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A NEW CLASS OF SUPERCONFORMAL SIGMA MODELS WITH THE WESS-ZUMINO ACTION

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I. INTRODUCTION

Nonlinear D = 2 sigma models with Wess-Zumino (WZ) terms[1-3] have a wide range of uses in string theories. One of the most important applications concerns the strings on group manifolds (see, e.g.[4] and references therein), where WZ sigma models provide a consistent description of group coordinates, ensuring D = 2 conformal symmetry at the full quantum level. Conformal invariance is achieved with a fixed ratio of the overall sigma model coupling constant and the coefficient of WZ term[2]. A characteristic feature of conformally invariant WZ sigma models is the presence of new symmetry under two commuting Kac-Moody gauge groups which are realized as left and right multiplications of the basic group element[2,5] $g(x)$:

$$g'(x^+,x^-)=g_L(x^-)g(x^+,x^-)g_R(x^+)\,, \quad x^{\pm}=\frac{1}{2}(x^0 \pm x^1)\,. \tag{1.1}$$

Promoting the group-manifold string action to a world-sheet supersymmetric one involves a proper supersymmetrization of conformally invariant bosonic WZ action.

Superextensions of the group space WZ sigma models explored so far[5-7] possess N = 1 D = 2 supersymmetry. Internal symmetry of corresponding actions commutes with supersymmetry, both the bosonic and fermionic fields are assigned to adjoint representation of internal symmetry group. At the points of conformal invariance, rigid D = 2 supersymmetry and internal symmetry are enlarged, respectively, to infinite-dimensional N = 1 , D = 2 superconformal symmetry and two commuting gauge symmetries which are a superextension of bosonic symmetries (1.1)[5]. At these points the physical component action is reduced to a sum of conventional bosonic WZ action and free fermionic actions[7,5]. Again, internal symmetry enters in a trivial combination with supersymmetry (now as a semi-direct product). With making use of general construction of Gates, Hall and Roček[8], WZ sigma models possessing other types of D = 2 supersymmetry become possible. However, the bosonic manifolds of these models cannot be immediately identified with any homogeneous group space and reveal a more complicated geometry.

Our purpose in the present paper is to introduce a new wide class of conformally invariant supersymmetric WZ sigma models on group manifolds. Their crucial distinctions from the models known previously consist, first, in that they possess in general N-extended D = 2 supersymmetry and, second, that the bosonic WZ sigma model one starts with is defined on the automorphism group $SO_-(N) \times SO_+(N)$ of above supersymmetry (indices $\pm$ refer to two independent light-cone directions in D = 2 Minkowski space). So, internal symmetry and supersymmetry do not commute even at the rigid level. When combined with D = 2 conformal symmetry, those close to give infinite-dimensional N-extended superconformal symmetry (of the type (N,N)). The latter is complete symmetry of the model in question for a given N. The bosonic sector of the model, apart from fields parametrizing the symmetric space $SO_-(N) \times SO_+(N)/SO(N)$, necessarily incorporates the dilaton $u(x)$ (it may possess either free of Liouville actions) and a number of further bosonic fields (for $N \geqslant 4$). Bosonic and fermionic fields are assigned to different representations of diagonal SO(N) and constitute an irreducible supermultiplet of N-extended superconformal symmetry. The fields on group space automatically enter with the correct conformally invariant WZ action because N-extended superconformal symmetry includes $SO_\mp(N)$ Kac-Moody symmetries/9,10/ which prove to be realized on the elements of coset $SO_-(N) \times SO_+(N)/SO(N)$ just by transformations (1.1).

The models we consider have a natural geometric description as nonlinear sigma models for infinite-dimensional N-extended superconformal groups C_N, with the quotient spaces $C_N/SO(1.1)\times SO(N)$ as target manifolds*) (here SO(1.1) is D = 2 Lorentz group). These manifolds are parametrized by coordinates of N-extended D = 2 superspace and by infinite sets of Nambu-Goldstone superfields. The nonlinear realization techniques combined with the covariant reduction method/12/ leave us with the finite number of $1+\frac{N(N-1)}{2}$ essential superfields. They turn out to be subjected to the universal irreducibility conditions which generalize the similar ones defining "twisted chiral" multiplets in N = 2 and N = 4 D = 2 supersymmetries/13-15,8/. The physical fields are leading components of these superfields. The superfield equations of motion also have a universal simple form for any N. They possess a zero-curvature representation on superalgebras

*) Another line of extending conventional D=2 geometric structures to an infinite-dimensional case, the construction of gauge theories for conformal supergroups, is worked out now by van Holten /11/.

$osp(N|2)$, thus indicating that the proposed models are classically integrable.

Our main incentives are to give a general characterization of the proposed class of sigma models and to study a number of simple instructive examples. We confine our analysis to the classical level. A complete quantum consideration will be given elsewhere.

