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ОБЪЕДИНЕННЫЙ ИНСТИТУТ ЯДЕРНЫХ ИССЛЕДОВАНИЙ

ЛАБОРАТОРИЯ ТЕОРЕТИЧЕСКОЙ ФИЗИКИ

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ON A SLIGHT MODIFICATION OF HORI'S STRONG-COUPLING METHOD

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1. Introduction

Performing some calculations by means of Hori's method[1] the author found the following difficulty [2],[3]: In order to deduce a physically acceptable Feynman-amplitude one has to operate with a large number of functional differentiations (with respect to the external sources) on the generating functional Ω of the vacuum-expectation-value of the S matrix. However Ω depends in interesting cases in a very complicated way on the external sources. Therefore one practically cannot calculate higher orders of Feynman-amplitudes with Hori's method. Here the author proposes to modify the Hori's method slightly. Do not calculate the generating functional Ω in a closed form, but write it as a Volterra series (with respect to the external sources). The coefficients of this series can be written down as functional integrals and calculated (in our example) exactly with help of distribution analysis. If you write the vacuum-expectation-value of the S -matrix also as a Volterra series, the coefficient functions of this series can be represented as infinite series of certain integrals over the coefficient functions of the Ω -functional. The main difficulty of the present method is to sum up this infinite series. Here we cannot solve this difficulty.

2. Volterra series for Ω

In Hori's work[1] the generating functional Ω is introduced; in some simple examples, Hori could give closed expressions for this functional by means of functional integration. Hori's definition of Ω reads as follows:

$$\Omega[\rho,\rho^+,j] = \int\int\int \exp[-i \int (g\, \psi^+\psi\, \phi - \rho^+\psi - \rho\psi^+ - j\phi)\, d^4x\,]\, D(\psi,\psi^+,\phi). \quad (1)$$

(Here and in all this work we use the example of one real (ϕ) and one complex (ψ , ψ^+) scalar field, interacting by the term $g\,\psi^+\psi\,\phi$ in the Lagrangian. The terms $\rho^+\psi$, $\rho\,\psi^+$, $j\,\phi$ represent the interaction of the fields with the external sources ρ , ρ^+, j . The free Lagrangian would read $L = \psi^+(\Box - m^2)\psi + \phi(\Box - \mu^2)\,\phi$. The symbol $D(\psi^+,\psi,\phi)$ in (1) stands for the integration element in the space of the functions ψ , ψ^+, ϕ). As already stated, we propose to represent Ω as a Volterra series (see, eg. Rzewuski[4]):

$$\Omega[\rho,\rho^+,j] = \sum_{\ell,m,n=0}^{\infty} \frac{(i)^{\ell+m+n}}{\ell!\; m!\; n!} \int\dots\int d^4x_1 \dots d^4x_n\, d^4y_1 \dots d^4y_\ell\, d^4z_1 \dots d^4z_m \cdot$$

$$\cdot\, \phi_{n,\ell,m}(x_1 \dots x_n;\, y_1 \dots y_\ell,\, z_1 \dots z_m)\, e_{n,\ell,m}(x_1 \dots z_m), \qquad (2)$$

with $e_{n,\ell,m}(x_1 \dots z_m) = \rho(x_1) \dots \rho(x_n)\, \rho^+(y_1) \dots \rho^+(y_\ell)\, j(z_1) \dots j(z_m)$.

The functions $\phi_{n,\ell,m}(x_1 \dots z_m)$ of course determine the functional Ω. They are symmetric with respect to the variables $x_1 \dots x_n$; $y_1 \dots y_\ell$ $z_1 \dots z_m$. (Here and in the following we assume, that our Volterra series converges uniformly at the "point" $\rho = \rho^+ = j = 0$; One is not sure if this is true. By comparing eqs (1) and (2) one gets the expressions:

$$(i)^{n+\ell+m} \cdot \phi_{n,\ell,m}(x_1 \dots z_m) =$$

$$= \lim_{\substack{\rho \to 0 \\ \rho^+ \to 0 \\ j \to 0}} \frac{\delta^{(n+\ell+m)}\, \Omega}{\delta\rho(x_1) \dots \delta\rho(x_n)\, \delta\rho^+(y_1) \dots \delta\rho^+(y_\ell)\, \delta j(z_1) \dots \delta j(z_m)} = \qquad (3)$$

$$= (i)^{n+\ell+m} \iiint \psi^+(x_1) \dots \psi^+(x_n)\, \psi(y_1) \dots \psi(y_\ell)\, \phi(z_1) \dots \phi(z_m) \exp(-ig \int \psi^+ \psi \phi\, d^4x)\, D(\psi^+, \psi, \phi)$$

