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E5-87-330

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ON THE DISSIPATIVE LAX - PHILLIPS SCATTERING THEORY

Submitted to "Journal of Mathematical Analysis and Applications"

1987

1. Introduction

In [4] C.Foias characterizes all possible scattering matrices occurring in the abstract framework of a dissipative Lax-Phillips scattering theory developed in [6]. The aim of this paper is to continue the investigation of the scattering matrix using a quite different approach to this object. The new approach forces a generalization of the notion of Darlington synthesis as defined in [3] to the case that the contraction-valued function is not an analytical one. This generalized notion which in the paper is called an analytically unitary synthesis of a contraction-valued function reduces to the notion of Darlington synthesis if the operator-valued function is an analytical one. Using this notion we find that a strongly measurable contraction-valued function can be regarded as the scattering matrix of a dissipative Lax-Phillips scattering theory if and only if the adjoint contraction-valued function admits an analytically unitary synthesis. Moreover, taking into account the above mentioned relation to the Darlington synthesis we find that a contraction-valued function arises from an orthogonal dissipative Lax-Phillips scattering theory if and only if the adjoint contraction-valued function is an analytical one and possesses a Darlington synthesis.

From this point of view the conditions (β), (β_*), (5.5.1) - (5.5.4) of C.Foias [4] characterizing the set of occurring scattering matrices in a necessary and sufficient manner are equivalent to the property that the adjoint contraction-valued function has an analytically unitary synthesis. If the adjoint function is an analytical one this means that (β), (β_*), (5.5.1) - (5.5.4) of [4] are necessary and sufficient conditions to guarantee the existence of a Darlington synthesis. At the end of this paper we give a direct proof of these conclusions.

Moreover, we believe that the present approach has the advantage of a great simplicity and transparency. Especially, this transparency appears in the reconstruction theorem which is based on the well-known and widely investigated reconstruction theorem of a conservative Lax-Phillips scattering theory [1,2,6].

In accordance with [4] we use a discret Lax-Phillips framework. For the convenience of the reader we repeat the assumptions of the dissipative Lax-Phillips scattering theory in a discret framwork. A triplet $\{T, \mathcal{D}_+, \mathcal{D}_-\}$ consisting of a contraction T on a separable Hilbert space $\mathcal{H}$ and two subspaces $\mathcal{D}_\pm$ of $\mathcal{H}$ is called a dissipative Lax-Phillips scattering theory if the following assumptions are fulfilled.

(h1) $T\mathcal{D}_+ \subseteq \mathcal{D}_+$, $T^*\mathcal{D}_- \subseteq \mathcal{D}_-$,

(h2) $T\restriction\mathcal{D}_+$ and $T^*\restriction\mathcal{D}_-$ are isometries,

(h3) $\bigcap_{n\in\mathbb{Z}_+} T^n\mathcal{D}_+ = \{0\} = \bigcap_{n\in\mathbb{Z}_+} T^{*n}\mathcal{D}_-$,

(h4) $P^{\mathcal{H}}_{\mathcal{H}\ominus\mathcal{D}_+} T^n \to 0$, $P^{\mathcal{H}}_{\mathcal{H}\ominus\mathcal{D}_-} T^{*n} \to 0$ strongly for $n\to+\infty$.

Let U on $\mathcal{K}$ be the minimal unitary dilation of T. Let

(1.1) $$\mathcal{H}_\pm = \bigvee_{n\in\mathbb{Z}} U^n\mathcal{D}_\pm .$$

Obviously, the subspaces $\mathcal{H}_\pm$ reduce the operator U. We set

(1.2) $$U_\pm = U\restriction\mathcal{H}_\pm .$$

The wave operators $W_\pm$ are defined by

(1.3) $$W_- = \operatorname{s-lim}_{n\to+\infty} T^n P^{\mathcal{K}}_{\mathcal{D}_-} U_-^{*n}$$

and

(1.4) $$W_+ = \operatorname{s-lim}_{n\to+\infty} T^{*n} P^{\mathcal{K}}_{\mathcal{D}_+} U_+^{n}.$$

The scattering operator S,

(1.5) $$S = W_+^* W_-,$$

acts from $\mathcal{H}_-$ into $\mathcal{H}_+$. The operators $U_\pm$ are bilateral shifts. Transforming these operators into their Fourier representations we find that in these representations the scattering operator S acts as a multiplication operator with a strongly measurable contraction-valued function which is called the scattering matrix of the dissipative Lax-Phillips scattering theory.

2. Conservative and nonconservative Lax-Phillips scattering theory

We say the triplet $\{T, \mathcal{D}_+, \mathcal{D}_-\}$ forms a conservative Lax-Phillips scattering theory [5] demanding in addition to (h1) - (h4) that T is a unitary operator. Usually, in this case the condition (h4) is replaced by

(2.1) $$\bigvee_{n\in\mathbb{Z}} T^n \mathcal{D}_{\pm} = \mathcal{H},$$

but it is not hard to see that (h4) and (2.1) are equivalent provided T is a unitary operator.

