ОБЪЕДИНЕННЫЙ ИНСТИТУТ ЯДЕРНЫХ ИССЛЕДОВАНИЙ

ЛАБОРАТОРИЯ ТЕОРЕТИЧЕСКОЙ ФИЗИКИ

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ON HELICITY AMPLITUDES

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Introduction

All relations of the relativistic kinematics become much simpler if the reactions are described in the three-dimensional velocity space, whose geometry is isomorphic to Lobachevsky's/1,2/. Such a description is especially convenient for studying the reactions involving the particles with spins. The method is simple because both the velocities and spins are considered as the three-dimensional vectors (not the 4-vectors). The relativistic effect reduces to a change in the metric of the velocity space, or, what is the same, in the vector addition rule.

One of the representations which has recently becomes very popular is that of Wick-Jacobi/3,4/. According to the latter the spin of the particle is characterized by its projection on the linear momentum of the particle itself in the centre-of-mass system.

At the first glance, such a definition is not covariant, since the scalar prodnct of a 4-spin and a 4-momentum vanishes by definition, and the projection of the vector part of the spin on the 3-momentum is not an invariant.

To make the covariant explicit it is more convenient to define the helicity as a projection of the 3-vector of the spin upon the direction of the relative velocity of the particle and the origin of the centre-of-mass system (we shall call it an (s system). This projection is, obviously, a relativistic scalar. It is also clear that one can choose, as a frame of reference, any other system (e.g. the rest system of any of the particles).

In the modern field theory we face the problem concerning the transition from the c.m.s. (s system) — to the c.m. system of the crossed reaction (t system, see below). This can be easily done by means of the method of kinematical graph.

It is worthwhile to note that the method of the velocity space is most suitable to describe the spin effects in the reactions, for the directions of the coordinate axes in such problems are defined by the vectors of the momenta of the particles themselves, while in the problem concerning Stark or Zeeman effects these directions are given in the coordinate space by the external fields.

2. Kinematical Graph and Notations

We consider the reaction of the type (s - channel)

$$'1' + '2' \to '3' + '4' \qquad (2.1)$$

(figures denote particles).

The conservation law in the s - channel is as follows:

$$p_1 + p_2 = p_3 + p_4 \qquad (22.)$$

(p_1 is the 4-momentum with the components ϵ_1 and $\vec{p}_1$ etc).

The particle velocities are described by the points on the upper sheet of the three-dimensional surface of a hyperboloid*.

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* The points on the lower of a hyperboloid correspond to antiparticles which are obtained from particles by a transformation $\tilde{p} = -p$ (by changing the signs of all the components of the 4-momentum). The scalar product of the particle momentum by the antiparticle one is equal, evidently, to $p\tilde{p} = -m^2$.

Since the hyperbolic cosine of the distance between the two points p_1/m and p_2/m in Lobachevsky's space equals the scalar 4-product of these two vectors, then the hyperbolic arc between the points corresponding to the particle and the antiparticle equals $\operatorname{argch}(-1) = i\pi$ Thus, the transition from the particle to the antiparticle is described by a hyperbolic rotation by the angle $i\pi$

If we consider the paper plane to be a map of this surface, then the particle velocities may be represented the points 1 ... 4 on this map. These points may be thought of as the ends of the vectors drawn from a coordinate origin whose exact position is of no importance and need not be indicated explicitly on the map. We are able to treat simultaneously all possible coordinate systems on the same figure, dealing only with the relative velocities of any two systems. The similar situation we have on the map of the world which gives all the distances between each pair of points on the earth's surface.

Transition between two coordinate systems is described, on our map of the velocity space, as the transition from the projections along one direction to those along the other.

Going back to Fig. 1, we note that the points of the intersection of two diagonals represent the c.m.s. velocity. It corresponds to the 4-velocity with the components

$$\left(\frac{\epsilon_1 + \epsilon_2}{\sqrt{s}} \; ; \; \frac{\vec{p}_1 + \vec{p}_2}{\sqrt{s}} \right) \tag{2.3}$$

where in usual notations

$$s = (p_1 + p_2)^2 = (\epsilon_1 + \epsilon_2)^2 - (\vec{p}_1 + \vec{p}_2)^2 \tag{2.4}$$

