(t)$, which corresponds to electron positron bound states. The levels of these states in the non-relativistic limit turn into the Coulomb ones.

4. Electron-positron scattering

For the vertex function and the Compton effect, the quantity F_λ in (1) is simply equal to $F(t)$, while for electron-positron scattering it is of the form:

$$F_\lambda = 2F(s) - 2F(u) + 2F(t), \tag{13}$$

where the variables s, u and t have the same meaning as in the previous Section.

The quantity M for electron-positron scattering in second order perturbation theory contains the terms $M_s^{(2)}$ and $M_t^{(2)}$ being poles at $s=0$ and $t=0$ respectively. As before, we shall take into account the terms of higher orders in these poles, which depend on the additional magnetic moment.

The other terms of the fourth order give the following contribution

$$M^{(4)} = 2(\Phi(s,t) - \Phi(u,t)\ )\ M_t^{(2)} + M_{ta}^{(4)} + \tag{14}$$

$$+ 2\ (\ \Phi(t,s) - \Phi(\ u,s\ )\ )\ M_s^{(2)} + M_{sa}^{(4)}\ ,$$

where

$$\Phi(s,t\ ) = \psi(s,t) - \psi(\ 0,t)\ , \tag{15}$$

$$\psi(s,t) = \frac{i\,a}{\pi^3} \int \frac{dk\ (p'p - (p\ 'k)(pk)\,|\,k^2)\ 2qk}{k^2(k^2 + 2p'k)(k^2 - 2pk\ )(q^2 - 2qk + i\epsilon)}\ , \tag{16}$$

p' and p are the electrn and positron momenta before (or after) the reaction, q is the momentum transfer.

The function Φ is of the form:

$$\Phi(s,t) = \frac{s}{\pi} \int_4^\infty \frac{Im\ \Phi(s',t)\ ds'}{s'(\ s' - s - i\epsilon)} \tag{17}$$

$$Im\ \Phi\ (\ s,t) = \frac{a}{2}\sqrt{\frac{s-4}{s}}\ \left[\ \frac{s-2}{s-4}\ \ln\ \frac{-t}{s-4} + ½\ \right]. \tag{18}$$

The functions $M_{sa}^{(4)}$ and $M_{ta}^{(4)}$ contain no pole terms and are analytical functions of the variables s, u and t,u respectively, and satisfy a Mandelstam representation.

The expression (14) can be rewritten in the form:

$$M^{(4)} = M_t^{(2)}[\ (\ \beta(s) - \beta(u)\ \ \ln(-\ t) + \epsilon(s) - \epsilon(u)\ ] \dotplus M_{ta}^{(4)}\ + \tag{19}$$

$$+ M_s^{(2)}\ [\ (\ \beta(t) - \beta(u)\ \ln(-s) + \epsilon(t) - \epsilon(u)\ ] + M_{sa}^{(4)}\ ,$$

where $\beta(t)$ is given by the formula (7) and $\epsilon(t)$ is of the form:

$$\epsilon(t) = \frac{a\,t}{\pi} \int_4^\infty \sqrt{\frac{t'-4}{t'}}\ \left[\frac{t'-2}{t'-4}\ \ \ln\ \frac{1}{t'-4}\ \ + ½\ \right] \frac{dt'}{t'(\,t' - t - i\epsilon\,)} \tag{20}$$

Repeating the arguments of the previous Section, for M we can write a representation as in equation (10), or the representation

$$M = \exp\ [\ (\ \beta(s) - \beta(u)\ \ln(-t)\ \ + \epsilon(s) - \epsilon(u)\ ]\ M_{ta}\ \ +$$

$$+ \exp [(\beta(t) - \beta(u) \ell n (-s) + \epsilon(t) - \epsilon(u)] M_{sa} . \qquad (21)$$