The functional integrals (3) can be calculated directly. The representation (2) for Ω enables one to perform any number of functional differentiations on Ω very easily. It is also very useful, because at the end of calculation of Feynman-amplitudes (which always is our aim, of course) the limiting process $\rho \to 0$, $\rho^+ \to 0$, $j \to 0$ has to be always performed. Let us first calculate ϕ_{000}, defined by

$$\phi_{000} = \iiint \exp(-ig \int \psi^+ \psi \Phi\, d^4x)\, D(\psi^+, \psi, \Phi)\,. \qquad (4)$$

Using the definition of the δ - functional

$$\delta[f(x)] = \int \exp(-2\pi i \int f(x)\, \Phi(x)\, d^4x)\, D(\Phi), \qquad (5)$$

we can at once perform the functional integration with respect to Φ. This gives:

$$\phi_{000} = \iint \delta\left[\frac{g}{2\pi} \psi^+ \psi\right] D(\psi^+, \psi)$$

If you write $\psi = re^{ia}$ and introduce the lattice space x_k instead of the continuous x - space, you have

$$\psi_{000} = \lim \prod_k \left(\int_0^\infty \int_0^{2\pi} \delta\left(\frac{g}{2\pi} r_k^2\right) r_k\, dr_k\, dx_k \right).$$

(The symbol $\lim \Pi_k$ means: Take the product over all points x_k of the

lattice-space, then go to the limit of the x - continuum). That is, we have

$$\phi_{000} = \lim \prod_k (\omega_k)\,;$$

$$\omega_k = 2\pi \int_0^\infty \delta\left(\frac{g}{2\pi} r_k^2\right) r_k\, dr_k = \frac{2\pi^2}{g} \int_0^\infty \delta(y)\, dy\,.$$

Now we write

$$\int_0^\infty \delta(y)\, dy = \int_{-\infty}^{+\infty} \theta(y)\, \delta(y)\, dy \qquad \text{when} \quad \theta(y) = \begin{cases} 1 & \text{for } y > 0 \\ 0 & \text{for } y < 0 \end{cases}$$

The product of the improper functions $\delta(y)$ and $\theta(y)$ in the integrand must be treated by means of distribution analysis. In the work [5] of Güttinger we find the rule

$$\theta(y)\, \delta(y) = -\, c_0\, \delta(y), \qquad (6)$$

where c_0 is a finite, but quite arbitrary constant. Therefore we get $\omega_k = -\frac{2\pi^2}{g} c_0$, and

$$\phi_{000} = \lim \prod_k \left(-\frac{2\pi^2}{g} c_0\right). \qquad (7)$$

From (7) one would like to conclude, that ϕ_{000} is a highly singular quantity. But using the rules for products of the type $\lim \prod_k f(x_k)$ stated in [2], we arrive at the expression:

$$\phi_{000} = \exp\left(c_1^2 \ln\left(-\frac{2\pi^2}{g} c_0\right)\right), \qquad (8)$$

where c_1 is another finite, arbitrary constant, which stems from Güttinger's rule (see [5])

$$\delta^2(y) = c_1\, \delta(y)\,. \qquad (9)$$

Remark please the singularity of ϕ_{000} for $g \to 0$ and arbitrary c_1, c_0 (see also [3])! Next let us discuss the function

$$\phi_{110}(x_1, y_1) = \iiint \psi^+(x_1)\, \psi(y_1) \exp(-ig \int \psi^+ \psi \Phi\, d^4x)\, D(\psi^+, \psi, \Phi)\,. \qquad (10)$$