In addition to s in the field theoretical calculations one makes use of two more variables

$$t = (p_1 - p_4)^2 = (p_2 - p_3)^2 \tag{2.5}$$

$$u = (p_1 - p_3)^2 = (p_2 - p_4)^2 \tag{2.6}$$

and

$$s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2 \tag{2.7}$$

with corresponding velocity 4-vectors

$$\left(\frac{\epsilon_1 - \epsilon_4}{\sqrt{t}} \; ; \; \frac{\vec{p}_1 - \vec{p}_4}{\sqrt{t}} \right) \tag{2.8}$$

$$\left(\frac{\epsilon_1 - \epsilon_2}{\sqrt{u}} \; ; \; \frac{\vec{p}_1 - \vec{p}_2}{\sqrt{u}} \right) \tag{2.9}$$

In many problems vectors (2.8) and (2.9) turn out to be space-like. In this case the points t and u which correspond to them in the velocity space will be imaginary, i.e. belong to the hypersurface of another hyperboloid $u_0^2 - u^2 = -1$, Such points are known in Lobachevsky's geometry, and we shall deal with them later on.

Let us calculate the lengths of sides and diagonals of a kinematical quadrangle (1423). Using that

$$p_1 p_2 = m_1 m_2 \, ch\,(12) \qquad \text{etc} \tag{2.10}$$

we get by squaring expressions (2.4), (2.5) and (2.6)

$$ch\,(14) = -(2\,m_1\,m_4)^{-1}\;(t - m_1^2 - m_4^2)$$

$$ch\,(23) = -(2\,m_2\,m_3)^{-1}\;(t - m_2^2 - m_3^2)$$

$$ch\,(42) = -(2\,m_4\,m_2)^{-1}(u - m_4^2 - m_2^2) \tag{2.11}$$

$$ch\,(31) = -(2\,m_3\,m_1)^{-1}\;(u - m_3^2 - m_1^2)$$

$$ch\,(12) = (2\,m_1\,m_2)^{-1}(s - m_1^2 - m_2^2)$$

$$ch\,(34) = (2\,m_3 m_4)^{-1}(s - m_3^2 - m_4^2)$$

Now we can determine all the angles of the graph by means of the cosine theorem.

For instance, for the angle (213), we find:

$$\cos\,(213) = \frac{ch(31)\;ch(12)\;-\;ch\,(23)}{sh\,(31)\;\;sh\,(12)} \tag{2.12}$$

The kinematical graph of Fig. 1 determines all the directions which serve usually as coordinate axes. Consider the nucleon-nucleon scattering as an example. The natural cases in the centre-of-mass system for this problem are $v_1 + v_1'$ and $v_1 - v_1'$, where v_1 and v_1' are the 3-velocity of one of the nucleons before and after collision. These directions are given on the kinematical graph by two orthogonal bisectors of the angles between the diagonals at the point s.

3. Transformation of Helicity Amplitudes

We write as usual the transition matrix element for spin particles in the s system (formula (1) in /3/)

$$< E'J'M'\lambda_3\,\lambda_4 |\, S \,|\, E\,J\,M\;\lambda_1\,\lambda_2> = \delta\,(E - E')\,\delta_{JJ'}\,\delta_{\mu\mu'}\times$$
$$\times < \lambda_3\,\lambda_4 |\, S^J(E) \,|\, \lambda_1\,\lambda_2> \tag{3.1}$$

where the helicity amplitude S^J (J is the orbital momentum) is connected with the amplitude for the spins quantized along an incident beam (the quantum numbers $\mu_1\,\mu_2\,\mu_3\,\mu_4$)

$$<\lambda_3\,\lambda_4 |\, S^J \,|\, \lambda_1\,\lambda_2> = \sum_{\mu_3\,\mu_4} <\mu_3\,\mu_4 |\, S^J \,|\, -\mu_1\,\mu_2> \times$$
$$\times\, D^{s_3}_{\mu_3\lambda_3}\,(-\phi, \theta, \phi)\, D^{s_4}_{\mu_4\,\lambda_4}\,(-\phi, \pi - \theta, \phi)\,. \tag{3.2}$$

Formula (3.2) has a simple meaning. The quantum numbers λ_1 ... characterize the spin projections on the four segments connecting point s in Fig. 1 with 4-vertices of a quadrangle. The direction of the incident beam in the same Figure is that from s towards '2'. In passing to the helicity amplitudes the projection of particle '1' will change sign, as the direction of the velocity '1' is that from s towards '1'. For particles in the final states it is necessary to rotate the axes of quantization from (s2) to (s3) for the particle '3' and from (12) to (34) for particle '4'. The rotation angles are, evidently, equal to the scattering ones θ and $\pi-\theta$, respectively. Two D factors in (3.2) correspond just to the rotation of the quantization axes by these angles.

