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ON CAUCHY-SCHWARZ INEQUALITIES FOR POSITIVE MAPS IN ALGEBRAS OF UNBOUNDED OPERATORS

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0. The study of positive linear maps between operator algebras is a difficult task even if the *-algebras under consideration are low-dimensional matrix algebras. Nevertheless, positive maps on C^*-algebras were the subject of many investigations (cf. /12/ for some review, /2/ and the references therein). In an early stage of the theory Stinespring /11/ discovered the extremely useful notion of completely positive maps and proved an important structure result about such maps. In the sequel it appeared that just this property of complete positivity excludes many pathologies.

An interesting range of problems concerns inequalities for positive maps. The mostly investigated inequalities are of Cauchy-Schwarz-type. Choi /2/ has given some more general inequalities which imply most of the known results, e.g. the now almost classical Cauchy-Schwarz-inequality for positive maps proved by Kadison in 1952.

The situation in the case of topological *-algebras of unbounded operators (or generally, non-normable topological *-algebras) is almost not at all investigated. To the authors best knowledge there are only two attempts to consider positive maps on such algebras. In /10/ Powers extended Stinespring's theorem and Arveson's result on extensions of completely positive maps. Moreover, he related completely positive maps with standard representations of *-algebras. The variant of Stinespring's theorem given by Powers is very general (in some sense too general) because there are considered completely positive maps from *-algebras into the space of all bilinear forms on a linear space. On the other hand, G. and G.A.Lassner /6/ proved Stinespring's theorem for completely positive maps from topological *-algebras in topological algebras of unbounded operators. They related their result with some special kinds of time evolution of physical systems.

The aim of the present paper is to prove some Cauchy-Schwarz-inequalities for positive maps between topological *-algebras of unbounded operators. We intend to investigate other properties of positive and completely positive maps in a forthcoming paper.

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1. Let us shortly fix the notions and notations used in what follows (see e.g. /4,5/). For a dense linear manifold $\mathcal{D}$ in a separable Hilbert space $\mathcal{H}$ the set of linear operators $\mathcal{L}^+(\mathcal{D}) = \{A : A\mathcal{D} \subset \mathcal{D}, A^* \mathcal{D} \subset \mathcal{D}\}$ is a $*$-algebra with respect to the usual operations and the involution $A \longrightarrow A^+ \equiv A^* \restriction \mathcal{D}$. An Op*-algebra $\mathcal{A}(\mathcal{D})$ is a $*$-subalgebra of $\mathcal{L}^+(\mathcal{D})$ containing the identity operator I. The set of bounded operators from $\mathcal{L}^+(\mathcal{D})$ is denoted by $\mathcal{L}^+(\mathcal{D})_b$. The graph topology $t_{\mathcal{A}}$ induced by $\mathcal{A}(\mathcal{D})$ on $\mathcal{D}$ is given by the family of seminorms

$$\mathcal{D} \ni \varphi \longrightarrow \|A\varphi\| \quad \text{for all } A \in \mathcal{A}(\mathcal{D}).$$

Denote $t_{\mathcal{L}^+(\mathcal{D})}$ simply by t. Thus, we have a rigged Hilbert space

$$\mathcal{D}[t] \subset \mathcal{H} \subset \mathcal{D}[t]'.$$

Among the many possible topologies on Op*-algebras we need only the uniform topology $\tau_{\mathcal{D}}$ given by the family of seminorms

$$\mathcal{A}(\mathcal{D}) \ni A \longrightarrow \|A\|_{\mathcal{M}} = \sup_{\varphi,\psi \in \mathcal{M}} |\langle \varphi, A\psi \rangle|,$$

where $\mathcal{M}$ runs over all bounded subsets of $\mathcal{D}[t_{\mathcal{A}}]$. Remark that $\tau_{\mathcal{D}}$ is also defined on $\mathcal{L}(\mathcal{D}, \mathcal{D}')$, especially on $\mathcal{B}(\mathcal{H})$, the set of bounded operators on $\mathcal{H}$.
To simplify the considerations we suppose that $\mathcal{L}^+(\mathcal{D})$ is selfadjoint, i.e.