Definition 2.1. Let $\{T, \mathcal{D}_+, \mathcal{D}_-\}$ be a dissipative Lax-Phillips scattering theory. If there exists a unitary operator U on $\mathcal{K} \supset \mathcal{H}$ as well as orthogonal incoming and outgoing subspaces G_- and G_+ of U such that the conditions

(2.2) $$P^{\mathcal{K}}_{\mathcal{H}} U \restriction \mathcal{H} = T$$

and

(2.3) $$\mathcal{K} = G_+ \oplus \mathcal{H} \oplus G_-$$

are fulfilled and $\{U, \mathcal{D}_+', \mathcal{D}_-'\}$, $\mathcal{D}_{\pm}' = \mathcal{D}_{\pm} \oplus G_{\pm}$, forms a conservative Lax-Phillips scattering theory, then we call $\{U, \mathcal{D}_+', \mathcal{D}_-'\}$, a conservative extension of $\{T, \mathcal{D}_+, \mathcal{D}_-\}$.

Proposition 2.2. Every dissipative Lax-Phillips scattering theory $\{T, \mathcal{D}_+, \mathcal{D}_-\}$ has a conservative extension.

Proof. Let U be the minimal unitary dilation of T on $\mathcal{K}$. Obviously, the condition (2.2) is fulfilled. We introduce the wandering subspaces $\mathcal{L} = ((U - T)\mathcal{H})^-$ and $\mathcal{L}_* = ((I - UT^*)\mathcal{H})^-$ in accordance with [7]. We set

(2.4) $$G_+ = M_+(\mathcal{L})$$

and

(2.5) $$G_- = M(\mathcal{L}_*) \ominus M_+(\mathcal{L}_*).$$

Taking into account the structure of a minimal unitary dilation we get

(2.6) $$\mathcal{K} = G_+ \oplus \mathcal{H} \oplus G_-.$$

Obviously, G_+ and G_- are outgoing and incoming subspaces of U.

Defining now the subspaces $\mathcal{D}_{\pm}'$ in accordance with Definition 2.1 the triplet $\{U, \mathcal{D}_+', \mathcal{D}_-'\}$ forms a conservative Lax-Phillips scattering theory if we establish the relation

(2.7) $$\mathcal{K} = \bigvee_{n\in\mathbb{Z}} U^n \mathcal{D}_{\pm}'.$$

But taking into account Lemma 3 of [4] we get

(2.8) $$\mathcal{K} = \mathcal{H}_+ \oplus M(\mathcal{L}) = \bigvee_{n\in\mathbb{Z}} U^n \mathcal{D}_+'$$

and

(2.9) $$\mathcal{K} = \mathcal{H}_- \oplus M(\mathcal{L}_*) = \bigvee_{n\in\mathbb{Z}} U^n \mathcal{D}_-'$$

which completes the proof. ∎

Let $\{U, \mathcal{D}_+', \mathcal{D}_-'\}$ be a conservative extension of the dissipative Lax-Phillips scattering theory $\{T, \mathcal{D}_+, \mathcal{D}_-\}$. Taking into account Definition 2.1 it is not hard to see that U is a unitary dilation of T.

Using this remark we obtain the invariance of the subspaces $\mathcal{D}_+$ and $\mathcal{D}_-$ with respect to U and U^*, respectively. Hence there are wandering subspaces $\mathcal{N}_\pm \subseteq \mathcal{D}_\pm$ with respect to U such that

(2.10) $$\mathcal{D}_+ = M_+(\mathcal{N}_+),$$

(2.11) $$\mathcal{D}_- = M(\mathcal{N}_-) \ominus M_+(\mathcal{N}_-)$$

and

(2.12) $$\mathcal{H}_\pm = M(\mathcal{N}_\pm).$$

Denoting by $\mathcal{L}$ and $\mathcal{L}_*$ the wandering subspaces of the outgoing and incoming subspaces G_+ and G_-, respectively,

(2.13) $$G_+ = M_+(\mathcal{L})$$

and

(2.14) $$G_- = M(\mathcal{L}_*) \ominus M_+(\mathcal{L}_*),$$

it is not hard to see that the subspaces

(2.15) $$Q_+ = \mathcal{N}_+ \oplus \mathcal{L} \text{ and } Q_- = \mathcal{N}_- \oplus \mathcal{L}_*$$

are also wandering subspaces obeying

(2.16) $$\mathcal{D}'_+ = M_+(Q_+)$$

and

(2.17) $$\mathcal{D}'_- = M(Q_-) \ominus M_+(Q_-).$$

Because $\{U, \mathcal{D}'_+, \mathcal{D}'_-\}$ forms a conservative Lax-Phillips scattering theory we get

(2.18) $$\mathcal{H} = M(Q_\pm).$$

If $\Phi'_\pm$ denotes the Fourier transformation corresponding to the wandering subspaces $Q_\pm$ we find

(2.19) $$\Phi'_+ \mathcal{D}'_+ = H^2(Q_+)$$

and

(2.20) $$\Phi'_- \mathcal{D}'_- = L^2(Q_-) \ominus H^2(Q_-).$$

Moreover, we have

(2.21) $$\Phi'_+ \mathcal{D}_+ = H^2(\mathcal{N}_+),$$

(2.22) $$\Phi'_+ G_+ = H^2(\mathcal{L})$$

and

(2.23) $$\Phi'_- \mathcal{D}_- = L^2(\mathcal{N}_-) \ominus H^2(\mathcal{N}_-),$$

(2.24) $$\Phi'_- G_- = L^2(\mathcal{L}_*) \ominus H^2(\mathcal{L}_*).$$

Let S' be the scattering operator of the conservative extension of $\{T, \mathcal{D}_+, \mathcal{D}_-\}$. The operator $\Phi'_+ S' \Phi'^{-1}_-$ acts as a multiplication operator with a strongly measurable function $\{Q_-, Q_+, S'(t)\}$, values of which are isometries from Q_- onto Q_+ (conservative Lax-Phillips scattering theory!). Usually, this unitary-valued function is called the scattering matrix of the conservative Lax-Phillips scattering theory $\{U, \mathcal{D}'_+, \mathcal{D}'_-\}$.

