Quantum Mechanics Through the Lens of Finite Groups: Computer Algebra Insights

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Abstract

The use of non-constructive infinities in physical theories can lead to contradictions and non-physical artifacts. Quantum behavior can be fully described using only finite subgroups of the general unitary group — specifically, the Weyl–Heisenberg group and its extension, the Clifford group. By formulating quantum theory in terms of these groups, we completely eliminate the need for the continuous unitary group, which leads to important empirical consequences. Crucially, this approach provides a natural explanation for the observed absence of quantum entanglement and interference between distinct types of elementary particles. Moving away from continuum-based mathematics also requires redefining the concept of quantum states: the continuous projective Hilbert space should be replaced by some combinatorial structure. Using computer algebra calculations, we study a potential framework for building constructive quantum states, governed by a fixed set of physically motivated criteria.

Keywords

cyclic group, Weyl–Heisenberg group, Clifford group, quantum evolution, constructive quantum states

1. Introduction

In standard quantum mechanics, the evolution of a closed system is described by a continuous one-parameter unitary group generated by a Hamiltonian: $U_t = \mathrm{e}^{-\mathbf{i}\frac{H}{\hbar}t} = \left(\mathrm{e}^{-\mathbf{i}\frac{H}{\hbar}}\right)^t = \mathrm{E}^t$. Any continuous one-parameter group is isomorphic to the unitary group $\mathsf{U}(1)$, usually realized as the unit circle in the complex plane. Without loss of describing physical reality, we can assume that time t is an integer parameter, and the operator E is an element of a representation of a finite cyclic group $\mathbb{Z}_N$, where N is a large natural number. In [1], assuming that time t is given in Planck units, estimates are provided $N \sim \begin{cases} \mathrm{Exp}(\mathrm{Exp}(20)) & \text{for } 1 \text{ cm}^3 \text{ of matter,} \\ \mathrm{Exp}(\mathrm{Exp}(123)) & \text{for the entire Universe.} \end{cases}$

In applications, the one-parameter group $\mathsf{U}(1)$ can be used as a continuum approximation of the discrete group $\mathbb{Z}_N$ as $N \to \infty$. However, this approximation fails to capture certain empirically observed fundamental quantum phenomena that depend on the number-theoretic properties of N. Key example: the primary decomposition of $\mathbb{Z}_N$ (via the Chinese remainder theorem) implies the decomposition of a N-dimensional quantum system into completely decoupled subsystems, i.e. there is nether quantum entanglement nor energy interaction between them.

Hermann Weyl was the first to discover [6] that finite groups $\mathbb{Z}_N$, $\mathbb{Z}_N \times \mathbb{Z}_N$, and a central extension of the latter are needed to describe quantum behavior in a finite-dimensional Hilbert space $\mathcal{H}_N$. Although unitary evolution can be fully described by the cyclic group, the product becomes necessary to incorporate the concept of observation into the theoretical framework. The factors of the product group are associated with *mutually unbiased bases* – a concept introduced by Julian Schwinger [4] that provides a mathematical refinement of Bohr's complementarity principle. Subsequently, the ideas of Weyl and Schwinger have been actively developed in

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various fields, including the foundations of quantum theory, quantum information theory [2], and signal processing theory [5].

2. Weyl–Schwinger formalism

Noting that the canonical Heisenberg commutation relation $[\hat{x}, \hat{p}] = \mathbf{i}\hbar\,\mathbb{1}$, – and hence standard quantum mechanics as a whole – can only be realized in an infinite-dimensional Hilbert space, Weyl proved that in physically more meaningful finite-dimensional case, the commutation relation in the Hilbert space $\mathcal{H}_N$ must be of the form $ZX = \omega XZ$, where X is an operator of cyclic permutation of N basis vectors in $\mathcal{H}_N$, Z is the Pontryagin dual of X, ω is a Nth primitive root of unity. Later, Schwinger proved that the eigenbases of operators X and Z in $\mathcal{H}_N$ constitute a mutually unbiased pair. Following 't Hooft's terminology, we will refer to the basis vectors cyclically permuted by the operator X as *ontic* vectors. The pair of operators X and Z generates a projective representation of the group $\mathbb{Z}_N \times \mathbb{Z}_N$ in $\mathcal{H}_N$ — this representation underlies the description of quantum behavior. In short, the main elements of the formalism are as follows.