The angle ϕ may be put equal to 0, as far as all the directions are in the same plane. The helicity quantization is usually described in the frame of reference s – One may transfer the quantization axes to any other frame of reference; for this purpose it is necessary to do parallel displacement between the corresponding points of the velocity space. So, any transformation of the axes will involve their parallel displacement to another point and usual rotation at a new point. The best way for performing the parallel displacement is to rotate first the axis of quantization so that its new direction is that of the tangent to the geodesical line which connects the points representing velocity of both systems on the kinematical graph. The spin amplitude in this new representation becomes obviously the helicity amplitude. The parallel displacement does not add anything new since the direction of tangent remains invariant under this displacement.

This gives just the geometrical interpretation of the helicity quantization. For example, the helicity λ of partical '1' will be the same both in system s and the rest systems of particle '1' or particle '2', just as in any other system on the non-Euclidian line connecting points '1' and '2'. Fig.2.

It is worthwhile to mention that we have not used the fact that the velocity of particles are less than that of light. Since in Lobachevsky's projection plane the point at infinity is an ordinary one, then all the results hold for photons or neutrinos; only the angle for the corresponding vertex at infinity becomes zero. The photon spin (its polarization) is also quantized along the relative velocity of a photon and the coordinate origin. Since the vertex angle is zero, the direction of quantization remains invariant.

So, we see that spins may be considered independently of the coordinate parts of the wave functions, the particles may be 'stripped' from spins just as in non-relativistic problems. The only difference in our case is that the rotation angles are computed by the formulas of the hyperbolic geometry and not of the euclidian one as in the non-relaticistic kinematics.

4. s and t System. Crossing-Transformations

Let us consider a somewhat more complex case of transformation-the transition to the crossed reaction. The reaction obtained from (2.1) by interchanging particles '2' and '3' with the corresponding substitution of particles for antiparticles is called crossed reaction. For instance, if reaction (2.1) is the reaction

$$p + \pi^- \to n + \pi^\circ$$

then the crossed reaction will be

$$p + \approx{n} \to \pi^+ + \pi^\circ$$

How are the helicity amplitudes of the crossed reaction related to those of the original reaction ?

In order to answer this question we construct a centre-of-mass system for the crossed reaction – the t -system. Since the conservation law for the crossed reaction assumes the form

$$p_1 - p_4 = - p_2 + p_3 \tag{4.1}$$

this system will have a 4-momentum $p_1 - p_4$ We have already pointed out that this vector usually is space-like, so that $t = (p_1 - p_4)^2 < 1$ as in the case of elastic scattering of identical particles. The velocity of the t -system will be in such a case greater than that of light. But this fails to introduce any difficulties whatsoever, as long as it is not the point t we are interested in, but only the direction to it. The directions to the point located 'farther than infinity' in Lobachevsky's plane may be seen from the same kinematical graph.

Like system s is at the point of intersection of the straight lines defined by pairs (1.2) and (3.4), the system t is given by the intersection (real or imaginary) of the straight lines defined by pairs of points (1.4) and (2.3).

The intersection of straight lines (1.4) and (2.3) should be thought of as an 'external' intersection. This implies that these straight lines are regarded as arcs of circles with an infinitely large radius; one of the intersecting segments goes from point 4 to infinity. Then it comes back 'crossing the infinity' to point 1. Similarly the segment crosses points (2 , ∞ 3).

If the particles have the same masses the transition to the crossed reaction is described by the following transformation

$$\begin{aligned} s &\to -t \\ t &\to -s \end{aligned} \tag{4.2}$$

If the masses of the particles are different the transformation is given by reversing all the four components of the linear impulse of the corresponding particle. By such transformations in Fig. 1 the point t is displaced inside the quadrangle and point s will come out of it, so that they change their roles .

It is clear now how of to pass to the new helicity amplitudes. First of all we displace the spin from the s -system to the rest system 1. Then we must rotate the quantization axis to the direction (41) , i.e. the angle (412) (cf. Fig. 3). The rotation angle must be expressed in terms of s and t . It is obvious that after crossing transformation the new amplitude will be transformed into normal helicity amplitude of the crossed reaction.