$$\mathcal{D} = \mathcal{D}_* \equiv \bigcap_{A \in \mathcal{L}^+(\mathcal{D})} \mathcal{D}(A^*)$$

and that $\mathcal{D}[t]$ is an (F)-space.
The set $\mathcal{B}(\mathcal{D}) = \{T : T\mathcal{H} \subset \mathcal{D}, T^*\mathcal{H} \subset \mathcal{D}\}$ is a two-sided $*$-ideal in $\mathcal{L}^+(\mathcal{D})$ with important properties. We mention only those used here /3, 13/:
i) $B \geq 0$, $B \in \mathcal{B}(\mathcal{D})$ implies $B^r \in \mathcal{B}(\mathcal{D})$ for all $r > 0$.
ii) $\mathcal{B}(\mathcal{D})$ is $\tau_{\mathcal{D}}$-dense in $\mathcal{L}(\mathcal{D}, \mathcal{D}')$, hence in $\mathcal{L}^+(\mathcal{D})$ as well as in $\mathcal{B}(\mathcal{H})$.
iii) The topology $\tau_{\mathcal{D}}$ can be given by the set of seminorms

$$A \longrightarrow \|BAB\| \quad \text{for all } B \in \mathcal{B}(\mathcal{D}),\ B \geq 0.$$

Remark that $\mathcal{L}(\mathcal{D}, \mathcal{D}')[\tau_{\mathcal{D}}]$ is complete /5/.
Concerning positive and completely positive maps between Op*-algebras there are several possible definitions. This is caused by the well-known fact that one has two natural notions of positivity of elements of operator algebras which does not coincide in general (i.e. in the non-C*-case). A short discussion in the context of positive maps was given in /6/. Let $\mathcal{A}(\mathcal{D})$ be a $*$-subalgebra of $\mathcal{L}^+(\mathcal{D})$, maybe without unit I and put

$$\mathcal{P}(\mathcal{A}(\mathcal{D})) = \{\sum_{\text{finite}} A_i^+ A_i,\ A_i \in \mathcal{A}(\mathcal{D})\}$$

$$\mathcal{K}(\mathcal{A}(\mathcal{D})) = \{A \in \mathcal{A}(\mathcal{D}) : \langle A\varphi, \varphi \rangle \geq 0 \text{ for all } \varphi \in \mathcal{D}\}.$$

Both sets are cones, $\mathcal{P}(\mathcal{A}(\mathcal{D})) \subset \mathcal{K}(\mathcal{A}(\mathcal{D}))$, and these sets do not coincide in general.
In case $\mathcal{A}(\mathcal{D}) = \mathcal{L}^+(\mathcal{D})$ we use the notations $\mathcal{P}(\mathcal{D})$, $\mathcal{K}(\mathcal{D})$ resp. .
With respect to these cones one has different notions of positive maps between Op*-algebras. For example, a linear map $\Phi : \mathcal{A}(\mathcal{D}_1) \longrightarrow \mathcal{A}(\mathcal{D}_2)$ is $(\mathcal{P}(\mathcal{A}(\mathcal{D}_1)), \mathcal{P}(\mathcal{A}(\mathcal{D}_2)))$-positive, if $\Phi(A) \in \mathcal{P}(\mathcal{A}(\mathcal{D}_2))$ for all $A \in \mathcal{P}(\mathcal{A}(\mathcal{D}_1))$. Analogous to linear functionals one uses the notion strongly positive linear maps for such ones which map $\mathcal{K}(\mathcal{A}(\mathcal{D}_1))$ into $\mathcal{K}(\mathcal{A}(\mathcal{D}_2))$.
Again analogous to linear functionals one has the following result:

<u>Lemma 1</u>

Let Φ be a continuous linear map from $\mathcal{L}^+(\mathcal{D}_1)[\tau_{\mathcal{D}_1}]$ into $\mathcal{L}^+(\mathcal{D}_2)[\tau_{\mathcal{D}_2}]$. Then for Φ all possible positivity notions in the context of the cones defined above coincide.

<u>Proof:</u>
One uses the fact that the cone $\mathcal{P}(\mathcal{B}(\mathcal{D}_1))$ is $\tau_{\mathcal{D}_1}$-dense in $\mathcal{K}(\mathcal{D}_1)$ /8/. This implies that $(\mathcal{P}(\mathcal{D}_1), \mathcal{K}(\mathcal{D}_2))$-positive maps are also $(\mathcal{K}(\mathcal{D}_1), \mathcal{K}(\mathcal{D}_2))$-positive. All other implications are trivial.

q.e.d.

