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# TIME MINIMIZING DESIGN OF AN EXPERIMENT FOR THE SPECIFICATION OF A SINGLE PHASE SHIFT

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## Introduction

The phase shift analysis[1] is a method of reconstructing the scattering matrix from the experimental data. A single set of phase shifts is now determined for the energies from 23 to 310 MeV. The specification of phase shifts (or mixing parameters) arises as an important problem for these energies. For instance the precise determination of the NN-scattering mixing parameter $\epsilon_1$ makes it possible to estimate the contribution of tensor forces to the NN-interaction[2].

In the phase shift analysis the least square method gives a set of phase shifts from the experimental data and the corresponding error matrix. We have to choose such an additional experiment, which allowes to estimate one phase shift (say the $\ell$ -th) with a given small error.

The proposed method (the measurement-time minimizing method) is a generalisation and improvement of previously suggested solutions of the problem[3,4].

The method can also be used for other measurements (not only in NN-scattering experiments). It also makes it possible to design experiments specifying some function of the parameters (phase shifts).

The computing time needed by the method is relatively short.

It is proved, that in the general case it is not possible to estimate a phase shift from a single experiment quite precisely. A limit error exists (when the measurement time tends to infinity). The value of this limit error is given (corollary 2).

## The Description of the Method

Let us assume, that the theoretical dependence of the experimental data $y_i$ on the phase shifts $\delta_1, \dots, \delta_m$ is given by the equations:

$$y_i = \eta_i(\delta) \qquad i = \overline{1, n+1} \quad . \tag{1}$$

To evaluate the error matrix we must take a linear aproximation:

$$y_i = \sum_{\alpha=1}^{m} f_{i\alpha}\,\delta_\alpha + f_{i0}\ , \tag{2}$$

where

$$f_{i\alpha} = \left[\frac{\partial \eta_i(\vec{\delta})}{\partial \delta_\alpha}\right]_{\vec{\delta}=\vec{\delta}_0}\ ;\ f_{i0} = \eta_i(\vec{\delta}_0) \tag{3}$$

and $\vec{\delta}_0$ is the vector of the least-squares estimates. The first $n$ experiments are the original ones and the $(n+1)$-st is the additional experiment (the planned experiment).

By $t$ we shall denote the measurement time of the additional experiment. As is known/3,4/ the element of the matrix reciprocal to the error matrix has the form:

$$D^{-1}_{\alpha\beta}(t) = \sum_{i=1}^{n+1} f_{i\alpha}\, w_i\, f_{i\beta} = D^{-1}_{\alpha\beta}(0) + t\lambda\, f_{n+1,\alpha}\, f_{n+1,\beta}, \tag{4}$$

where $w_i$ denotes the weight of the $i$-th experiment and $\lambda$ is the efficiency of the additional experiment/5,6/ / $w_{n+1} = 1/\lambda t$ /.

The expression (4) indicates a "direct method" of solving a similar problem. We fix the time $t$ and substitute all possible additional experiments into (4). For each additional experiment we compute the reciprocal matrix. Thus we find the experiment which gives $\delta_\ell$ with the minimal error.

To get an analytical solution of the problem we used the following lemma:

**Lemma.**

If $A, B$ are two $m\times m$ matrices with the ranks $r(A)=m$, $r(B)=1$ then:

$$\det(A+B) = (\det A)\left(1+\sum_{p,s=1}^{m} A^{-1}_{ps} B_{sp}\right). \tag{5}$$

**Proof.**

We denote the $i$-the column of the matrix $A$ / $B$ / by $\alpha_i$ / $\beta_i$ /. We have:

$$\begin{aligned}\det(A+B) &= \det(\alpha_1+\beta_1, \ldots, \alpha_m+\beta_m) = \\ &= \det(\alpha_1, \alpha_2, \ldots, \alpha_m) + \det(\beta_1, \alpha_2, \ldots, \alpha_m) + \ldots + \\ &\quad + \det(\alpha_1, \alpha_2, \ldots, \beta_m).\end{aligned} \tag{6}$$

In (6) we drop the determinants containing two or more columns of the matrix $B$. They are equal to zero, since $r(B)=1$. We decompose the determinants in (6) in terms of the elements of the $\beta$-s (Laplace theorem/7/) and obtain (5).

Theorem x/.