<u>Proposition 2.3.</u> Let $\{\mathcal{N}_-, \mathcal{N}_+, S(t)\}$ be the scattering matrix yielded by a dissipative Lax-Phillips scattering theory $\{T, \mathcal{D}_+, \mathcal{D}_-\}$. If $\{Q_-, Q_+, S'(t)\}$ denotes the scattering matrix of the conservative extension of $\{T, \mathcal{D}_+, \mathcal{D}_-\}$, then both scattering matrices are related by

$$(2.25) \qquad S(t) = P^{Q_+}_{\mathcal{N}_+} S'(t) \restriction \mathcal{N}_-,$$

$t \in [0, 2\pi)$ a.e..

<u>Proof.</u> Let $W'_\pm$ be the wave operators of the conservative extension defined by

$$(2.26) \qquad W'_\pm = \operatorname*{s-lim}_{n \to \pm\infty} U^{-n} P^{\mathcal{K}}_{\mathcal{D}'_\pm} U^n .$$

Obviously, we have

$$(2.27) \qquad W_\pm = P^{\mathcal{K}}_{\mathcal{H}} W'_\pm \restriction \mathcal{H}_\pm ,$$

which implies

$$(2.28) \qquad P^{\mathcal{K}}_{\mathcal{H}_+} S' \restriction \mathcal{H}_- = S.$$

But (2.28) immediately yields (2.25). ■

In such a way Proposition 2.3 shows us that every scattering matrix of a dissipative Lax-Phillips scattering theory can be regarded as the compression of the scattering matrix of its conservative extension.

<u>3. Scattering matrix and analytically unitary synthesis</u>

Every strongly measurable contraction-valued function can be dilated to a strongly measurable unitary-valued function. Further, it is well-known that every strongly measurable unitary-valued function can be regarded as the scattering matrix of a conservative Lax-Phillips scattering theory. Hence the conjecture seems to be true that in virtue of Proposition 2.3 every strongly measurable contraction-valued function can be thought as the scattering matrix of a dissipative Lax-Phillips scattering theory. But this conjecture is false. The point is that the scattering matrix of a conservative extension obeys some additional properties description of which is the contents of the following

<u>Proposition 3.1.</u> Let $\{U, \mathcal{D}'_+, \mathcal{D}'_-\}$ be a conservative extension of the dissipative Lax-Phillips scattering theory $\{T, \mathcal{D}_+, \mathcal{D}_-\}$. If $\{Q_-, Q_+, S'(t)\}$ denotes the scattering matrix of $\{U, \mathcal{D}'_+, \mathcal{D}'_-\}$, then the contraction-valued functions $\{\mathcal{L}, Q_-, S'(t)^* \restriction \mathcal{L}\}$ and $\{Q_+, \mathcal{L}_*, P^{Q_-}_{\mathcal{L}_*} S'(t)^*\}$ are analytic ones. Moreover, if U is a minimal unitary dilation

of T, then the analytic contraction-valued function $\{\mathcal{L}, \mathcal{L}_*, \theta(\lambda)\}$ defined by

$$\theta(e^{it}) = P^{Q_-}_{\mathcal{L}_*} S'(t)^* \restriction \mathcal{L} \tag{3.1}$$

for a.e. $t \in [0, 2\pi)$ coincides with the characteristic function of T.

Proof. Taking into account the definition of the wave and scattering operators we find

$$P^{\mathcal{K}}_{G_+} S' \restriction \mathcal{D}'_- = P^{\mathcal{K}}_{G_+} \restriction \mathcal{D}'_- = 0. \tag{3.2}$$

But (3.2) yields

$$S'(t)f(t) \perp H^2(\mathcal{L}) \tag{3.3}$$

for every $f \in L^2(Q_-) \ominus H^2(Q_-)$. Hence we obtain

$$S'(t)^* f(t) \perp L^2(Q_-) \ominus H^2(Q_-) \tag{3.4}$$

for every $f \in H^2(\mathcal{L})$. Consequently, $\{\mathcal{L}, Q_-, S'(t)^* \restriction \mathcal{L}\}$ forms an analytical contraction-valued function.

Using the relation

$$P^{\mathcal{K}}_{G_-} S'^* \restriction \mathcal{D}'_+ = 0 \tag{3.5}$$

we similarly conclude that $\{Q_+, \mathcal{L}_*, P^{Q_-}_{\mathcal{L}_*} S'(t)^*\}$ is an analytical contraction-valued function.