Thus, we must not distinguish between different positivity notions if we consider $\tau_{\mathcal{D}}$-continuous maps between maximal Op*-algebras.
Next we repeat the notion of completely positive maps. Let M_n denote the algebra of all (nxn)-matrices. Then for any Op*-algebra $\mathcal{A}(\mathcal{D})$ the tensor product $\mathcal{A}(\mathcal{D})^{(n)} \equiv \mathcal{A}(\mathcal{D}) \otimes M_n$ is the Op*-algebra of (nxn)-matrices with entries from $\mathcal{A}(\mathcal{D})$ defined on $\mathcal{D} \oplus \ldots \oplus \mathcal{D}$ (n-fold sum). A linear map $\Phi : \mathcal{A}(\mathcal{D}_1) \longrightarrow \mathcal{A}(\mathcal{D}_2)$ extends to a linear map $\Phi_n : \mathcal{A}(\mathcal{D}_1)^{(n)} \longrightarrow \mathcal{A}(\mathcal{D}_2)^{(n)}$, namely, $\Phi_n = \Phi \otimes I$, i.e. $\Phi_n((A_{ij})) = (\Phi(A_{ij}))$, $1 \leq i,j \leq n$.
Φ is called n-positive, if Φ_n is positive. Φ is called completely positive if it is n-positive for all natural n. Clearly, these positivity notions must be specified by indicating the corresponding cones.
Finally, Φ is unital if $\Phi(I) = I$.

Next we describe a restriction-extension procedure for positive, $\tau_{\mathcal{D}}$-continuous maps (cf. /8/ for the analogous procedure for functionals). We formulate it only for maps on $\mathcal{L}^+(\mathcal{D})$ to simplify the notations. The generalization to maps between different maximal Op^*-algebras is obvious.

Lemma 2

Let Φ be a linear, positive $\tau_{\mathcal{D}}$-continuous map on $\mathcal{L}^+(\mathcal{D})$. Then there exists a unique linear, positive $\tau_{\mathcal{D}}$-continuous map $\tilde{\Phi}$ on $\mathcal{B}(\mathcal{H})$ with $\tilde{\Phi}(A) = \Phi(A)$ for all $A \in \mathcal{L}^+(\mathcal{D})_b$ if and only if $\Phi(I)$ is bounded. $\tilde{\Phi}$ is called the extension of Φ to $\mathcal{B}(\mathcal{H})$.

Proof:

Clearly, if $\tilde{\Phi}$ exists, $\tilde{\Phi}(I)$ must be bounded. Now let $\Phi(I)$ be a bounded operator. Let $\overline{\Phi}$ denote the continuation of Φ onto $\mathcal{L}(\mathcal{D}, \mathcal{D}')$. If $\tilde{\Phi}$ exists, it must coincide with $\overline{\Phi}$ on $\mathcal{B}(\mathcal{H})$. So we have to prove that $\overline{\Phi}(A)$ is bounded and positive for any bounded, positive $A \in \mathcal{B}(\mathcal{H})$. Since Φ is $\tau_{\mathcal{D}}$-continuous, for any t-bounded $\mathcal{M} \subset \mathcal{D}$ there is a $B \geq 0$, $B \in \mathcal{B}(\mathcal{D})$ so that

$$\|\Phi(A)\|_{\mathcal{M}} \leq \|BAB\| \quad \text{for all } A \in \mathcal{L}^+(\mathcal{D}).$$

By continuity,

$$\|\overline{\Phi}(A)\|_{\mathcal{M}} \leq \|BAB\| \quad \text{for all } A \in \mathcal{L}(\mathcal{D}, \mathcal{D}').$$

For $B = \int_0^b \lambda \, dE_\lambda$ we put $P_{n,B} = \int_{1/n}^b dE_\lambda$ and $A_{n,B} = P_{n,B} A P_{n,B}$.