The matter is organized as follows. In Sect.2 we construct nonlinear realizations of general N-extended D=2 superconformal symmetry, deduce the superfield irreducibility conditions and equations of motion, find the transformation laws both of superspace coordinates and the basic Nambu-Goldstone superfields. In Sect.3,4 specific examples are considered, the new models with N=3 and N=4. These are the first examples of D=2 superconformal Lagrangian field models enjoying symmetries with noncanonical generators. We find the relevant component actions and superconformal transformations leaving these actions invariant. In Sect.4 we also discuss the reduction to a special version of N=4 model constructed earlier by two of us/14,15/. As the new result, the invariant action for this theory is found. The examples illustrate the most of basic features of new superconformal WZ sigma models. These features are summarized in Sect.5, where we also speculate upon a possible contact with string theories and indicate a generalization to the case of D=2 heterotic supersymmetry. In Appendix A we quote the structure relations of general N-extended D=2 superconformal algebra and of its N=3 and N=4 subalgebras. Appendix B treats the physical on-shell field content of the WZ supermultiplet for arbitrary N.

2. NONLINEAR REALIZATIONS OF D = 2 SUPERCONFORMAL SYMMETRIES

To construct a nonlinear sigma model for N-extended D=2 superconformal group C_N, we take advantage of standard nonlinear realization techniques augmented by the covariant reduction method/12/. This method allows one to define the group action in infinite-dimensional coset spaces doing with a finite number of essential parameters - (super) fields. The scheme we apply has already been used in/13-15/ to deduce N=2 and N=4 superextensions of the D=2 Liouville equation.

So, let us consider a nonlinear realization of infinite-dimensional conformal supergroup C_N with the superalgebra $\mathcal{G}_N = \mathbb{K}_-(1|N) \oplus \mathbb{K}_+(1|N)$ (see Appendix A) in the coset space C_N/H_N, $H_N = SO(1,1) \times SO(N)$. Here, SO(1.1) and SO(N) are generated, respectively, by $U = L_{0+} - L_{0-}$ and $T^{ij} = L^{ij}_{0+} + L^{ij}_{0-}$ (this choice of the stability subgroup will be substantiated later on). We take for the coset the following parametrization:

$$g = e^{i x^{\pm} L_{-1\pm}}\, e^{\theta^{\pm i} G^{i}_{-\frac{1}{2}\pm}}\, e^{i \sum_{n=1}^{\infty} a_n^{\pm} L_{n\pm}}\, e^{\sum_{r=1/2}^{\infty} \xi_r^{\pm} G_{r\pm}} \times$$

$$\times\, e^{i u (L_{0+} + L_{0-})}\, e^{i \varphi^{ij} (L^{ij}_{0+} - L^{ij}_{0-})} \; ; \{g\} = C_N / H_N . \tag{2.1}$$

Here

$$a_n^{\pm} L_{n\pm} \equiv \sum_{i,j,k,\ldots} a_n^{\pm[ij\ldots k]} L_{n\pm}^{[ij\ldots k]} \, , \quad \xi_r^{\pm} G_{r\pm} \equiv \sum_{i,j\ldots k} \xi_r^{\pm[ij\ldots k]} G_{r\pm}^{[ij\ldots k]}$$

(the indexing is explained in Appendix A). Note that we limit ourselves to a subgroup of complete $D = 2$ superconformal group corresponding to $n \geqslant -1$, $r \geqslant -1/2$ (cf. [11]). The transformation laws given below immediately extend to the whole group.

The coset element (2.1) is parametrized by superspace coordinates $Z^{\pm} \equiv \{x^+, x^-, \theta^{+i}, \theta^{-j}\}$; $(i,j = 1,2,\ldots N)$ and an infinite number of Nambu-Goldstone superfields $u(Z)$, $\varphi^{ij}(Z)$, $\xi_r^{\pm i}(Z), a_n^{\pm i}(Z), \ldots$. Fortunately, by the general theorem of ref. [16] all those can be expressed in terms of a finite set of essential superfields by imposing proper constraints on relevant Cartan's 1-forms. An inspection of the structure relations of superalgebra $\mathcal{G}_N$ (see Appendix A) from the standpoint of aforementioned theorem shows that our choice for the stability subgroup guarantees a minimal set of unremovable parameters. These are superdilaton $u(z)$ and superfields $\varphi^{ij}(z)$ which parametrize the coset $SO_+(N) \times SO_-(N)/SO(N)$. For further convenience we combine them into a single $N \times N$ matrix superfield $q^{ij}(Z)$:

$$q^{ij}(Z) = \left(e^{-u(Z) I - 2i \varphi^{kl}(Z) \tau^{kl}} \right)^{ij} \equiv e^{-u(Z)} \tilde{q}^{ij}(Z) , \tag{2.2}$$

where $(\tau^{kl})^{ij}$ are $SO(N)$ generators in vector representation. The transformation properties of $q^{ij}(Z)$ and $Z^{\pm}$ can be found by resorting to the fact that C_N is realized in the space C_N/H_N by left shifts

$$g_0 \cdot g = g' \cdot h' \; ; \; g_0 \in C_N \; ; \; g, g' \in C_N/H_N \; ; \; h' \in H_N , \tag{2.3}$$