Let $\mathbb{K}_n$ denote the group of nth roots of unity. The element $\tau = -\,\mathrm{e}^{\pi\mathbf{i}/N}$ generates $\mathbb{K}_N$ if $N = 2k+1$ and $\mathbb{K}_{2N}$ if $N = 2k$. The **Weyl–Heisenberg group** is defined as $\mathrm{WH}(N) = \langle \tau, X, Z \rangle$. The order of $\mathrm{WH}(N)$ is N^3 or $2N^3$ depending on the parity of N. Quantum evolutions are generated by the *displacement operators* $\mathsf{D}_{\mathbf{p}} = \tau^{p_1 p_2} X^{p_1} Z^{p_2}$, $\mathbf{p} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \in \mathbb{Z}^2$, which form the **projective Weyl–Heisenberg group** $\mathrm{PWH}(N) \cong \mathbb{Z}_N \times \mathbb{Z}_N$ of order N^2. The composition $\mathsf{D}_{\mathbf{p}}\mathsf{D}_{\mathbf{q}} = \tau^{\langle \mathbf{p},\mathbf{q} \rangle}\mathsf{D}_{\mathbf{p}+\mathbf{q}}$ contains the symplectic form $\langle \mathbf{p}, \mathbf{q} \rangle = p_2 q_1 - p_1 q_2$. The symmetry group of this form, the symplectic group $\mathrm{Sp}(2, \mathbb{Z}_N)$, is the outer automorphism group of $\mathrm{WH}(N)$.

Combining inner and outer automorphisms, we arrive at a semidirect product called the **Clifford group** $\mathrm{CL}(N) = \mathrm{Aut}(\mathrm{WH}(N)) \cong \mathrm{WH}(N) \rtimes \mathrm{Sp}(2, \mathbb{Z}_N)$.

Traditionally, the Clifford group is defined as the normalizer of the Weyl–Heisenberg group in the unitary group $\mathsf{U}(N)$. The need for $\mathsf{U}(N)$, which remains a relic of continuous theory, follows neither from the description of quantum evolution by finite cyclic groups nor from Weyl's arguments. We will consider the Clifford group exclusively as the symmetry group of the Weyl–Heisenberg group without resorting to a reference to the continuous group $\mathsf{U}(N)$.

In terms of generators, the Clifford group can be presented as $\mathrm{CL}(N) = \langle X, F, S \rangle$, where F is the Fourier transform matrix and $S = \mathrm{diag}\big(\tau^{i(i+N)}, i = 0, \ldots, N-1\big)$.

The projective Clifford group — the quotient group of $\mathrm{CL}(N)$ by its center $\mathrm{Z}(\mathrm{CL}(N))$ — is generated by the same elements, but matrices that differ only by a phase factor are considered equivalent: $\mathrm{PCL}(N) = \langle X, F, S \rangle / \mathrm{Z}(\mathrm{CL}(N))$.