It is easy to see that $\{A_{n,B} : n \in \mathbb{N},\ B \geq 0,\ B \in \mathcal{B}(\mathcal{D})\} \equiv \{A_\alpha\}$ forms a net converging to A with respect to $\tau_{\mathcal{D}}$. Hence,

$$\overline{\Phi}(A_\alpha) \longrightarrow \overline{\Phi}(A)$$

in view of the $\tau_{\mathcal{D}}$-continuity of Φ, $\overline{\Phi}$. For $A \in \mathcal{B}(\mathcal{H})$, $A \geq 0$ one has $0 \leq A \leq cI$. Moreover, $0 \leq A_\alpha \leq cI$, i.e. $0 \leq \overline{\Phi}(A_\alpha) \leq c\,\Phi(I)$. Hence, $\overline{\Phi}(A) = \tau_{\mathcal{D}}\text{-}\lim \Phi(A_\alpha) \leq c\,\Phi(I)$, $\overline{\Phi}(A)$ is positive. To see that $\overline{\Phi}(A)$ is indeed from $\mathcal{B}(\mathcal{H})$ one can consider e.g. the positive quadratic forms or sesquilinear forms generated by the operators $\Phi(A_\alpha)$. Then these forms converge to a bounded form (because of the uniform boundedness of $\{\Phi(A_\alpha)\}$ i.e. to a continuous form and that form is generated by an operator from $\mathcal{B}(\mathcal{H})$, namely by $\overline{\Phi}(A)$. Thus, $\overline{\Phi}(A) = \tilde{\Phi}(A) \in \mathcal{B}(\mathcal{H})$. But this implies $\tilde{\Phi}(X)$ bounded for all $X \in \mathcal{B}(\mathcal{H})$.

q.e.d.

Remarks 3

i) We start with the trivial remark that $\Phi(I) \in \mathcal{L}^+(\mathcal{D})_b$ implies $\Phi(\mathcal{L}^+(\mathcal{D})_b) \subset \mathcal{L}^+(\mathcal{D})_b$.

ii) The proof of Lemma 2 contains also the argument for the fact that the cone $\mathcal{P}(\mathcal{B}(\mathcal{D})) = \mathcal{K}(\mathcal{B}(\mathcal{D}))$ is $\tau_{\mathcal{D}}$-dense in $\mathcal{P}(\mathcal{B}(\mathcal{H})) = \mathcal{K}(\mathcal{B}(\mathcal{H}))$.

iii) For (positive) linear functionals one has also the "inverse" procedure to that described in Lemma 2. Namely, any $\tau_{\mathcal{D}}$-continuous linear functional on $\mathcal{B}(\mathcal{H})$ gives rise to a $\tau_{\mathcal{D}}$-continuous (positive) linear functional on $\mathcal{L}^+(\mathcal{D})$, so that both coincide on $\mathcal{L}^+(\mathcal{D})_b$. Concerning $\tau_{\mathcal{D}}$-continuous (positive) maps on $\mathcal{B}(\mathcal{H})$ this seems in general to be not true. We were unable to give natural additional conditions which imply such an extension procedure, i.e. conditions so that the extension $\overline{\Phi}$ of Φ to $\mathcal{L}(\mathcal{D}, \mathcal{D}')$ leaves $\mathcal{L}^+(\mathcal{D})$ invariant.

iv) Lemma 2 implies also that $\Phi(A^+) = \Phi(A)^+$. To see this, let $\tilde{\Phi}$ be the extension of Φ onto $\mathcal{B}(\mathcal{H})$. Then $\tilde{\Phi}(A^*) = \tilde{\Phi}(A)^*$ for all $A \in \mathcal{B}(\mathcal{H})$, hence $\Phi(A^+) = \Phi(A)^+$ for all $A \in \mathcal{L}^+(\mathcal{D})_b$. The existence of $A_\alpha \in \mathcal{B}(\mathcal{D})$ with $A_\alpha \longrightarrow A$ and $A_\alpha^+ \longrightarrow A^+$ implies $\Phi(A_\alpha^+) = \Phi(A_\alpha)^+$, hence $\Phi(A^+) = \Phi(A)^+$ for all $A \in \mathcal{L}^+(\mathcal{D})$.

2. Now we give some variants of the Cauchy-Schwarz inequality. We start with the generalization of the classical result of Kadison.