(Remark that all functions of type $\phi_{n,0,0}\,(n \neq 0)$ and $\phi_{0,\ell,0}\,(\ell \neq 0)$ and $\phi_{n,\ell,0}\,(\ell \neq 0)$ (with $n \neq \ell \neq 0$) are zero because they contain at least one integral of the type $\int_0^{2\pi} e^{ia}\, da$). The function ϕ_{110} is zero for all $x_1 \neq y_1$ because in those cases also appears the integral $\int_0^{2\pi} e^{ia}\, da$, if you introduce $\psi = re^{ia}$. For $x_1 = y_1$, however, you get with $\psi(x_1) = r_1 e^{ia_1}$:

$$\phi_{110}(x_1, x_1) = \lim \prod_k{}' \left(\int_0^\infty \int_0^{2\pi} \delta\left(\frac{g}{2\pi} r_k^2\right) r_k\, dr_k\, da_k \right) \cdot \int_0^\infty \int_0^{2\pi} r_1^2\, \delta\left(\frac{g}{2\pi} r_1^2\right) r_1\, dr_1\, da_1$$

(The symbol $\prod_k{}'$ means: Take the product over all points x_k of lattice space with the exception of the point x_1). That means, we can write:

$$\phi_{110}(x_1,x_1) = \phi_{000} \frac{\int_0^\infty r^2 \delta(\frac{g}{2\pi} r^2)\, d(r^2)}{-\frac{c_0 \cdot 2\pi}{g}} .$$

Now we have $\int_0^\infty r^2\, \delta(\frac{g}{2\pi} r^2)\, d(r^2) = (\frac{2\pi}{g})^2 \int_{-\infty}^{+\infty} y\theta(y)\, \delta(y)\, dy .$

Once more we use formula (6) and get for our integral $\int \delta(y) \cdot y \cdot dy = 0$.
So we arrive at the result:

$$\phi_{110}(x_1,y_1) = 0 . \qquad (11)$$

By quite a similar calculation we can show, that

$$\phi_{n,n,0}(x_1 \ldots y_n) = 0 \qquad \text{for} \quad n \neq 0 . \qquad (12)$$

Next we are interested in the function

$$\phi_{002}(z_1,z_2) = \iiint \Phi(z_1)\Phi(z_2) \exp(-ig \int \psi^+ \psi \Phi d^4x) D(\psi,\psi^+,\Phi) . \qquad (13)$$

Let us first suppose $z_1 \neq z_2$. Then we get

$$\phi_{002} = \lim \prod_k{}'' (-\frac{2\pi^2}{g} c_0^2) \, (\frac{-i\pi}{g} \int_0^\infty \delta^{(1)}(y)\, dy)^2 , \qquad \text{x)}$$

where

$$\delta^{(1)}(y) = \frac{d}{dy} \delta(y) , \quad \text{i.e.}$$

$$\int \Phi_k \exp(-2\pi i y \Phi_k)\, d\Phi_k = \frac{i}{2\pi} \frac{d}{dy} \delta(y) .$$

Now we have $\int_0^\infty \delta^{(1)}(y)\, dy = \int_{-\infty}^{+\infty} \theta(y) \delta^{(1)}(y)\, dy =$

$$= - c_0 \int_{-\infty}^{+\infty} \delta^{(1)}(y)\, dy - c_1 \int_{-\infty}^{+\infty} \delta(y)\, dy = - c_1 \qquad (\text{see}^{/5/}).$$

This leads to

$$\phi_{002}(z_1,z_2) = \phi_{000} \; (-\frac{c_1^2}{4\pi^2 c_0^2}) \quad \text{for} \quad z_1 \neq z_2 . \qquad (14a)$$

For $z_1 = z_2$ however we get

x) $\prod_k''$ means: Let out the points $x_k = z_1$ and $x_k = z_2$ from the product!

$$\phi_{002}(z_1,z_1) = \lim \prod_k{}' (-\frac{2\pi^2}{g} c_0)(-\frac{1}{2g} \int_0^\infty \delta^{(2)}(y)\, dy) .$$

With$^{/5/}$ we have

$$\delta^{(2)}(y)\theta(y) = -c_0 \delta^{(2)}(y) - c_1 \delta^{(1)}(y) - c_2 \delta(y)$$