Using the explicit form of thr function (13) we may represent the matrix element M_λ in the form

$$M_\lambda = \exp [(\beta(s) - \beta(u) \ \ell n \frac{-t}{\lambda} + 2F(t)] M_{ta} + \qquad (22)$$

$$+ \exp [(\beta(t) - \beta(u) \ \ell n \frac{-s}{\lambda} + 2F(s)] M_{sa} .$$

[illegible] consider the imaginary part of the power of the first exponent of this expression in the physical region $s > 4$, $t < 0$, $u < 0$. It is equal to

$$i \, Im \, \beta(s) \ \ell n \frac{-t}{\lambda} = i a \frac{s-2}{\sqrt{s(s-4)}} \ell n \frac{-t}{\lambda} . \qquad (23)$$

We see that the imaginary part of the singular expression in the power of the first exponent in (22) leads to a divergent phase

$$\exp [\, ia \frac{E^2 + p^2}{pE} \ell n \frac{2p \sin (\theta | 2)}{\sqrt{\lambda}}] \qquad (24)$$

(in c.m.s.) which, in the non-relativistic limit, coincides with the divergent phase of the scattering amplitude in the Coulomb field in non-relativistic theory /5/.

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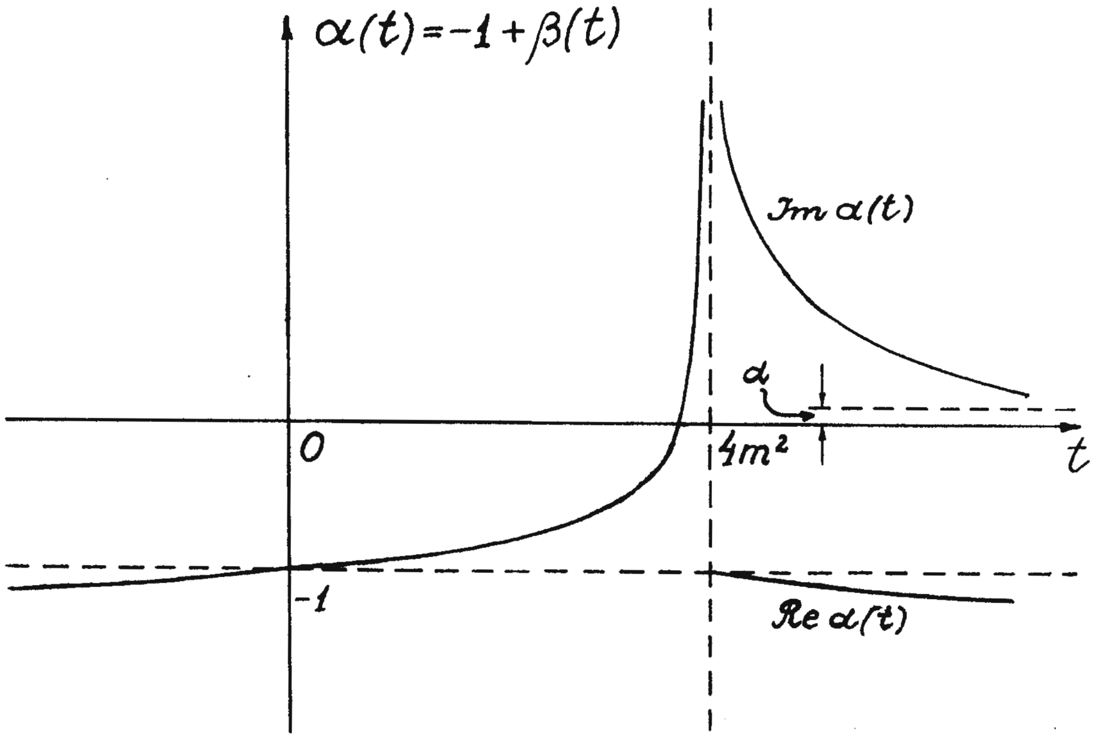
$\alpha(t) = -1 + \beta(t)$
$\mathcal{I}m\, \alpha(t)$
α
0
$4m^2$
t
-1
$\mathcal{R}e\, \alpha(t)$