Let us denote by $[C^{(\alpha,\beta)}(t)]^{-1}$ the submatrix of $D^{-1}(t)$ which we obtain by removing the $\alpha$-th row and the $\beta$-th column of $D^{-1}(t)$. The following equality is valid:

$$D_{\alpha\beta}(t)=D_{\alpha\beta}(0)\,\frac{1+t\lambda\sum\limits_{\substack{s=1\\ s\neq\alpha}}\sum\limits_{\substack{p=1\\ p\neq\beta}} f_{n+1,s}C^{(\alpha,\beta)}_{sp}(0)f_{n+1,p}}{1+t\lambda\sum\limits_{s=1}^{m}\sum\limits_{p=1}^{m} f_{n+1,s}D_{sp}(0)f_{n+1,p}}\,. \tag{7}$$

Proof.

According to a formula well known from algebra/7/ we have:

$$D_{\alpha\beta}(t)=\frac{\det[C^{(\alpha,\beta)}(t)]^{-1}}{\det D^{-1}(t)}\,. \tag{8}$$

Now (4) written in a matrix form gives:

$$D^{-1}(t)=D^{-1}(0)+t\lambda F\,, \tag{9}$$

where

$$F_{\alpha\beta}=f_{n+1,\alpha}\,f_{n+1,\beta}$$

and $r(F)=1$ because $F$ is a product of two vectors. Using the lemma we can write:

$$\det D^{-1}(t)=(\det D^{-1}(0))(1+t\lambda\sum_{p,s=1}^{m} D_{sp}(0)f_{n+1,p}\,f_{n+1,s}). \tag{10}$$

In the same way we also transform $\det[C^{(\alpha,\beta)}(t)]^{-1}$. We substitute the obtained result and (10) into (8) and obtain (7). The theorem is proved.

Corollary I.

If a variable $z$ is given by a function $z=z(\delta_1,\ldots,\delta_m)$ which allows a linear approximation at the point $\vec{\delta}=\vec{\delta}_0$, then the time dependence of the squared

x/ The author found out only after this paper was done, that a theorem similar to the given one, was proved in a paper by G.E.P.Box and Wiliam G. Hunter: Sequential Design of Experiments for Nonlinear Models; Proceedings of the IBM Scientific Computing Symposium on Statistics 113, Oct. 1963.

error of $z$ is given as:

$$\sigma_z^2(t)=\sigma_z^2(0)\ \frac{1+t\lambda k}{1+t\lambda\sum_{s=1}^{m}\sum_{p=1}^{m} f_{n+1,s} D_{sp}(0) f_{n+1,p}}\ , \qquad (11)$$

where $k$ denotes some quantity constant in time.

Proof.

According to a known equality[3]:

$$\sigma_z(t)=\sum_{\alpha,\beta=1}^{m} g_\alpha D_{\alpha\beta}(t)\, g_\beta\ , \qquad (12)$$

where

$$g_\alpha=\left[\frac{\partial z}{\partial \delta_\alpha}\right]_{\vec{\delta}=\vec{\delta}_0}\ .$$

By the substitution of (7) into (12) we obtain (11).

It is evident from the proof how we can obtain the numerical value of the constant $k$.

Corollary 2.

The squared error of the phase shift $\delta_\ell$ has the following time dependence:

$$\sigma_\ell^2(t)=\sigma_\ell^2(0)\ \frac{1+t\lambda\sum_{\substack{s=1\\ s\neq\ell}}^{m}\sum_{\substack{p=1\\ p\neq\ell}}^{m} f_{n+1,s}\, C_{sp}^{(\ell,\ell)}(0)\, f_{n+1,p}}{1+t\lambda\sum_{s=1}^{m}\sum_{p=1}^{m} f_{n+1,s}\, D_{sp}(0)\, f_{n+1,p}} \qquad (13)$$

and its limit (the least possible error) is expressed by:

$$\sigma_{\ell\,\min}^2=\sigma_\ell^2(0)\ \frac{\sum_{\substack{s=1\\ s\neq\ell}}^{m}\sum_{\substack{p=1\\ p\neq\ell}}^{m} f_{n+1,s}\, C_{sp}^{(\ell,\ell)}(0)\, f_{n+1,p}}{\sum_{s=1}^{m}\sum_{p=1}^{m} f_{n+1,s}\, D_{sp}(0)\, f_{n+1,p}}\ . \qquad (14)$$

The expression (13) is a special case of (7) (for $\alpha=\beta=\ell$) and the expression (14) follows from the equality:

$$\sigma_{\ell\,\min}^2=\lim_{t\to\infty}\sigma_\ell^2(t)\ .$$

Corollary 3.