As follows from the structure relations of $\mathcal{G}_N$ (A.5), it suffices to know the supersymmetric transformations with generators $G^{i}_{r\pm}$, all the remaining ones are recovered by commuting those among themselves. Infinitesimally, they are

$$\begin{aligned}
\delta x^{\pm} &= i(\theta^{\pm} \mu^{\pm}) \\
\delta \theta^{\pm i} &= \mu^{\pm i} + i(\theta^{\pm} \mu'^{\pm}) \theta^{\pm i} \\
\delta q^{ij} &= q^{ik} W_+^{kj} + W_-^{ik} q^{kj} \\
W_+^{ij} &= \frac{1}{2} \left(D_+^i \delta\theta^{+j} - D_+^j \delta\theta^{+i} - \frac{2\delta^{ij}}{N} D_+^k \delta\theta^{+k} \right) \\
W_-^{ij} &= \frac{1}{2} \left(D_-^j \delta\theta^{-i} - D_-^i \delta\theta^{-j} - \frac{2\delta^{ij}}{N} D_-^k \delta\theta^{-k} \right) .
\end{aligned} \tag{2.4}$$

Here $\mu^{\pm i}(x^{\pm})$ are anticommuting parameters-functions and $D^{i}_{\pm}$ are covariant spinor derivatives

$$D_{\pm}^{i} = \frac{\partial}{\partial \theta^{\pm i}} + i\theta^{\pm i} \frac{\partial}{\partial x^{\pm}} , \; \{D_{\pm}^{i}, D_{\pm}^{j}\} = 2i\delta^{ij} \frac{\partial}{\partial x^{\pm}} , \; \{D_{+}^{i}, D_{-}^{j}\} = 0 . \tag{2.5}$$

General superconformal variation of q^{ij} is of the same form as in eq.(2.4). Corresponding functions $W^{ij}_{\pm}$ involve as the leading terms the parameters of induced Weyl transformations and of two Kac-Moody $SO_{\pm}(N)$ -symmetries

$$W_{\pm}^{ij}(Z^{\pm}) \Big|_{\theta = 0} = -\frac{1}{2} f^{\pm\prime}(x^{\pm}) \delta^{ij} \pm \lambda^{\pm[ij]}(x^{\pm}) .$$

Extension to the whole $D = 2$ superconformal group having in addition an infinite set of generators with negative dimensions goes simply by allowing arbitrary inverse powers of $x^{\pm}$ in the x -decompositions of group parameters $\mu^{\pm i}(x^{\pm})$, $\lambda^{\pm[ij]}(x^{\pm})$, etc. Note that the left and right indices of q^{ij} are rotated by independent groups $SO_-(N)$ and $SO_+(N)$. We denote these indices by the same letters with the hope this will not lead to any misunderstanding.

So far, $q^{ij}(Z)$ was not subject to any constraints besides the purely algebraic ones $\tilde{q}^{ij} \cdot \tilde{q}^{lj} = \tilde{q}^{ji} \cdot \tilde{q}^{jl} = \delta^{il}$ (orthogonality conditions) following from the definition (2.2). To get the dynamical equations for q^{ij} we have to carry out the covariant reduction of coset space C_N/H_N to its subspace $OSp(N|2)/H_N$. Here, $OSp(N|2)$ is generated by diagonal combinations of generators of the left and right finite-dimensional conformal superalgebras $osp_{\pm}(N|2) \subset K_{\pm}(1|N)$:

$$\begin{aligned}
R_{\pm} &= L_{-1\pm} + m^2 L_{1\mp} \, , \qquad G^{i}_{\pm} = G^{i}_{-\frac{1}{2}\pm} \pm m G^{i}_{\frac{1}{2}\mp} \\
U &= L_{0+} - L_{0-} \, , \\
T^{ij} &= L^{ij}_{0+} + L^{ij}_{0-} \, , \qquad [m] = cm^{-1} .
\end{aligned} \tag{2.6}$$

This reduction goes as follows. Given Cartan's 1-forms defined from the beginning over infinite-dimensional superalgebra $\mathcal{G}_N$:

$$\Omega = g^{-1} dg = \omega^{\pm} L_{\pm} + \mu^{\pm} G_{\pm} + \dots , \tag{2.7}$$

one imposes on them the covariant constraint

$$\Omega = \Omega^{Red} \in osp(N|2) \tag{2.8}$$

which means that all the components of Ω except for those entering with the generators of subalgebra (2.6) are equated to zero. Covariance of the condition (2.8) is seen by the following simple reasoning. The only 1-forms among (2.7) that transform inhomogeneously are those before the generators of $H_N = SO(1,1) \times SO(N)$. These forms remain non-zero since $H_N \in OSp(N|2)$.