Theorem 4

Let Φ be a unital, positive continuous map from $\mathcal{L}^+(\mathcal{D}_1)[\tau_{\mathcal{D}_1}]$ into $\mathcal{L}^+(\mathcal{D}_2)[\tau_{\mathcal{D}_2}]$. Then for all $A = A^+ \in \mathcal{L}^+(\mathcal{D}_1)$ the following Cauchy-Schwarz inequality holds:

$$\Phi(A^2) \geq \Phi(A)^2.$$

Proof:

It must be shown that for all $\psi \in \mathcal{D}_2$: $\langle \Phi(A^2)\psi, \psi \rangle \geq \langle \Phi(A)^2\psi, \psi \rangle$

Now let $\psi \in \mathcal{D}_2$ be given. Choose a t-bounded set $\mathcal{M} \subset \mathcal{D}_2$ with the following two properties:

i) $\psi \in \mathcal{M}$

ii) $\mathcal{M}$ is absolutely convex and $\bigcup_{\lambda > 0} \{\lambda \mathcal{M}\}$ is $\| \ \|$-dense in $\mathcal{H}_2$, the completion of $\mathcal{D}_2$.

Since Φ is continuous, there is a $B \in \mathcal{B}(\mathcal{D}_1)$, $B \geq 0$ with

$$\|\Phi(A)\|_{\mathcal{M}} \leq \|BAB\| \quad \text{for all } A \in \mathcal{L}^+(\mathcal{D}_1). \qquad (1)$$

Let $B = \int_0^b \lambda dE_\lambda$ and put $P_n = \int_{1/n}^b dE_\lambda$. Then $P_n \in \mathcal{B}(\mathcal{D}_1)$ and $A_n = P_n A P_n \in \mathcal{B}(\mathcal{D}_1)$ as well as $\| B(A_n - A)B \| \longrightarrow 0$ for $n \longrightarrow \infty$.

Thus, (1) implies

$$\| \Phi(A_n - A) \|_{\mu} \longrightarrow 0, \text{ especially } \langle \Phi(A_n)\psi, \psi \rangle \longrightarrow \langle \Phi(A)\psi, \psi \rangle$$

Property ii) gives us

$$\langle \Phi(A_n)\psi, \chi \rangle \longrightarrow \langle \Phi(A)\psi, \chi \rangle \tag{2}$$

for all χ from a $\| \ \|$-dense subset of $\mathcal{H}_2$. Because of $A_n = A_n^*$ the Cauchy-Schwarz inequality for $\tilde{\Phi}$ on $\mathcal{B}(\mathcal{H}_1)$ can be applied and one gets (remember $\Phi = \tilde{\Phi}$ on $\mathcal{B}(\mathcal{D}_1)$):

$$\langle \Phi(A_n^2)\psi, \psi \rangle \geq \langle \Phi(A_n)^2\psi, \psi \rangle \tag{3}$$

Therefore

$$\| \Phi(A_n)\psi \|^2 \leq \langle \Phi(A_n^2)\psi, \psi \rangle \leq \| \Phi(A_n^2) \|_{\mu} \leq \| BA_n^2 B \| \leq$$

$$= \| BP_nAP_nAP_nB \| \leq \| BA \| \cdot \| AB \| .$$

Thus, $(\Phi(A_n)\psi)$ is a $\| \ \|$-bounded sequence and this means together with (2) that $(\Phi(A_n)\psi)$ convergens weakly to $\Phi(A)\psi$. Now we combine this fact with the lower semicontinuity of the norm (see e.g. /9/):

For given $\varepsilon > 0$ there is an $N_1(\varepsilon)$ so that for all $n \geq N_1(\varepsilon)$:

$$\langle \Phi(A_n)^2\psi, \psi \rangle = \| \Phi(A_n)\psi \|^2 \geq \underline{\lim} \| \Phi(A_n)\psi \|^2 - \varepsilon$$
$$\geq \| \Phi(A)\psi \|^2 - \varepsilon \tag{4}$$