To prove the remaining part of the proposition we

remark that the triplet $\{U, G_+, G_-\}$ forms another kind of nonconservative Lax-Phillips scattering theory which is usually called a Lax-Phillips scattering theory with losses. This scattering theory is an orthogonal one which in distinction from the conservative scattering theory does not fulfil the completeness condition (2.1). The wave operators $\tilde{W}_\pm$ of this scattering theory with losses are defined by

$$\tilde{W}_\pm = \operatorname*{s-lim}_{n\to\pm\infty} U^{-n} P^{\mathcal{K}}_{G_\pm} U^n. \tag{3.6}$$

Obviously, we have

$$\tilde{W}_\pm = W'_\pm \restriction G_\pm. \tag{3.7}$$

Hence the scattering operator $\tilde{S} = \tilde{W}^*_+ \tilde{W}_-$ admits the representation

$$\tilde{S} = P^{\mathcal{K}}_{G_+} S' \restriction G_-. \tag{3.8}$$

Taking into account the incoming and outgoing spectral representations given by (2.22) and (2.24) we obtain

$$\tilde{S}(t) = P^{Q_+}_{\mathcal{L}} S'(t) \restriction \mathcal{L}_*, \tag{3.9}$$

where $\{\mathcal{L}_*, \mathcal{L}, \tilde{S}(t)\}$ denotes the scattering matrix of $\{U, G_+, G_-\}$. But it is well-known [1] that by virtue of the minimality of U this scattering matrix coincides with the adjoint characteristic function $\{\mathcal{L}_*, \mathcal{L}, \theta_T(\lambda)^*\}$ of T, i.e.

$$(3.10) \qquad \tilde{S}(t) = \theta_T(e^{it})^*$$

for a.e. $t \in [0,2\pi)$. ■

On the basis of Proposition 3.1 the introduction of the following definition seems to be useful.

<u>Definition 3.2.</u> Let $\{\mathfrak{G}_0, \mathfrak{H}_0, R(t)\}$ be a strongly measurable operator-valued function values of which are contractions acting from the separable Hilbert space $\mathfrak{G}_0$ into the separable Hilbert space $\mathfrak{H}_0$. We say $\{\mathfrak{G}_0, \mathfrak{H}_0, R(t)\}$ admits an analytically unitary synthesis if there exist three analytical contraction-valued functions $\{\mathfrak{G}_1, \mathfrak{H}_0, Z(\lambda)\}$, $\{\mathfrak{G}_0, \mathfrak{H}_1, Y(\lambda)\}$ and $\{\mathfrak{G}_1, \mathfrak{H}_1, X(\lambda)\}$, where $\mathfrak{G}_1$ and $\mathfrak{H}_1$ are separable Hilbert spaces, such that the contraction-valued function $R'(t)$,

$$(3.11) \qquad R'(t) = \begin{pmatrix} X(e^{it}) & Y(e^{it}) \\ Z(e^{it}) & R(t) \end{pmatrix} : \begin{matrix} \mathfrak{G}_1 \\ \oplus \\ \mathfrak{G}_0 \end{matrix} \longrightarrow \begin{matrix} \mathfrak{H}_1 \\ \oplus \\ \mathfrak{H}_0 \end{matrix},$$

forms a unitary-valued function for a.e. $t \in [0,2\pi)$.

We remark that if $\{\mathfrak{G}_0, \mathfrak{H}_0, R(t)\}$ is also an analytical function, then Definition 3.2 coincides with the definition of the Darlington synthesis given in [3].

Now Proposition 3.1 can be formulated as follows.

<u>Theorem3.3.</u> Let $\{\mathcal{N}_-, \mathcal{N}_+, S(t)\}$ be the scattering matrix of a dissipative Lax-Phillips scattering theory. Then the adjoint contraction-valued function $\{\mathcal{N}_+, \mathcal{N}_-, S(t)^*\}$ admits an analytically unitary synthesis.

<u>Proof.</u> By $\{Q_-, Q_+, S'(t)\}$ we denote the scattering matrix of a conservative extension. Taking into account (2.25) and (3.1) we obtain

$$(3.12) \qquad S(t)^* = P^{Q_-}_{\mathcal{N}_-} S'(t)^* \upharpoonright \mathcal{N}_+$$

and

$$(3.13) \qquad \theta(e^{it}) = P^{Q_-}_{\mathcal{L}_*} S'(t)^* \upharpoonright \mathcal{L}$$

for a.e. $t \in [0,2\pi)$. Further we set

$$(3.14) \qquad C(e^{it}) = P^{Q_-}_{\mathcal{N}_-} S'(t)^* \upharpoonright \mathcal{L}$$

and

$$(3.15) \qquad C_*(e^{it}) = P^{Q_-}_{\mathcal{L}_*} S'(t)^* \upharpoonright \mathcal{N}_+,$$

$t \in [0,2\pi)$ a.e.. Because of Proposition 3.1 the contraction-valued functions $\{\mathcal{L}, \mathcal{N}_-, C(\lambda)\}$ and $\{\mathcal{N}_+, \mathcal{L}_*, C_*(\lambda)\}$ are analytical ones. Consequently, the block-matrix representation

$$(3.16) \qquad S'(t)^* = \begin{pmatrix} \theta(e^{it}) & C_*(e^{it}) \\ C(e^{it}) & S(t)^* \end{pmatrix} : \begin{matrix} \mathcal{L} \\ \oplus \\ \mathcal{N}_+ \end{matrix} \longrightarrow \begin{matrix} \mathcal{L}_* \\ \oplus \\ \mathcal{N}_- \end{matrix}$$

defines an analytically unitary synthesis of the adjoint contraction-valued function $\{\mathcal{N}_+, \mathcal{N}_-, S(t)^*\}$. ■