From the geometric point of view, the constraint (2.8) reduces C_N/H_N to its fully geodesic hypersurface $OSp(N|2)/H_N$. Indeed, any motions of Cartan's moving frame along the directions orthogonal to that subspace are covariantly forbidden by eq. (2.8). Note that the parameter m has the meaning of inverse "radius" of pseudosphere $SO(1,2)/SO(1,1)$ contained as a subspace in $OSp(N|2)/SO(1,1) \times SO(N)$ /12/. Setting $m = 0$ in eq.(2.8) gives rise to an alternative reduction to the flat superspace $P(N|2)/H_N$, where $P(N|2)$ is N-extended $D = 2$ Poincaré supergroup with generators $\{ L_{-1\pm}, G^{i}_{-\frac{1}{2}\pm}, U, T^{ij} \}$ following from $OSp(N|2)$ by contraction $m \to 0$. Covariant reductions to other subspaces of C_N/H_N are as well possible. However, in all these cases the number of essential superfields increases. We limit our consideration to the constraint (2.8) and its flat $m = 0$ version.

Without entering into detailes, we mention that the practical role of eq.(2.8) is to express all the superfield parameters in terms of $q^{ij}(Z)$ and its spinor and ordinary derivatives. Furthermore, it imposes the following covariant differential constraints on $q^{ij}(Z)$:

$$D^i_+ q^{mj} + D^j_+ q^{mi} = \frac{2}{N} \delta^{ij} D^{\ell}_+ q^{m\ell} ,$$

$$D^i_- q^{jm} + D^j_- q^{im} = \frac{2}{N} \delta^{ij} D^{\ell}_- q^{\ell m} \tag{2.9}$$

and

$$D^i_- (q^{-1} D^j_+ q)^{k\ell} = i m (\delta^{j\ell} q^{ik} + \delta^{k\ell} q^{ij} - \delta^{jk} q^{i\ell}) . \tag{2.10}$$

For completeness, we have written down also the nonlinear orthogonality condition for q^{ij}:

$$q^{ij} q^{mj} = q^{ji} q^{jm} = \frac{1}{N} \delta^{im} (q^{k\ell} q^{k\ell}) = \delta^{im} e^{-2u} . \tag{2.11}$$

It leaves in q^{ij} just $1 + \frac{N(N-1)}{2}$ independent superfield components in agreement with eq.(2.2).

The superfield system (2.9)-(2.11) defines the sought nonlinear sigma model for N-extended D=2 superconformal group C_N. Dynamics is concentrated just in eq.(2.10) which incorporates correct equations of motion for physical component fields (see below). Constraints (2.9) are the kinematical irreducibility conditions. It is a simple exercise to check their compatibility with eqs.(2.10). For $m \neq 0$, they can be independently derived by applying spinor derivatives on both sides of eq. (2.10). Remarkably, they do not imply this equation, i.e. are fulfilled off-shell and in fact irrespective of specific value of m . This leaves a room for constructing more general sigma models on the basis of q^{ij} (with, as well as without, conformal invariance). These constraints directly generalize the Grassmann analyticity conditions of $N = 2$ and $N = 4$ cases /13-15,8/, which single out there the relevant analytic representations ("twisted chiral" by terminology of ref. /8/).

Exposing an irreducible off-shell field content of q^{ij} for arbitrary N is technically hard problem. Here we explicitly solve it only for $N \leq 4$. On the other hand, on-shell and with $m = 0$ this analysis becomes much simpler and can be brought to the end for any N. By continuity in m, the result equally applies to general situation with $m \neq 0$.

Equation (2.10) for $m = 0$ can be easily solved*):

$$q^{mj}(z) = q_L^{m\ell}(z^-) q_R^{j\ell}(z^+), \tag{2.12}$$

where the left- and right-moving superfields $q_L^{m\ell}$ and $q_R^{j\ell}$ comprise the multiplets of two independent light-cone components of supergroup C_N. Inserting this general solution in the system (2.9), one splits the latter into a pair of unrelated equations:

*)Superfield solution (2.12) encompasses familiar classical solutions for the free field $u(x) \equiv u(z)|_{\theta=0}$ and for the WZ field $\tilde{q}_0^{mj}(x) \equiv \tilde{q}^{mj}(z)|_{\theta=0}$:
$u(x) = u_L(x^-) + u_R(x^+)$, $\tilde{q}_0^{mj}(x) = \tilde{q}_{L_0}^{m\ell}(x^-) \tilde{q}_{R_0}^{j\ell}(x^+)$ /2/.

$$D_-^i q_L^{j\ell}(z^-) + D_-^j q_L^{i\ell}(z^-) = \frac{2}{N}\,\delta^{ij} D_-^m q_L^{m\ell}(z^-) \qquad (a)$$

$$D_+^i q_R^{j\ell}(z^+) + D_+^j q_R^{i\ell}(z^+) = \frac{2}{N}\,\delta^{ij} D_+^m q_R^{m\ell}(z^+)\,. \qquad (b) \qquad (2.13)$$