(c_0 , c_1 , c_2 finite, arbitrary constants), so that $\int_0^\infty \delta^{(2)}(y)\, dy = - c_2$. At the end we find:

$$\phi_{002}(z_1,z_1) = - \phi_{000} \cdot \frac{c_2}{4\pi^2 c_0} . \qquad (14b)$$

As the last example for the calculation of coefficients from the Volterra series(2) we take

$$\phi_{112}(x_1;y_1;z_1,z_2) = \iiint \psi^+(x_1)\psi(y_1)\Phi(z_2) \exp(-ig \int \psi^+ \psi \Phi d^4x) \times D(\psi^+,\psi,\Phi) . \qquad (15)$$

The functional integration over Φ can be performed in the same manner as in the function ϕ_{002} . Because of integration over the phases of the functions ψ^+ and ψ we get the result $\phi_{112} = 0$ for $x_1 \neq y_1$.
Calculations similar to those performed earlier show us, that ϕ_{112} is different from zero only, if $x_1 = y_1 = z_1$ or $x_1 = y_1 = z_2$ or even $x_1 = y_1 = z_1 = z_2$.
More exactly we get:

$$\phi_{112}(x_1;y_1;z_1,z_2) = \begin{cases} \phi_{000} \cdot (-\frac{c_1}{2\pi g c_0}) & \text{for} \begin{cases} x_1 = y_1 = z_1 & \text{or} \\ x_1 = y_1 = z_2 & \text{or} \\ x_1 = y_1 = z_1 = z_2 \end{cases} \\ 0 & \text{else} \end{cases} \qquad (16)$$

Higher ϕ functions can be calculated in a very similar manner; all of them can be calculated exactly.

3. Connection between the ϕ -functions and Feynman-amplitudes

It turns out to be useful to develop also the vacuum-expectation-value of the S -matrix in a Volterra series:

$$S_{vac}[\rho,\rho,j] = \sum_{n,\ell,m=0}^{\infty} \frac{(i)^{n+\ell+m}}{n!\,\ell!\,m!} \int \ldots \int d^4x_1 \ldots d^4z_m \, . \; T_{n,\ell,m}(x_1 \ldots z_m)\, e_{n,\ell,m}(x_1 \ldots z_m) . \qquad (17)$$

The quantities $e_{n,\ell,m}$ are the same as in eq. (2). The functions $T_{n,\ell,m}$

of course determine the functional $S_{vac}[\rho, \rho^+, j]$. They are (up to a constant) already the Feynman-amplitudes χ, because by definition e.g.:

$$\chi(x_1, y_1) = \lim_{\substack{\rho \to 0 \\ \rho^+ \to 0 \\ j \to 0}} S_{vac}^{-1} \cdot \frac{\delta^2 S_{vac}}{\delta \rho^+(x_1) \delta\rho(y_1)} = - \frac{T_{110}(x_1, y_1)}{T_{000}} . \qquad (18)$$

(Here one can see once more, that the development (17) is quite useful).

To establish the relation between the coefficients $T_{n,\ell,m}$ and $\phi_{n,\ell,m}$ we start from the equation

$$S_{vac} = \exp(-i\omega)\,\Omega , \qquad (19)$$

with

$$\omega = \iint \left(\frac{\delta}{\delta\rho(x)} G_m(x,y) \frac{\delta}{\delta\rho^+(y)} + \frac{\delta}{\delta j(x)} G_\mu(x,y) \frac{\delta}{\delta j(y)} \right) d^4x\, d^4y$$

and

$$G_\lambda(x,y) = \delta(x-y)(\Box_y - \lambda^2)$$

(see/1/).

Consider now the functional

$$F_{n,\ell,m}[\rho,\rho^+,j] = \int \dots \int d^4x_1 \dots d^4z_m\, \phi_{n,\ell,m}(x_1 \dots z_m)\, e_{n,\ell,m}(x_1 \dots z_m) .$$