3. Decomposition of a *N*-dimensional quantum system in subsystems

Let $N = \prod_i n_i$ be a factorization of N into pairwise coprime integers $\{n_i\}$. For concreteness, we assume that all factors take the form $n_i = p_i^{\ell_i}$, where $\left\{p_i^{\ell_i}\right\}$ are prime powers with distinct primes. The cyclic group $\mathbb{Z}_N$ can be decomposed into a direct product of *primary cyclic groups* $\mathbb{Z}_N \cong \prod_i \mathbb{Z}_{n_i}$. This isomorphism provides a natural way to decompose the N-dimensional quantum system into subsystems. Specifically, the *global* Hilbert space admits a decomposition as a tensor product of *local* Hilbert spaces $\mathcal{H}_N = \bigotimes_i \mathcal{H}_{n_i}$. Using the shorthand notation GH for element-wise group action and the tensor product identity $AX \otimes BY = (A \otimes B)\,(X \otimes Y)$, the equivalence class of this decomposition — accounting for the freedom in choosing Hilbert space coordinates — can be formally expressed as:

$$G(N)\,\mathcal{H}_N \simeq \prod_i G(n_i) \bigotimes_i \mathcal{H}_{n_i}, \tag{1}$$

where $G(d)$ denotes the symmetry group acting on a d-dimensional Hilbert space. Since the product of local groups is a subgroup of the global group, $\prod_i G(n_i) \leq G(N)$, description (1) reduces to $G(N)\,\mathcal{H}_N \simeq \bigotimes_i \mathcal{H}_{n_i}$, implying that all decompositions related by the action of $G(N)$ are equivalent. The equivalence class is uniquely fixed by the coprime factorization $N = \prod_i n_i$.

The assumption $G(N) = \mathsf{U}(N)$ can lead to artifacts, since the continuous group $\mathsf{U}(N)$ freely "mixes" states between different components of the tensor product, which would lead to non-observable in nature entanglement between fundamental particles of different types.

The assumption $G(N) = \mathsf{CL}(N)$ does not cause such problems, since in the global Clifford group does not contain transformations that mix states between local Hilbert spaces of coprime dimensions. Mathematically, this is expressed by the fact, proved using the Chinese remainder theorem, that the global Clifford group decomposes into a direct product of local ones: $\mathsf{CL}(N) = \prod_i \mathsf{CL}(n_i)$. The absence of quantum entanglement between subsystems means that during any evolutions of the global system, only quantum states that are tensor products of the states of the subsystems (or their classical combinations, called separable states) are possible.

By applying the Chinese remainder theorem to the eigenvalues of the Hamiltonian for the cyclic evolution of the global system [3], we derive the additive decomposition of energy levels between the global quantum system and its local subsystems:

$$E_{k/N} = \sum_i E_{k_i/n_i}\,, \tag{2}$$

where $E_\nu = h\nu$ represents Planck's energy-frequency relation. Equality (2) demonstrates that:

1. The energy of the global system equals the sum of the energies of its subsystems, and
2. No interaction energies exist between components.

This result implies that subsystems of coprime dimensions are completely decoupled, showing no quantum entanglement or energy exchange, and thus permit fully independent investigation.

In prime-dimensional quantum systems, quantum interference can occur. However, they cannot exhibit quantum entanglement due to absence of proper subsystems. Entanglement is possible only in a system of non-trivial prime power dimension.

Consequently, the most physically significant systems for study are:

1. Systems of prime dimensions ($N = p$), only quantum interference is possible, and
2. Systems of prime power dimensions ($N = p^\ell,\ \ell > 1$) where quantum entanglement emerges.

4. Constructive quantum states

In continuous quantum mechanics, the set of pure states in a N-dimensional Hilbert space is the complex projective space $\mathbb{P}(\mathcal{H}_N) = \mathbb{CP}^{N-1}$, which is a homogeneous space of the unitary group $\mathsf{U}(N)$, i.e., $\mathbb{CP}^{N-1}$ is the orbit of an arbitrary unit vector, e.g., $|0\rangle$, under the action of the unitary group: $\mathbb{CP}^{N-1} \cong \mathsf{Orb}_{\mathsf{U}(N)}(|0\rangle) = \mathsf{U}(N)\,|0\rangle$.