Both equations are identical by form, so it sufficies to study one of them, say for $q_L^{j\ell}(Z^-)$. As is shown in Appendix B, $q_L^{j\ell}(Z^-)$ contains in general $(2^{N-1} + 2^{N-1})$ independent components. Their array starts from a $SO_-(N)$ -singlet $u_L(x^-)$ and further includes the components with the following type of symmetry:

$$\square + \boxed{\begin{smallmatrix} \; \\ \; \end{smallmatrix}} + \ldots + \left.\boxed{\begin{smallmatrix} \; \\ \; \\ \vdots \\ \; \end{smallmatrix}}\right\} N\,, \qquad (2.14)$$

where the even and odd numbers of cells correspond, respectively, to bosons and fermions (in the special $N = 4$ case a further reduction is possible, see Sect.4). In general, only $1 + \frac{N(N-1)}{2}$ bosonic and $N + \frac{N!}{3!(N-3)!}$ fermionic components entering q^{ij} as coefficients of zero and first degrees of Grassmann variables have from the beginning a correct physical dimension (cm^0 for bosons and $cm^{-1/2}$ for fermions). Starting from $N = 4$, physical fields appear also with higher degrees of θ. So, to achieve physical dimensions one is led to take appropriate degrees of derivatives ∂_- off these fields (respectively, ∂_+ off the right movers), that is to pass to "potentials". The necessity of passing to potentials roots already in the off-shell conditions (2.9) which for $N \geqslant 4$ entail the notoph type differential constraints on the higher dimension components. For $N = 4$ such a constrained is easily solved in terms of potential of physical dimension (Sect.4). However for $N > 4$ solving them may be a matter of serious difficulty (due to the explicit presence of the WZ field $\tilde{q}_{o}^{ij}(x) = \tilde{q}^{ij}(Z)|_{\theta=0}$ in these constraints).

Before turning to the examples we mention an important property of the system (2.9), (2.10), its classical integrability. This system is equivalent to the zero curvature condition for the $OSp(N|2)$ valued 1 -superform Ω^{Red} (2.8) and therefore can be interpreted as the compatibility condition for some matrix linear problem. To see this, recall that the original 1-superform Ω (2.7) by construction satisfied the Maurer-Cartan equation on the whole infinite-dimensional superalgebra $\mathcal{G}_N$. This causes the reduced 1-form Ω^{Red} to satisfy an analogous equation on superalgebra $OSp(N|2)$ (2.6). The existence of zero-curvature representation for the system (2.9), (2.10) means that the latter can be explicity solved for any value of parameter m . Further

discussion of integrability aspects is beyond the scope of present work.

3. EXAMPLES: N = 1,2,3

N = 1. In this simplest case indices i,j take only one value 1, and the superfield q^{ij} involves one component $q^{11}(Z) = e^{-u(Z)}$. Constraint (2.9) is satisfied identically, eq. (2.10) is nothing else than the $N = 1$ supersymmetric Liouville equation /17/.

$$D_+ D_- u = i m e^{-u}.$$

N = 2. This model possesses abelian internal symmetry $SO_+(2) \times SO_-(2)$. Correspondingly, $q^{ij}(Z)$ is a 2×2 matrix: $q^{ij} = e^{-u}(e^{\varphi\varepsilon})^{ij}$, $\varepsilon^{ij} = -\varepsilon^{ji}$. Equations (2.9), (2.10) take the most readable form with making use of the complex U(I)-notation:

$$\bar{D}_+ (u + i\varphi) = 0\,, \quad D_-(u + i\varphi) = 0 \qquad (3.1)$$

$$\bar{D}_- D_+ (u + i\varphi) = -4 i m\, e^{-u + i\varphi} \qquad (3.2)$$

$$D_\pm = D_\pm^1 + i D_\pm^2\,, \quad \bar{D}_\pm = D_\pm^1 - i D_\pm^2\,.$$

Constraints (3.1) are Grassmann U(I) -analyticity conditions /18/ and eq.(3.2) is $N = 2$ supersymmetric Liouville equation /13/. In this case there appears for the first time a prototype of the WZ field, the scalar field $\varphi(x) = \varphi(Z)|_{\theta=0}$ coordinatizing the coset $SO_+(2) \times SO_-(2)/SO(2)$. In the limit $m = 0$, when Yukawa couplings to fermions vanish, this field becomes free /13/.

N = 3. It is the first model with the nonabelian internal symmetry group $SO_+(3) \times SO_-(3)$ and, correspondingly, with the WZ action in the bosonic sector. Furthermore, it is the first example of $D = 2$ Lagrangian model respecting invariance under superconformal symmetry which has noncanonical generators in addition to the canonical ones.