On the other hand $A_n^2 = AP_nA + AP_nA(P_n - I) + (P_n - I)AP_nAP_n$

Hence

$$\langle \Phi(A_n^2)\psi, \psi \rangle = |\langle \Phi(A_n^2)\psi, \psi \rangle| \leq |\langle \Phi(AP_nA)\psi, \psi \rangle| +$$
$$+ |\langle \Phi((AP_nA)(P_n - I))\psi, \psi \rangle| + |\langle \Phi((P_n - I)AP_nAP_n)\psi, \psi \rangle| \leq$$
$$\leq \langle \Phi(A^2)\psi, \psi \rangle + \| BAP_nA(P_n - I)B \| + \| B(P_n - I)AP_nAP_nB \| . \tag{5}$$

The first term on the right-hand side is a consequence of $\langle AP_nA\psi, \psi \rangle \leq \langle A^2\psi, \psi \rangle$ and the positivity of Φ, the rest is implied by (1). Now we show that the second and third term on the right-hand side of (5) go to zero for $n \longrightarrow \infty$. This is easy to see: $\| BAP_nA(P_n - I)B \| \leq \| BA \| \cdot \| P_n \| \cdot \| AB^{1/2} \| \cdot \| (P_n - I)B^{1/2} \|$ and this last factor goes to zero. The third term is estimated similarly.

Thus for $n \geq N_2(\varepsilon)$ we obtain

$$\langle \Phi(A_n^2)\psi, \psi \rangle \leq \langle \Phi(A^2)\psi, \psi \rangle + \varepsilon . \tag{6}$$

Putting (4) and (6) together, it follows that for $n \geq N(\varepsilon) = \max(N_1(\varepsilon), N_2(\varepsilon))$:

$$\langle \Phi(A^2)\psi, \psi \rangle + \varepsilon \geq \langle \Phi(A)^2\psi, \psi \rangle - \varepsilon .$$

Since ε was arbitrary, the desired inequality is proved.

q.e.d.

Next we generalize an interesting result of Choi /2/ which says that for any 2-positive map on $\mathcal{B}(\mathcal{H})$ the Cauchy-Schwarz inequality

$\Phi(A^*A) \geq \Phi(A^*)\Phi(A)$ is valid for all $A \in \mathcal{B}(\mathcal{H})$.

<u>Theorem 5</u>

Let Φ be a 2-positive unital continuous linear map from $\mathcal{L}^+(\mathcal{D}_1)[\tau_{\mathcal{D}_1}]$ to $\mathcal{L}^+(\mathcal{D}_2)[\tau_{\mathcal{D}_2}]$. Then

$$\Phi(A^+A) \geq \Phi(A)^+\Phi(A) \quad \text{for all } A \in \mathcal{L}^+(\mathcal{D}_1).$$

For the proof we need the following version of the Theorem above.

<u>Proposition 6</u>

Let Φ be a 2-positive linear map from $\mathcal{L}^+(\mathcal{D}_1)$ to $\mathcal{L}^+(\mathcal{D}_2)$ so that $\Phi(I)$ is bounded and has a bounded inverse. Then for all $A \in \mathcal{L}^+(\mathcal{D}_1)_b$ the Cauchy-Schwarz inequality $\Phi(A^+A) \geq \Phi(A)^+\Phi(A)$ is valid.

<u>Proof:</u>

Here ideas from /2/ are used. We repeat them to make the proof self-consistent. Let $R, S, T \in \mathcal{L}^+(\mathcal{D}_1)_b$, $T \geq 0$, T^{-1} exists as a bounded operator from $\mathcal{B}(\mathcal{H}_1)$. First remark that

$$\begin{pmatrix} T & S \\ S^* & R \end{pmatrix} \geq 0 \quad \text{iff} \quad R \geq S^*T^{-1}S . \tag{7}$$

To see this, put

$$P = \begin{pmatrix} T & S \\ S^* & R \end{pmatrix}, \quad Q = \begin{pmatrix} I & 0 \\ 0 & R - S^*T^{-1}S \end{pmatrix}, \quad X = \begin{pmatrix} T^{1/2} & T^{-1/2}S \\ 0 & I \end{pmatrix}.$$

Then it is easy to check that X^{-1} exists and $P = X^* Q X$. Hence $P \geq 0$ if and only if $Q \geq 0$ and this is clearly the case if and only if $R - S^*T^{-1}S \geq 0$. Consequently, $P \geq 0$ iff $R = S^*T^{-1}S$, i.e. (7).