It can be proved easily, that

$$\frac{\delta F_{n,\ell,m}}{\partial\rho(x)} = n \int \dots \int d^4x_2 \dots d^4z_m\, \phi_{n,\ell,m}(x_1 x_2 \dots z_m)\, e_{n-1,\ell,m}(x_2 \dots z_m)$$

and

$$\iint \frac{\delta}{\delta\rho(x)} G(x,y) \frac{\delta}{\delta\rho^+(y)} d^4x\, d^4y\, F_{n,\ell,m} =$$

$$= n \cdot \ell \int \dots \int d^4x_2 \dots d^4x_n\, d^4y_2 \dots d^4y_\ell\, d^4z_1 \dots d^4z_m\, d^4x\, d^4y .$$

$$\cdot G(x,y) \cdot \phi_{n,\ell,m}(x, x_2 \dots x_n; y, y_2 \dots y_\ell; z_1 \dots z_m)\rho(x_2)\dots\rho(x_n)\rho^+(y_2)\dots\rho^+(y_\ell) j(z_1) j(z_m).$$

Using this formula one can write up easily for each $T_{n,\ell,m}$ a series of the following type:

$$T_{000} = \phi_{000} + \iint G_m(x_1,y_1)\phi_{110}(x_1,y_1)\, d^4x_1\, d^4y_1 + \qquad (20)$$

$$+ i\iint G_\mu(z_1,z_2)\phi_{002}(z_1,z_2)\, d^4z_1\, d^4z_2 -$$

$$- \iiiint G_m(x_1,y_1) G_\mu(z_1,z_2)\phi_{112}(x_1;y_1;z_1,z_2)\, d^4x_1\, d^4y_1\, d^4z_1\, d^4z_2 + \qquad (20)$$

$$+ \dots\dots\dots .$$

$$T_{110} = \phi_{110}(x_2;y_1) + i\iint G_m(x_2,y_2)\phi_{220}(x_1,x_2;y_1,y_2)\, d^4x_2\, d^4y_2 +$$

$$+ i\iint G_\mu(z_1,z_2)\phi_{112}(x_1;y_1;z_1,z_2)\, d^4z_1\, d^4z_2 - \qquad (21)$$

$$- \tfrac{1}{2} \iiiint G_m(x_2,y_2) G_m(x_3,y_3)\phi_{330}(x_1,x_2,x_3;y_1,y_2,y_3)\, d^4x_2\, d^4x_3\, d^4y_2\, d^4y_3 -$$

$$- \iiiint G_m(x_2,y_2) G_\mu(z_1,z_2)\phi_{222}(x_1,x_2;y_1,y_2;z_1,z_2)\, d^4x_2\, d^4y_2\, d^4z_1\, d^4z_2 -$$

$$- \tfrac{1}{2} \iiiint G_\mu(z_3,z_4) G_\mu(z_1,z_2)\phi_{114}(x_1;y_1,z_1 \dots z_4)\, d^4z_1\, d^4z_2\, d^4z_3\, d^4z_4 +$$

$$+ \dots\dots .$$

The rule for the construction of these infinite series is easily visualized. It has no great value to write up the general series for $T_{n,\ell,m}$, but of course this could be done.

4. Discussion

In order to examine the value of the presently proposed formalism, now we introduce into (21) the ϕ -functions from chapter 2 of the present paper and remember, that T_{110} has (up to a constant, see (18)) the meaning of the propagation function for the ψ - particles.

In the series (21) the zeroth-order-term vanishes ($\phi_{110} \equiv 0$). From the two first-order-terms only one does not vanish (ϕ_{112}), but it is different from zero only for $x_1 = y_1$. That means, we do not have any propagation. This is well understandable: Propagation is done by the term L_0 of the Lagrangian. We have in Ω however only the term L_w of L.

Therefore propagation should appear only in higher orders. In the next order the term $\phi_{330} \equiv 0$. ϕ_{114} is only different from zero, if $x_1 = y_1$. However ϕ_{222} is different from zero also for $x_1 \neq y_1$, namely for $x_1 = y_2$, $x_2 = y_1$.

But performing the integrations over x_2 and y_2 the factor $\delta(x_2 - y_2)$ appears in $G_m(x_2, y_2)$. That means, only the points $x_2 = y_2$ survive and T_{110} is also in this order different from zero only for $x_1 = y_1$.

It seems to the author, that one can prove the same property of T_{110} also in higher orders. If this is true, it would mean that one can get a real propagation function first after summing up the whole infinite series (21), at least partly.

The author hopes, that he will be able to come back to these problems later.

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