To reveal the off-shell irreducible field content of $N = 3$ superfield q^{ij} $(i,j = 1,2,3)$ we take advantage of the projection method. The constraint (2.9) leaves in q^{ij} 8 + 8 independent components:

$$\begin{array}{ll} \underline{\text{Fermions}} & \underline{\text{Bosons}} \\ \psi_+^i(x) \equiv \psi_+^i(z)\big|_{\theta=0} = \frac{i}{3}(q^{-1}D_+^m q)^{im}\big|_{\theta=0} & e^{-u(x)}\tilde{q}_0^{ij}(x) = q^{ij}(z)\big|_{\theta=0} \\ \psi_-^i(x) \equiv \psi_-^i(z)\big|_{\theta=0} = \frac{i}{3}(D_-^m q\cdot q^{-1})^{mi}\big|_{\theta=0} & A(x) = D_-^i \psi_+^j(z) q^{ij}(z)\big|_{\theta=0} = \bar{A}(x) \\ \chi_+(x) = -\frac{i}{6}\varepsilon^{ijk}(q^{-1}D_+^i q)^{jk}\big|_{\theta=0} & B^\ell(x) = D_-^i \psi_+^k(z) q^{jk}(z)\varepsilon^{\ell ij}\big|_{\theta=0} = \\ \chi_-(x) = -\frac{i}{6}\varepsilon^{ijk}(q D_-^i q^{-1})^{jk}\big|_{\theta=0} & = \bar{B}^\ell(x) \end{array} \tag{3.3}$$

(this definition is most convenient though not unique). The fields $A(x)$, $B^\ell(x)$ are auxiliary, they are eliminated by the dynamical equation (2.10)

$$A(x) = -3m\, e^{-2u(x)}\ , \quad B^\ell(x) = 0\ . \tag{3.4}$$

For the physical fields eq.(2.10) yields the following system:

$$\begin{aligned} &\partial_-(\tilde{q}_0^{-1}\partial_+\tilde{q}_0)^{ij} = im\left[(\psi_-\tilde{q}_0)^j\psi_+^i - (\psi_-\tilde{q}_0)^i\psi_+^j\right]e^{-u} \\ &\partial_-\partial_+ u = im\, e^{-u}(\psi_-\tilde{q}_0\psi_+) + m^2 e^{-2u} \\ &\partial_-\psi_+^j = m(\psi_-\tilde{q}_0)^j e^{-u}\ , \quad \partial_-\chi_+ = 0 \\ &\partial_+\psi_-^i = -m(\tilde{q}_0\psi_+)^i e^{-u}\ , \quad \partial_+\chi_- = 0\ . \end{aligned} \tag{3.5}$$

Its bosonic sector embodies the Liouville equation for dilaton $u(x)$ and the equations of conformally invariant WZ sigma model on the coset space $SO_+(3) \times SO(3)/SO(3)$ modified by Yukawa couplings with fermions. Thus, in the framework of this model there comes about at once $N = 3$ supersymmetrization both of the Liouville equation (or the free one for $m = 0$) and the equations of conformally invariant $SO_+(3) \times SO_-(3)/SO(3)$ - WZ sigma model. It is worth remarking that one may put $m = 0$ in eqs. (3.5) without losing any invariance properties of them. Couplings with fermions are switched off in this limit (simultaneously with vanishing of the potential term of dilaton), and the system (3.5) splits into the pure WZ sigma model equation and free equations for dilaton and fermion fields (this phenomenon is quite analogous to the trivialization of ordinary $N = 1$ supersymmetric WZ sigma model at the points of conformal invariance/5,7/).

Equations (3.5) are derivable from the action

$$\begin{aligned} S = \frac{1}{f^2}\int d^2x \Big\{ &\frac{1}{2}\partial_+ u\,\partial_- u + \frac{1}{4}\mathrm{tr}(\partial_+\tilde{q}_0^{-1}\partial_-\tilde{q}_0) + \\ + \frac{1}{4}\int_0^1 dt\, \mathrm{tr}\big\{&\tilde{q}_0^{-1}\dot{\tilde{q}}_0\left[(\tilde{q}_0^{-1}\partial_+\tilde{q}_0)(\tilde{q}^{-1}\partial_-\tilde{q}_0) - (+\leftrightarrow -)\right]\big\} - \\ -\frac{i}{2}\psi_-^i\partial_+\psi_-^i - \frac{i}{2}\psi_+^i\partial_-\psi_+^i &- \frac{i}{2}\chi_-\partial_+\chi_- - \frac{i}{2}\chi_+\partial_-\chi_+ - \\ - \frac{m^2}{2}e^{-2u} - im\,\psi_-^i\tilde{q}_0^{ij}\psi_+^j e^{-u}\Big\}\ , &\quad [f] = cm^0\ , \end{aligned} \tag{3.6}$$

where $\tilde{q}_0(t = 1) \equiv \tilde{q}_0$, $\tilde{q}_0(t = 0) = I$. It is straightforward to check that the kinetic and potential parts of (3.6) are invariant separately with respect to the $N = 3$ superconformal transformations (for simplicity, we write down merely supersymmetric transformations from the right component of C_3):

$$\begin{aligned} &(\tilde{q}_0^{-1}\delta\tilde{q}_0)^{ij} = -i\mu^{+k}(x^+)(\chi_+\varepsilon^{kij} + \delta^{ki}\psi_+^j - \delta^{kj}\psi_+^i)\ , \\ &\delta u = -i\mu^{+k}(x^+)\psi_+^k\ , \\ &\delta\psi_+^i = \partial_+\mu^{+i}(x^+) + i\mu^{+k}(x^+)\varepsilon^{kij}\chi_+\psi_+^j + (\tilde{q}_0^{-1}\partial_+\tilde{q}_0)^{ij}\mu^{+j}(x^+) - \partial_+ u\,\mu^{+i}(x^+)\ , \\ &\delta\psi_-^i = -m\,\tilde{q}_0^{ik}\mu^{+k}(x^+)e^{-u}\ , \quad \delta\chi_- = 0 \\ &\delta\chi_+ = \nu_+(x^+) - \frac{1}{2}\mu^{+k}(x^+)\varepsilon^{kij}(\tilde{q}_0^{-1}\partial_+\tilde{q}_0)^{ij} - \frac{i}{2}\mu^{+k}(x^+)\varepsilon^{kij}\psi_+^i\psi_+^j\ . \end{aligned} \tag{3.7}$$