In our approach, the group of symmetries of quantum systems is the finite Clifford group, which acts on the set of quantum states non-transitively, splitting it into disjoint orbits. Replacing $\mathsf{U}(N)$ with $\mathsf{CL}(N)$ as the group of symmetries, we assume that the constructive set of pure quantum states, which we denote as $\mathsf{CQS}(N)$, consists of elements of the form $|a\rangle = \sum_{i=0}^{N-1} \varphi_i \alpha_i\,|i\rangle$, where $\alpha_i \in \mathbb{R}$, $\sum_{i=0}^{N-1} \alpha_i^2 = 1$, $\varphi_i \in \mathsf{Z}(\mathsf{CL}(N))$, i.e., the phase factors belong to the center of the Clifford group $\mathsf{CL}(N)$. The set $\mathsf{CQS}(N)$ must 1) be $\mathsf{CL}(N)$-invariant; 2) contain ontic vectors; 3) consist only of elements with rational Born probabilities of transitions between themselves; 4) contain all superpositions of vectors with phase factors from $\mathsf{Z}(\mathsf{CL}(N))$ that satisfy the rationality requirement.

To study the properties of the quantum states corresponding to these requirements, we implemented a procedure to sequentially construct the orbits that make up the set $\mathsf{CQS}(N)$. At

first, the orbit $\mathcal{O}_1 = \mathrm{Orb}_{\mathrm{CL}(N)}(|0\rangle)$ is constructed. In dimensions $N = p^\ell$ this orbit consists of $N(N+1)$ vectors, forming a complete set of $N+1$ mutually unbiased bases and, in particular, contains the ontic basis. Next, we construct other orbits using superpositions of already existing elements. The results of computer experiments possibly indicate the formation of a dense subset in $\mathbb{CP}^{N-1}$.

4.1. Computations in dimensions 2 and 3

Table 1
Generators, centers, and sizes of the Clifford groups in dimensions 2 and 3, $\omega = -\frac{1}{2} + \mathbf{i}\frac{\sqrt{3}}{2}$.

N	X	F	S	$\mathrm{Z}(\mathrm{CL}(N))$	$\lvert\mathrm{CL}(N)\rvert$
2	$\begin{pmatrix}0&1\\1&0\end{pmatrix}$	$\frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix}$	$\begin{pmatrix}1&0\\0&\mathbf{i}\end{pmatrix}$	$\mathbb{K}_8$	192
3	$\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}$	$\frac{1}{\sqrt{3}}\begin{pmatrix}1&1&1\\1&\omega&\omega^2\\1&\omega^2&\omega\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&\omega^2&0\\0&0&\omega^2\end{pmatrix}$	$\mathbb{K}_{12}$	2592

For quantum states $|a\rangle, |b\rangle \in \mathrm{CQS}(N)$, we define the distance function $\mathrm{Dist}(a,b) = 1 - \mathbf{P}(a,b) \equiv \sin^2 \mathrm{D_{FS}}(a,b)$, where $\mathbf{P}(a,b) = |\langle a \mid b\rangle|^2$ is the Born probability of transition between states, and $\mathrm{D_{FS}}(a,b)$ is the length of the geodesic line in the natural for $\mathbb{CP}^{N-1}$ Fubini–Study metric.

To estimate the density of states for a subset $S \subset \mathrm{CQS}(N)$ in the complex projective space, we define the function $\Delta(S) = \max_{a\in S} \min_{b\in S\setminus\{a\}} \mathrm{Dist}(a,b)$ that computes, for each point in S, the distance to its nearest neighbor and then takes the maximum of these minimal distances.