Considering now an orthogonal dissipative Lax-Phillips scattering theory ($\mathcal{D}_+ \perp \mathcal{D}_-$) we obtain the following

Corollary 3.4. Let $\{\mathcal{N}_-, \mathcal{N}_+, S(t)\}$ be the scattering matrix yielded by an orthogonal dissipative Lax-Phillips scattering theory. Then the adjoint scattering matrix $\{\mathcal{N}_+, \mathcal{N}_-, S(t)^*\}$ is an analytical contraction-valued function, which admits a Darlington synthesis.

Proof. Because of the orthogonality we find that the conservative extension is an orthogonal conservative Lax-Phillips scattering theory ($\mathcal{D}'_+ \perp \mathcal{D}'_-$). But this implies that the adjoint scattering matrix $\{Q_+, Q_-, S'(t)^*\}$ of the conservative extension is an inner function of both sides. Applying Proposition 2.3 we complete the proof. ∎

4. Reconstruction

Our next aim is to prove the converse to Theorem 3.3.

Theorem 4.1. Let $\{\mathcal{N}_-, \mathcal{N}_+, S(t)\}$ be a strongly measurable contraction-valued function. If the adjoint function $\{\mathcal{N}_+, \mathcal{N}_-, S(t)^*\}$ admits an analytically unitary synthesis, then $\{\mathcal{N}_-, \mathcal{N}_+, S(t)\}$ can be regarded as the scattering matrix of a dissipative Lax-Phillips scattering theory.

Proof. In accordance with our assumptions we suppose that are separable Hilbert spaces $\mathcal{L}$ and $\mathcal{L}_*$ as well as analytical contraction-valued functions $\{\mathcal{L}, \mathcal{L}_*, \Theta(\lambda)\}$, $\{\mathcal{N}_+, \mathcal{L}_*, C_*(\lambda)\}$ and $\{\mathcal{L}, \mathcal{N}_-, C(\lambda)\}$ such that (3.16) defines an analytically unitary synthesis of $\{\mathcal{N}_+, \mathcal{N}_-, S(t)^*\}$. With the help of the unitary-valued function $\{Q_-, Q_+, S'(t)\}$, $Q_- = \mathcal{N}_- \oplus \mathcal{L}_*$ and $Q_+ = \mathcal{N}_+ \oplus \mathcal{L}$,

$$(4.1) \qquad S'(t) = \begin{pmatrix} \Theta(e^{it})^* & C(e^{it})^* \\ C_*(e^{it})^* & S(t) \end{pmatrix} : \begin{matrix} \mathcal{L}_* & & \mathcal{L} \\ \oplus & \longrightarrow & \oplus \\ \mathcal{N}_- & & \mathcal{N}_+ \end{matrix}$$

we construct a conservative Lax-Phillips scattering theory in the following way. We set $\mathcal{K} = L^2(Q_+)$, $\mathcal{D}'_+ = H^2(Q_+)$ and $\mathcal{D}'_- = S'(L^2(Q_-) \ominus H^2(Q_-))$, where S' denotes the multiplication operator from $L^2(Q_-)$ into $L^2(Q_+)$ induced by the unitary-valued function $\{Q_-, Q_+, S'(t)\}$. Denoting by U the multiplication operator induced by e^{it} on $\mathcal{K} = L^2(Q_+)$, it is not hard to see that the triplet $\{U, \mathcal{D}'_+, \mathcal{D}'_-\}$ forms a conservative Lax-Phillips scattering theory scattering matrix of which coincides with $\{Q_-, Q_+, S'(t)\}$.

Next we define the contraction T. To this end we introduce the subspaces $G_+ = H^2(\mathcal{L})$ and $G_- = S'(L^2(\mathcal{L}_*) \ominus H^2(\mathcal{L}_*))$. Taking into account the properties of the analytically unitary synthesis (4.1) we find that the subspaces G_+ and G_- are orthogonal, i.e. $G_+ \perp G_-$. Moreover, the subspaces G_+ and G_- are invariant with respect to U and U^*, respectively. Consequently, introducing the subspace $\mathcal{H} = \mathcal{K} \ominus (G_+ \oplus G_-)$ the relation

$$(4.2) \qquad T = P^{\mathcal{K}}_{\mathcal{H}} U \restriction \mathcal{H}$$

defines a contraction on $\mathcal{H}$. The operator U is a unitary dilation of T.

The following aim is to define the invariant subspaces $\mathcal{D}_+$ and $\mathcal{D}_-$. We set $\mathcal{D}_+ = H^2(\mathcal{N}_+)$ and $\mathcal{D}_- = S'(L^2(\mathcal{N}_-) \ominus H^2(\mathcal{N}_-))$. Obviously, we have $\mathcal{D}_+ \perp G_+$ and

$\mathcal{D}_+ \perp G_-$ which implies $\mathcal{D}_+ \subseteq \mathcal{H}$. Similarly, we obtain $\mathcal{D}_- \perp G_-$ and $\mathcal{D}_- \perp G_+$ which implies $\mathcal{D}_- \subseteq \mathcal{H}$.