As was already mentioned in the preceding section, realization of the remaining C_3 -transformations can be found by commuting (3.7) among themselves. Note the presence of additional $SO_+(3)$ - singlet spinor parameter $\nu_+(x^+)$ in (3.7). The coefficients in its x^+ -decomposition are group parameters associated with an infinite set of noncanonical spinor generators present in $N = 3$ superconformal algebra - (see Appendix A). Respectively, under the action of these generators the field χ_+ and its derivatives of any rank undergo pure shifts and have thus a meaning of Nambu-Goldstone fermions corresponding to spontaneous breakdown of these noncanonical supersymmetries (an analogous interpretation is valid as well for χ_- which is shifted under the action of noncanonical generators from the left branch of C_3). It should be stressed that all other fields in the action (3.6) are also of the Nambu-Goldstone nature, just as in ordinary sigma models. These fields and an infinite set of their x-derivatives enter as leading components into the superfield parameters of infinite-dimensional quotient space C_3/H_3 (with the inverse Higgs phenomenon constraints taken into account). The group-theoretical meaning of fields $u(x)$ and $\tilde{q}_0^{ij}(x)$ is obvious. The fields ψ_+^i, ψ_-^j and their derivatives of any rank are group coordinates associated with an infinite set of special conformal supersymmetries whose constant parameters appear as

Equations (3.5) are derivable from the action

$$S = \frac{1}{f^2}\int d^2x \Big\{ \frac{1}{2}\partial_+ u\, \partial_- u + \frac{1}{4}\mathrm{tr}\,(\partial_+ \tilde{q}_0^{-1}\partial_- \tilde{q}_0) + $$
$$+ \frac{1}{4}\int_0^1 dt\, \mathrm{tr}\big\{ \tilde{q}_0^{-1}\dot{\tilde{q}}_0 \big[(\tilde{q}_0^{-1}\partial_+\tilde{q}_0)(\tilde{q}^{-1}\partial_-\tilde{q}_0) - (+ \leftrightarrow -)\big]\big\} -$$
$$- \frac{i}{2}\psi_-^i \partial_+ \psi_-^i - \frac{i}{2}\psi_+^i\partial_-\psi_+^i - \frac{i}{2}\chi_-\partial_+\chi_- - \frac{i}{2}\chi_+\partial_-\chi_+ - \tag{3.6}$$
$$- \frac{m^2}{2} e^{-2u} - im\,\psi_-^i \tilde{q}_0^{ij}\psi_+^j e^{-u} \Big\}, \quad [f] = cm^0,$$

where $\tilde{q}_0(t=1) \equiv \tilde{q}_0$, $\tilde{q}_0(t=0) = I$. It is straightforward to check that the kinetic and potential parts of (3.6) are invariant separately with respect to the $N = 3$ superconformal transformations (for simplicity, we write down merely supersymmetric transformations from the right component of C_3):

$$(\tilde{q}_0^{-1}\delta\tilde{q}_0)^{ij} = -i\mu^{+k}(x^+)\,(\chi_+\varepsilon^{kij} + \delta^{ki}\psi_+^j - \delta^{kj}\psi_+^i),$$
$$\delta u = -i\mu^{+k}(x^+)\,\psi_+^k,$$
$$\delta\psi_+^i = \partial_+\mu^{+i}(x^+) + i\mu^{+k}(x^+)\varepsilon^{kij}\chi_+\psi_+^j + (\tilde{q}_0^{-1}\partial_+\tilde{q}_0)^{ij}\mu^{+j}(x^+) - \partial_+ u\,\mu^{+i}(x^+), \tag{3.7}$$
$$\delta\psi_-^i = -m\,\tilde{q}_0^{ik}\mu^{+k}(x^+)\,e^{-u}, \quad \delta\chi_- = 0$$
$$\delta\chi_+ = \nu_+(x^+) - \frac{1}{2}\mu^{+k}(x^+)\varepsilon^{kij}(\tilde{q}_0^{-1}\partial_+\tilde{q}_0)^{ij} - \frac{i}{2}\mu^{+k}(x^+)\,\varepsilon^{kij}\psi_+^i\psi_+^j.$$