4.1.1. *N* = 2

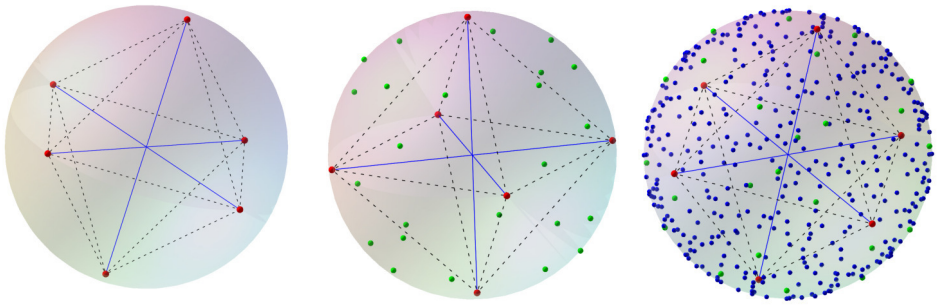
The results permit geometric visualization: pure states reside in the complex projective line $\mathbb{CP}^1$, which is isomorphic to the Riemann sphere (Bloch sphere). The projective Clifford group $\mathrm{PCL}(2) = \mathrm{CL}(2)/\mathbb{K}_8$ has order 24. We calculated — initial steps are shown in Fig. 1 — a subset of states $S \subset \mathrm{CQS}(2)$, which is the union of 986 orbits with a total number of elements 23646. The maximum distances between neighboring states for the initial orbit and for the entire computed set of states are, respectively, $\Delta(\mathcal{O}_1) = 1/2$ and $\Delta(S) = 1/1515 \approx 10^{-3}$.

The initial orbit $\mathcal{O}_1 = \mathrm{Orb}_{\mathrm{CL}(2)}(|0\rangle)$ consists of six vectors, orthogonal pairs of which form a complete set of three mutually unbiased bases:

$$\mathcal{O}_1 = \left\{ |0\rangle, |1\rangle;\ \frac{|0\rangle + |1\rangle}{\sqrt{2}}, \frac{|0\rangle - |1\rangle}{\sqrt{2}};\ \frac{|0\rangle + \mathbf{i}\,|1\rangle}{\sqrt{2}}, \frac{|0\rangle - \mathbf{i}\,|1\rangle}{\sqrt{2}} \right\}. \tag{3}$$

4.1.2. *N* = 3

The order of the projective group $\mathrm{PCL}(3) = \mathrm{CL}(3)/\mathbb{K}_{12}$ is 216. The initial orbit forms a complete set of four mutually unbiased bases: $\mathcal{O}_1 = \left\{ |0\rangle, |1\rangle, |2\rangle; \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}, \frac{1}{\sqrt{3}}\begin{pmatrix}1\\\omega\\\omega^2\end{pmatrix}, \frac{1}{\sqrt{3}}\begin{pmatrix}1\\\omega^2\\\omega\end{pmatrix}; \frac{1}{\sqrt{3}}\begin{pmatrix}1\\\omega^2\\\omega^2\end{pmatrix}, \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\\omega\end{pmatrix}, \frac{1}{\sqrt{3}}\begin{pmatrix}1\\\omega\\1\end{pmatrix}; \frac{1}{\sqrt{3}}\begin{pmatrix}1\\\omega\\\omega\end{pmatrix}, \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\\omega^2\end{pmatrix}, \frac{1}{\sqrt{3}}\begin{pmatrix}1\\\omega^2\\1\end{pmatrix} \right\}$. The set of states $S \subset \mathrm{CQS}(3)$ that we calculated consists of 169 orbits that contain a total of 27237 vectors. The maximum distances between neighboring states for the initial orbit and for the entire calculated set of states are, respectively, $\Delta(\mathcal{O}_1) = 2/3$ and $\Delta(S) = 1/99 \approx 10^{-2}$.



(a) (b) (c)

Figure 1: Initial steps in generating constructive quantum states in dimension 2:
(a) the vectors of orbit (3) form the vertices of an octahedron, whose spatial diagonals represent the three mutually unbiased bases;
(b) pairwise interferences of the vectors in (a) with rational transition probabilities add one orbit of size 24;
(c) pairwise interferences of the vectors in (b) add 16 orbits of size 24 to the set of constructive quantum states.

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