Further we show that $\{T, \mathcal{D}_+, \mathcal{D}_-\}$ forms a dissipative Lax-Phillips scattering theory. Obviously, the subspaces $\mathcal{D}_+$ and $\mathcal{D}_-$ are invariant with respect to U and U^*, respectively. But this implies the invariance of $\mathcal{D}_+$ and $\mathcal{D}_-$ with respect to T and T^*, respectively. Moreover, we get $T \restriction \mathcal{D}_+ = U \restriction \mathcal{D}_+$ and $T^* \restriction \mathcal{D}_- = U^* \restriction \mathcal{D}_-$. But this implies (h2) and (h3).

To prove (h4) we note the relation

$$(4.3) \qquad \mathfrak{K} = L^2(\mathcal{N}_+) \oplus L^2(\mathcal{L}) = \\ = \bigvee_{n \in \mathbb{Z}} U^n \mathcal{D}_+ \oplus \bigvee_{n \in \mathbb{Z}} U^n G_+ .$$

Now for every $m \in \mathbb{Z}$ and every $f \in H^2(\mathcal{N}_+)$ we find

$$(4.4) \qquad \text{s-}\lim_{n \to +\infty} P^{\mathfrak{K}}_{\mathcal{H} \ominus \mathcal{D}_+} U^n U^m f = 0,$$

which implies

$$(4.5) \qquad \text{s-}\lim_{n \to +\infty} P^{\mathfrak{K}}_{\mathcal{H} \ominus \mathcal{D}_+} U^n f = 0$$

for every $f \in L^2(\mathcal{N}_+)$. Similarly, for every $m \in \mathbb{Z}$ and every $g \in H^2(\mathcal{L})$ we get

$$(4.6) \qquad \text{s-}\lim_{n \to +\infty} P^{\mathfrak{K}}_{\mathcal{H} \ominus \mathcal{D}_+} U^n U^m g = 0$$

But (4.6) yields

$$(4.7) \qquad \text{s-}\lim_{n \to +\infty} P^{\mathfrak{K}}_{\mathcal{H} \ominus \mathcal{D}_+} U^n g = 0$$

for every $g \in L^2(\mathcal{L})$. Consequently, taking into account (4.3), (4.5) and (4.7) we obtain $\text{s-}\lim_{n \to +\infty} P^{\mathfrak{K}}_{\mathcal{H} \ominus \mathcal{D}_+} U^n h = 0$ for every $h \in \mathfrak{K}$. Hence we find $\text{s-}\lim_{n \to +\infty} P^{\mathcal{H}}_{\mathcal{H} \ominus \mathcal{D}_+} T^n = 0$. Similarly, we prove $\text{s-}\lim_{n \to +\infty} P^{\mathcal{H}}_{\mathcal{H} \ominus \mathcal{D}_-} T^{*n} = 0$.

Obviously, the triplet $\{U, \mathcal{D}'_+, \mathcal{D}'_-\}$ is a conservative extension of the dissipative Lax-Phillips scattering theory $\{T, \mathcal{D}_+, \mathcal{D}_-\}$. Taking into account Proposition 2.3 and (4.1) we obtain that the scattering matrix of $\{T, \mathcal{D}_+, \mathcal{D}_-\}$ coincides with $\{\mathcal{N}_-, \mathcal{N}_+, S(t)\}$. ∎

Theorem 4.1 implies the following

<u>Corollary 4.2.</u> Let $\{\mathcal{N}_-, \mathcal{N}_+, S(t)\}$ be a strongly measurable contraction-valued function. If the adjoint function $\{\mathcal{N}_+, \mathcal{N}_-, S(t)^*\}$ is an analytical one and admits a Darlington synthesis, then $\{\mathcal{N}_-, \mathcal{N}_+, S(t)\}$ can be regarded as the scattering matrix of an orthogonal dissipative Lax-Phillips scattering theory.

<u>Proof.</u> Using the considerations of Theorem 4.1 it remains to show that the subspaces $\mathcal{D}_+ = H^2(\mathcal{N}_+)$ and $\mathcal{D}_- = S'(L^2(\mathcal{N}_-) \ominus H^2(\mathcal{N}_-))$ are orthogonal. But this is obvious in virtue of the analyticity of $\{\mathcal{N}_+, \mathcal{N}_-, S(t)^*\}$. ∎

<u>5. Analytically unitary synthesis and the solution of C.Foias</u>

An obvious consequence of Theorem 4.1 is the following

<u>Proposition 5.1.</u> The strongly measurable contraction-valued function $\{\mathcal{N}_-, \mathcal{N}_+, S(t)\}$ can be regarded as the scat-

tering matrix of a dissipative Lax-Phillips scattering theory if and only if there exist analytical contraction-valued functions $\{\mathfrak{L}, \mathcal{N}_-, C(\lambda)\}$, $\{\mathcal{N}_+, \mathfrak{L}_*, C_*(\lambda)\}$ and $\{\mathfrak{L}, \mathfrak{L}_*, \theta(\lambda)\}$ such that the relations

$$(5.1) \qquad I = \theta(e^{it})\theta(e^{it})^* + C_*(e^{it})C_*(e^{it})^*,$$

$$(5.2) \qquad 0 = \theta(e^{it})C(e^{it})^* + C_*(e^{it})S(t),$$

$$(5.3) \qquad I = C(e^{it})C(e^{it})^* + S(t)^*S(t)$$

and

$$(5.4) \qquad I = \theta(e^{it})^*\theta(e^{it}) + C(e^{it})^*C(e^{it}),$$

$$(5.5) \qquad 0 = C_*(e^{it})^*\theta(e^{it}) + S(t)C(e^{it}),$$

$$(5.6) \qquad I = C_*(e^{it})^*C_*(e^{it}) + S(t)S(t)^*$$

are fulfilled for a.e. $t \in [0,2\pi)$.