If it is possible to obtain the phase shift $\delta_\ell$ after the additional measurement with the requested error $\sigma'_\ell$, then the necessary measurement time is:

$$t = \frac{1-\mu^2}{\mu^2} \cdot \frac{1}{\lambda \sum_{s,p=1}^{m} f_{n+1,s} R_{sp} f_{n+1,p}} , \tag{15}$$

where

$$R_{sp} \begin{cases} = D_{sp}(0) - \dfrac{C_{sp}^{(\ell,\ell)}(0)}{\mu^2} & \text{if} \quad s \neq \ell, \; p \neq \ell \\ = D_{sp}(0) & \text{if} \quad s = \ell \quad \text{or} \quad p = \ell \end{cases} \tag{16}$$

and

$$\mu = \frac{\sigma'_\ell}{\sigma_\ell(0)} .$$

We obtain (15) directly from (13) and (16).

Naturally, if the efficiency is not equal but only proportional to the reciprocal value of the variance of the additional experiment in a time unit, then we must multiplicate the right side of (15) by a constant.

The equality (15) makes it possible to find the optimal additional experiment minimizing the time $t$ in the set of all possible additional experiments. We must emphasize, that only the efficiency $\lambda$ and the coefficients $f_{n+1,a}$ depend on the additional experiment. The matrix $R$ can be computed once and for all before the planning so that the computation is shorter than in the direct method.

If we want to minimize the price of the experiment we must minimize the value:

$$v = h_0 + h_1 t , \tag{17}$$

where $h_0$ is the price of the experimental equipement, $h_1$ is the price of the accelerator run per time unit and $t$ is taken from (15).

## Comparison with other Methods

The method most similar to the given one is the continuous planning of experiments /4/. The results of /4/ and of the proposed method coincide if the measurement time is infinitesimally small. In this sense the proposed method is a generalization of /4/.

We shall discuss another method in detail – the fixed parameter method. This method is based on intuitive considerations and was exploited by the experimentalists in the JINR.

The additional experiment can be estimated from the original experiments. The squared variance of this estimate is given by /3/:

$$\sigma^2_{n+1} = \sum_{s,p=1}^{m} f_{n+1,s} D_{sp}(0) f_{n+1,p} \quad . \tag{18}$$

Let us assume, that the phase shift $\delta_\ell$ is known exactly (the fixed parameter). Then we obtain the variance (18) in the form:

$$\sigma^2_{fix,\,n+1} = \sum_{\substack{s=1\\ s\neq\ell}}^{m} \sum_{\substack{p=1\\ p\neq\ell}}^{m} f_{n+1,s} D_{sp}(0) f_{n+1,p} \quad . \tag{19}$$

According to this method the optimal additional experiment is the "most sensitive" one to the fixing of the parameter. Thus, the optimal additional experiment maximizes the expression:

$$\Delta\sigma^2 = \sigma^2_{n+1} - \sigma^2_{fix,\,n+1} = \sum_{s,p=1}^{m} f_{n+1,s} Q_{sp} f_{n+1,p} \quad , \tag{20}$$

where

$$Q_{sp} \begin{cases} = D_{sp}(0) - D_{sp}(0) = 0 & \text{if } s\neq\ell,\ p\neq\ell \\ = D_{sp}(0) & \text{if } s=\ell \text{ or } p=\ell \quad . \end{cases} \tag{21}$$

Let us compare the reciprocal value of (15) with $\Delta\sigma^2$ from (20) and the matrix $R_{sp}$ from (16) with $Q_{sp}$ from (21). If the phase shift estimates are uncorrelated, then $D_{sp}(0) = C^{(\ell\ell)}_{sp}(0)$ / $s\neq\ell,\ p\neq\ell$ /. Moreover, if $\mu \to 1$ and $\lambda$ is a constant, then (16) coincides with (21) and the planning of the experiment by both methods coincides too. We have proved the following statement :

If:

1. The phase shift estimates are uncorrelated .

2. All possible additional experiments have the same efficiency.

3. We only want to find a locally optimal experiment (that is optimal in the beginning of the measurement) ,

then the design of experiment by the time-minimizing method coincides with the fixed parameter method.

It can be proved, that the two methods give the same results only under these assumptions, so that the fixed parameter method can be applied only in this case.

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