Particle '2' has been quantized along the direction from s to '2'. In order to change this direction to that from t to '2', we have to perform rotation by the angle $\pi - (321)$. Similarly we must rotate in their rest system the amplitudes es of particles '3' and '4'. As a result we get all the amplitudes to be quantized in the direction 'from t to their rest system'.

To summarize, we can say; if the transformation of spinless amplitude (stripped amplitude) is given by the same transformation of the arguments

$$S \to S^{cr} \tag{4.3}$$

then the complete amplitude (with spins) transforms according to the formula

$$\langle \lambda_3 \lambda_2 | S^{cr} | \lambda_1 \lambda_4 \rangle = \\ = \sum_{\lambda'} D^{s_1}_{\lambda_1 \lambda_1'}(0, \theta_1, 0)\, D^{s_2}_{\lambda_2 \lambda_2'}(0, \theta_2, 0)\, D^{s_3}_{\lambda_3 \lambda_3'}(0, \theta_3, 0)\, D^{s_4}_{\lambda_4 \lambda_4'}(0, \theta_4, 0) \\ \langle \lambda_3' \lambda_4' | S | \lambda_1' \lambda_2' \rangle \tag{4.4}$$

where the angles θ are calculated by means of the kinematical graph (fig. 2 and 3):

$$\cos\theta_1 = -\cos(412); \qquad \cos\theta_2 = -\cos(123); \qquad \cos\theta_3 = -\cos(234);$$

$$\cos\theta_4 = -\cos(341); \qquad \cos(abc) = \frac{ch(ba)\, ch(bc) - ch(ac)}{sh(ba)\, sh(bc)}$$

5. Spin Addition

It is worth to add few remarks on the addition of spins. Consider $p-p$ scattering as an example. Let us suppose that siuglet and triplet amplitudes are given in the s -channel. Then the spin part of the triplet scattering amplitude is given by the scalar product of triplet spin functions of initial and final states. It is easy to see that this scalar product results in the summation of spins of four particles in the following order $(\vec{1}+\vec{2})+(\vec{3}+\vec{4})=0$

The spin amplitudes in crossed channel will correspond to the summation $(\vec{1}+\vec{4})+(\vec{3}+\vec{2})=0$

These two summations are connected by the usual Recah coefficients, which serve as transformation coefficients in the transition from $p\,\bar{p}$-p scattering in the s -channel, to p-p scattering in t-channel.

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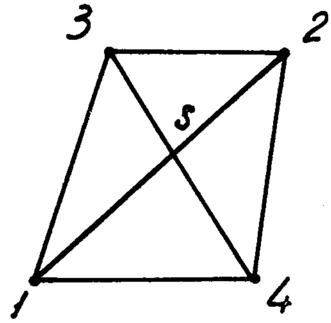

Fig 1

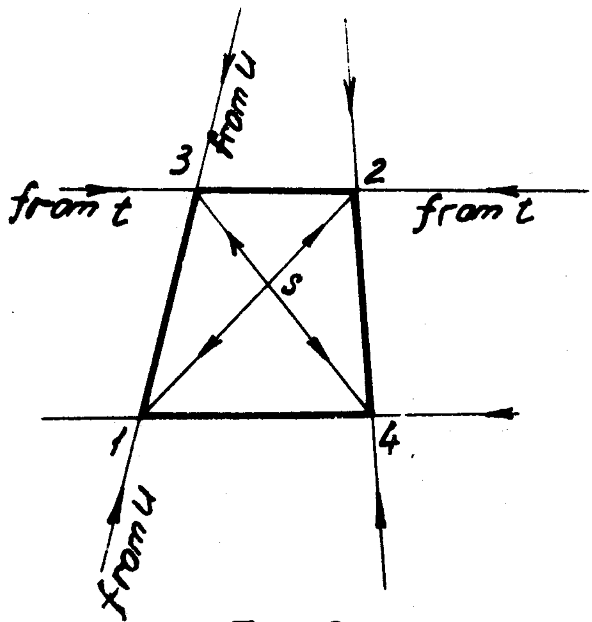

Fig 2

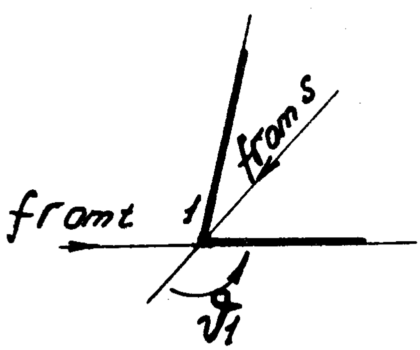

Fig 3