Proof. Let $\{\mathcal{N}_-, \mathcal{N}_+, S(t)\}$ be the scattering matrix of a dissipative Lax-Phillips scattering theory. Then on account of Theorem 3.3 there are analytical functions $\{\mathfrak{L}, \mathcal{N}_-, C(\lambda)\}$, $\{\mathcal{N}_+, \mathfrak{L}_*, C_*(\lambda)\}$ and $\{\mathfrak{L}, \mathfrak{L}_*, \theta(\lambda)\}$ such that (4.1) forms a unitary-valued function. Consequently, we have $S'(t)^*S'(t) = I_{\mathfrak{L}_* \oplus \mathcal{N}_-}$ and $S'(t)S'(t)^* = I_{\mathfrak{L} \oplus \mathcal{N}_+}$ for a.e. $t \in [0,2\pi)$. But these relations imply (5.1) - (5.6).

Conversely, if there are analytical contraction-va-

lued functions such that (5.1) - (5.6) are fulfilled, then we easily check, that the operator-valued function $\{\mathfrak{L}_* \oplus \mathcal{N}_-, \mathfrak{L} \oplus \mathcal{N}_+, S'(t)\}$ performed in accordance with (4.1) is a unitary-valued one. Taking into account Theorem 4.1 we complete the proof. ∎

Proposition 5.1 immediatly yields Proposition 4, Proposition 5 and Proposition 6 of C.Foias [4]. In order to show Proposition 4 and Proposition 5 of [4] we introduce the canonical and $*$-canonical factorizations of the analytical contraction-valued functions $\{\mathcal{N}_+, \mathfrak{L}_*, C_*(\lambda)\}$ and $\{\mathfrak{L}, \mathcal{N}_-, C(\lambda)\}$, respectively. We set $C_*(\lambda) = \mathfrak{B}(\lambda) \cdot B_*(\lambda)$ and $C(\lambda) = B(\lambda)\mathfrak{A}(\lambda)$, where $\{\mathcal{N}_+, P_*, B_*(\lambda)\}$ and $\{P, \mathcal{N}_-, B(\lambda)\}$ are outer and $*$-outer functions, respectively, and $\{P_*, \mathfrak{L}_*, \mathfrak{B}(\lambda)\}$ and $\{\mathfrak{L}, P, \mathfrak{A}(\lambda)\}$ are inner and $*$-inner functions, respectively. Taking into account these factorizations we obtain that (5.3) and (5.6) imply (β) and (β_*) of Proposition 4 of [4]. Introducing in accordance with (5.4.1) and (5.4.7) of [4] the contraction-valued function $\{P, P_*, S_{red}(t)\}$ and using (5.5) we get

$$(5.7) \qquad 0 = D_{S(t)^*}\{\omega_*(t)\mathfrak{B}(e^{it})^*\theta(e^{it}) + \\ + S(t)\omega(t)\mathfrak{A}(e^{it})\}$$

for a.e. $t \in [0,2\pi)$. Because of $S(t)(\mathrm{ima}(D_{S(t)}))^- \subseteq \subseteq (\mathrm{ima}(D_{S(t)^*}))^-$ for a.e. $t \in [0,2\pi)$ we obtain

$$(5.8) \qquad 0 = \omega_*(t)\mathfrak{B}(e^{it})^*\theta(e^{it}) + S(t)\omega(t)\mathfrak{A}(e^{it})$$

for a.e. $t \in [0,2\pi)$. On account of $\omega_*(t)^*\omega_*(t) = I_{P_*}$ and $\mathfrak{A}(e^{it})\mathfrak{A}(e^{it})^* = I_P$ for a.e $t \in [0,2\pi)$ we find

$$(5.9) \qquad S_{red}(t) = -\mathfrak{B}(e^{it})^*\theta(e^{it})\mathfrak{A}(e^{it})^*$$

for a.e. $t \in [0,2\pi)$, which implies (5.5.3) of [4]. The relation (5.5.4) follows from (5.1) and (5.4). It was pointed out in section 6.6 of [4] that the condition (5.5.1) is redundant, since (5.5.1) is a consequence of (β) of [4].