Now let $A \in \mathcal{L}^+(\mathcal{D}_1)_b$ arbitrary. Put

$$B = \begin{pmatrix} I & A \\ A^+ & A^+A \end{pmatrix}.$$

Then $B \geq 0$ and because Φ is 2-positive, also

$$\Phi(B) = \begin{pmatrix} \Phi(I) & \Phi(A) \\ \Phi(A^+) & \Phi(A^+A) \end{pmatrix} \geq 0 .$$

Since $\Phi(A)$, $\Phi(A^+)$ and $\Phi(A^+A)$ are bounded operators and $\Phi(I) = T$ fulfills the conditions for (7), this inequality can be applied and we get the desired result

$$\Phi(A^+A) \geq \Phi(A^+)\Phi(A)$$

q.e.d.

<u>Proof of Theorem 5:</u>

We must show that for all $A \in \mathcal{L}^+(\mathcal{D}_1)$ and all $\psi \in \mathcal{D}_2$

$$\langle \Phi(A^+A)\psi , \psi \rangle \geq \langle \Phi(A^+)\Phi(A)\psi , \psi \rangle .$$

So let $A \in \mathcal{L}^+(\mathcal{D}_1)$, $\psi \in \mathcal{D}_2$ be fixed. As in the proof of Theorem 4 let $\mathcal{M} \subset \mathcal{D}_2[t]$ be an absolutely convex set, bounded, $\psi \in \mathcal{M}$, $\bigcup_\lambda \{\lambda \mathcal{M}\}$ is norm-dense in $\mathcal{H}_2$. Then there is a $B \in \mathcal{B}(\mathcal{D}_1)$, $B \geq 0$ with

$$\|\Phi(A)\|_{\mathcal{M}} \leq \|BAB\| .$$

Let P_n, A_n have the same meaning as in the proof of Theorem 4. Then $\|B(A_n - A)B\| \longrightarrow 0$, $\|B(A_n^+ - A^+)B\| \longrightarrow 0$ and consequently $\|\Phi(A_n - A)\|_{\mathcal{M}} \longrightarrow 0$, $\|\Phi(A_n^+ - A^+)\|_{\mathcal{M}} \longrightarrow 0$.

Especially

$$\langle \Phi(A_n)\psi , \chi \rangle \longrightarrow \langle \Phi(A)\psi , \chi \rangle , \quad \langle \Phi(A_n^+)\psi , \chi \rangle \longrightarrow \langle \Phi(A^+)\psi , \chi \rangle \qquad (8)$$

for all χ from a norm-dense set in $\mathcal{H}_2$.

Since Φ is 2-positive, Proposition 6 applies for A_n, i.e.

$$\langle \Phi(A_n^+A_n)\psi , \psi \rangle \geq \langle \Phi(A_n^+)\Phi(A_n)\psi , \psi \rangle \quad \text{i.e.}$$

$$\|\Phi(A_n)\psi\|^2 \leq \|\Phi(A_n^+A_n)\|_{\mathcal{M}} \leq \|BA_n^+A_nB\| \leq \|BA^+\| \cdot \|AB\| .$$

That means $(\Phi(A_n)\psi)$ is a $\| \ \|$-bounded sequence. Replacing A, A_n by A^+, A_n^+ one gets that also $(\Phi(A_n^+)\psi)$ is $\| \ \|$-bounded. Together with (8) this implies that $\Phi(A_n)\psi \longrightarrow \Phi(A)\psi$, $\Phi(A_n^+)\psi \longrightarrow \Phi(A^+)\psi$ weakly on $\mathcal{H}_2$. Now the proof proceeds as that of Theorem 4. One has only to replace A^2, A_n^2 resp. by A^+A, $A_n^+A_n$ resp. Again the lower semi-continuity of the norm is used and so on.

q.e.d.

Let us mention some more inequalities for positive maps in C^*-algebras. Suppose that Φ is a unital positive linear map from $\mathcal{A}$ to $\mathcal{B}$.

i) $\Phi(A^*A) \geq \Phi(A^*)\Phi(A)$ and $\Phi(A^*A) \geq \Phi(A)\Phi(A^*)$ for every normal $A \in \mathcal{A}$, cf. /12/.

ii) If $T, A \in \mathcal{A}$ with $T \geq A^*A$ and $TA = AT$, then $\Phi(T) \geq \Phi(A^*) \cdot \Phi(A)$ and $\Phi(T) \geq \Phi(A) \cdot \Phi(A^*)$, /2/.