To prove Proposition 6 of [4] it is sufficient to show that under the assumptions of Proposition 6 of [4] there exist analytical contraction-valued functions $\{\mathcal{L}, \mathcal{N}_-, C(\lambda)\}$, $\{\mathcal{N}_+, \mathcal{L}_*, C_*(\lambda)\}$ and $\{\mathcal{L}, \mathcal{L}_*, \theta(\lambda)\}$ such that the relations (5.1) - (5.6) of Proposition 5.1 are fulfilled. Because $\{\mathcal{L}, \mathcal{L}_*, \theta(\lambda)\}$ is given by Proposition 6 of [4] it remains to define $\{\mathcal{L}, \mathcal{N}_-, C(\lambda)\}$ and $\{\mathcal{N}_+, \mathcal{L}_*, C_*(\lambda)\}$. We set

$$(5.10) \qquad C_*(\lambda) = -\mathfrak{B}(\lambda)B_*(\lambda)$$

and

$$(5.11) \qquad C(\lambda) = B(\lambda)\mathfrak{A}(\lambda),$$

$\lambda \in \{z \in \mathbb{C} : |z| < 1\}$. Because of ($\beta$) and ($\beta_*$) of [4] we obtain (5.3) and (5.6). From (5.5.3) of [4] we get

$$(5.12) \qquad \mathfrak{B}(e^{it})^*\theta(e^{it})\mathfrak{A}(e^{it}) = \omega_*(t)^*S(t)\omega(t)$$

for a.e. $t \in [0,2\pi)$. Multiplying on the right by $B(e^{it})^*$ we find

$$(5.13) \qquad \mathfrak{B}(e^{it})^*\theta(e^{it})C(e^{it})^* = \omega_*(t)^*S(t)D_{S(t)}$$

from which we conclude

$$(5.14) \qquad \mathfrak{B}(e^{it})^*\theta(e^{it})C(e^{it})^* = B_*(e^{it})S(t)$$

for a.e. $t \in [0,2\pi)$. But (5.14) yields

$$(5.15) \qquad \mathfrak{B}(e^{it})\mathfrak{B}(e^{it})^*\theta(e^{it})C(e^{it})^* = -C_*(e^{it})S(t)$$

for a.e. $t \in [0,2\pi)$. On account of (5.5.4) of [4] we find $\theta(e^{it})^*\ker(\mathfrak{B}(e^{it})^*) \subseteq \ker(\mathfrak{A}(e^{it}))$ for a.e. $t \in [0,2\pi)$. Using this conclusion we obtain (5.2) from (5.15). Similarly, we prove (5.5).

It remains to show (5.1) and (5.4). Taking into account (5.5.4) of [4] we find

$$(5.16) \qquad \mathfrak{B}(e^{it})^*\theta(e^{it})\theta(e^{it})^*\mathfrak{B}(e^{it}) =$$
$$\mathfrak{B}(e^{it})^*\theta(e^{it})\mathfrak{A}(e^{it})^*\mathfrak{A}(e^{it})\theta(e^{it})^*\mathfrak{B}(e^{it})$$

for a.e. $t \in [0,2\pi)$. By virtue of (5.5.3) of [4] we get

$$(5.17) \qquad \mathfrak{B}(e^{it})^*\theta(e^{it})\theta(e^{it})^*\mathfrak{B}(e^{it}) =$$
$$\omega_*(t)^*S(t)\omega(t)\omega(t)^*S(t)^*\omega_*(t)$$

for a.e. $t \in [0,2\pi)$. On account of (5.4.1) of [4] we conclude

$$(5.18) \qquad \mathcal{B}(e^{it})^* \theta(e^{it}) \theta(e^{it})^* \mathcal{B}(e^{it}) = \omega_*(t)^* S(t) S(t)^* \omega_*(t)$$

for a.e. $t \in [0,2\pi)$. But (5.18) and (5.4.1) of [4] imply

$$(5.19) \qquad \mathcal{B}(e^{it})^* \theta(e^{it}) \theta(e^{it})^* \mathcal{B}(e^{it}) + B_*(e^{it}) B_*(e^{it})^* = \omega_*(t)^* \{S(t)S(t)^* + D^2_{S(t)^*}\} \omega_*(t) = I$$

for a.e. $t \in [0,2\pi)$. Hence we find

$$(5.20) \qquad \mathcal{B}(e^{it}) \mathcal{B}(e^{it})^* D^2_{\theta(e^{it})^*} \mathcal{B}(e^{it}) \mathcal{B}(e^{it})^* = C_*(e^{it}) C_*(e^{it})^*$$

for a.e. $t \in [0,2\pi)$. Taking into account (5.5.4) of [4] it is not hard to see that (5.20) implies (5.1). Similarly, we prove (5.4).

In such a way we have seen that the conditions (β), (β_*), (5.5.2), (5.5.3) and (5.5.4) of [4] are equivalent to the assumptions of Proposition 5.1. Using the notion of analytically unitary synthesis this means that the conditions (β), (β_*), (5.5.2), (5.5.3) and (5.5.4) are equivalent to the existence of an analytically unitary

synthesis of the strongly measurable contraction-valued function $\{\mathcal{N}_+, \mathcal{N}_-, S(t)^*\}$. Hence if $\{\mathcal{N}_+, \mathcal{N}_-, S(t)^*\}$ is an analytical contraction-valued function, then these conditions are equivalent to the existence of a Darlington synthesis of $\{\mathcal{N}_+, \mathcal{N}_-, S(t)^*\}$. The Darlington synthesis is performed by (5.10), (5.11) and (3.16).

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Received by Publishing Department
on May 12, 1987.