Althrough i) is a special case of ii) we give both results to demonstrate the difficulties in generalizing them to Op^*-algebras. To generalize i) to $\mathcal{L}^+(\mathcal{D})$, the proof of Theorem 4 suggests to proceed as follows. Let $A \in \mathcal{L}^+(\mathcal{D})$ be normal, then for every $B \in \mathcal{B}(\mathcal{D})$, $B \geq 0$ we construct an appropriate sequence $(A_n) \subset \mathcal{B}(\mathcal{D})$ of normal operators so that $\|B(A - A_n)B\| \longrightarrow 0$. With the help of this sequence we try to reduce the problem to the bounded case. Unfortunately, we are not able to construct an approximating sequence consisting of normal operators.

To generalize ii) there would be two possibilities. A first one is to construct such approximating sequences (A_n), (T_n) from $\mathcal{B}(\mathcal{D})$ that A_n, T_n fulfil the assumptions of ii) for each n. But this seems to be yet much more hopeless than the game for normal operators.

A second possibility would be an adaption of the proof given in /2/. But this proof contains two features which are crucial in $\mathcal{L}^+(\mathcal{D})$. First, i) must be valid. Secondly, the proof works with square roots of operators. If $\mathcal{A}$ is a C^*-algebra, they are contained in $\mathcal{A}$. But this is not the case for $\mathcal{L}^+(\mathcal{D})$.

At the end let us demonstrate how proofs are simplified if one works with completely positive maps. Let us recall the generalization of Stinespring's theorem given in /6/.

Let $\mathcal{R}$ be a $*$-algebra with identity e, $\mathcal{A}(\mathcal{D})$ an Op^*-algebra. A $(\mathcal{P}(\mathcal{R}), \mathcal{K}(\mathcal{A}(\mathcal{D})))$-completely positive map Φ from $\mathcal{R}$ into $\mathcal{A}(\mathcal{D})$ is of the form

$$\Phi(a) = V^* \rho(a) V \quad \text{for all } a \in \mathcal{R} ,$$

where $a \longrightarrow \rho(a)$ is a $*$-representation of $\mathcal{R}$ onto an Op^*-algebra $\rho(\mathcal{R}) = \mathcal{A}_1$ on $\mathcal{D}_1 = \mathcal{D}_1[t_{\mathcal{A}_1}]$, $V \in \mathcal{L}(\mathcal{D}[t_{\mathcal{A}}], \mathcal{D}_1[t_{\mathcal{A}_1}])$ and V^* is the adjoint map to V, i.e. $V^* \in \mathcal{L}(\mathcal{D}_1[t_{\mathcal{A}_1}]^*, \mathcal{D}[t_{\mathcal{A}}]^*)$.

From the proof of this result as well as of that of the classical Stinespring theorem one can derive some additional information.

For example, if Φ is a unital map, then V, V^* are isometric maps (for V^* this means e.g. if restricted to $\mathfrak{D}_1$ which is contained in $\mathfrak{D}_1^*$ in a canonical way).
Now we get the Cauchy-Schwarz inequality almost trivially.

Proposition 7

Let $\Phi : \mathfrak{A} \longrightarrow \mathcal{L}^+(\mathfrak{D})$ be a unital completely positive map, then

$$\Phi(a^*a) \geq \Phi(a^*)\,\Phi(a) \quad \text{for all } a \in \mathfrak{A}.$$

Proof:

The isometry of V implies

$$\Phi(a^*)\,\Phi(a) = V^*\rho(a^*)V\,V^*\rho(a)V \leq V^*\rho(a^*a)V = \Phi(a^*a) .$$

q.e.d.

A short proof of result i) mentioned on page 9 is based just on the fact that positive maps on commutative C^*-algebras are completely positive.
Hence, it would be quite useful to extend this result to general Op^*-algebras, i.e.

Problem

Under which conditions a strongly positive unital linear map on a commutative Op^*-algebra $\mathcal{A}(\mathfrak{D})$ is completely positive?

One way to attack this problem could be to construct a suitable representation of the Op^*-algebra as function algebra and then proceed as in the commutative case for C^*-algebras.

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