As was already mentioned in the preceding section, realization of the remaining C_3-transformations can be found by commuting (3.7) among themselves. Note the presence of additional $SO_+(3)$ - singlet spinor parameter $\nu_+(x^+)$ in (3.7). The coefficients in its x^+-decomposition are group parameters associated with an infinite set of noncanonical spinor generators present in $N = 3$ superconformal algebra (see Appendix A). Respectively, under the action of these generators the field χ_+ and its derivatives of any rank undergo pure shifts and have thus a meaning of Nambu-Goldstone fermions corresponding to spontaneous breakdown of these noncanonical supersymmetries (an analogous interpretation is valid as well for χ_- which is shifted under the action of noncanonical generators from the left branch of C_3). It should be stressed that all other fields in the action (3.6) are also of the Nambu-Goldstone nature, just as in ordinary sigma models. These fields and an infinite set of their x-derivatives enter as leading components into the superfield parameters of infinite-dimensional quotient space C_3/H_3 (with the inverse Higgs phenomenon constraints taken into account). The group-theoretical meaning of fields $u(x)$ and $\tilde{q}_0^{ij}(x)$ is obvious. The fields ψ_+^i, ψ_-^j and their derivatives of any rank are group coordinates associated with an infinite set of special conformal supersymmetries whose constant parameters appear as

coefficients in the x-expansions of functions $\mu^{+i}(x^+)$ and $\mu^{-i}(x^-)$ (starting from the linear terms). An analogous situation persists in models with higher N.

To complete this Section we quote the coupling constant quantization formula needed to make meaningful[2,3] the quantum theory associated with the action (3.6). With our normalization of the WZ term, the quantization condition is as follows

$$\frac{K f^2}{8\pi} = 1, \tag{3.8}$$

where K is an arbitrary integer, $K \in \mathbb{Z}$.

4. N = 4 SUPERCONFORMAL SIGMA MODELS

The case $N = 4$ is distinguished in that it admits two nonequivalent infinite-dimensional superconformal algebras $\mathcal{G}_4$ and $\tilde{\mathcal{G}}_4$ having, respectively, $osp_-(4|2) \oplus osp_+(4|2)$ and $su_-(2|1,1) \oplus su_+(2|1,1)$ as the maximal finite-dimensional subalgebras[9] (see Appendix A). The second superalgebra is minimal in the sense that all its generators are canonical (just as in the N = 1 and N = 2 cases). It forms a subalgebra of the first, more extensive, N = 4 superalgebra. The latter involves in addition an infinite set of noncanonical bosonic and fermionic generators and is a straightforward extension of N = 3 superalgebra treated in the preceding Section. The superconformal sigma model for the superalgebra of the second kind, with the WZ - $SU_-(2)\times SU_+(2)/SU(2)$ -sigma model in the bosonic sector, was constructed by two of us (E.I. and S.K.) in[14,15] (it was called "N = 4 supersymmetric Liouville equation", because at $m \neq 0$ it incorporates, just as its $N \leq 3$ prototypes, the Liouville equation for dilaton). Here we construct a sigma model for the N = 4 superconformal algebra of the first kind, discuss its peculiarities and reduction to the N = 4 model studied previously.

In accord with the general algorithm of Sect. 2, we should covariantly reduce the space C_4/H_4 to its finite-dimensional geodesic subspace $OSp(4|2)/H_4$. Specificity of this case is the presence of one-parameter family of diagonal superalgebras coefficients in the x -expansions of functions $\mu^{+i}(x^+)$ and $\mu^{-i}(x^-)$ (starting from the linear terms). An analogous situation persists in models with higher N .

To complete this Section we quote the coupling constant quantization formula needed to make meaningful[2,3] the quantum theory associated with the action (3.6). With our normalization of the WZ term, the quantization condition is as follows

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$osp(4|2)$ in $\mathcal{G}_4 = \mathbb{K}_-(1|4) \oplus \mathbb{K}_+(1|4)$:

$$\begin{aligned} osp^{(\alpha)}(4|2) &= \{ R_{\pm}^{(\alpha)}, U, T^{ij}, G_{\pm}^{i(\alpha)} \} \\ R_{\pm}^{(\alpha)} &= L_{-1\pm} + m^2 (L_{1\mp} + \alpha\,\Delta_{1\mp}) \\ G_{\pm}^{i(\alpha)} &= G^i_{-\frac{1}{2}\pm} \pm m\,(G^i_{\frac{1}{2}\mp} - \alpha\,\Gamma^i_{\frac{1}{2}\mp}). \end{aligned} \tag{4.1}$$

Though the general construction of Sect. 2 corresponded to the choice of $\alpha = 0$ in eqs.(4.1), it can be easily adapted to the case of $\alpha \neq 0$ too (the equations of motion undergo a minor modification). For special values $\alpha = \pm 1/2$, the generators of one of two SU(2)'s contained in SO(4) $\propto \{T^{ij}\}$ drop out from the r.h.s. of basic anticommutator $\{G_+^i, G_-^j\}$ and superalgebra (4.1) contracts into $su(2|1,1)$. The latter enters as a subalgebra into the minimal N = 4 superconformal algebra $\tilde{\mathcal{G}}_4$ (eq. (A.10) in Appendix A). This fact will be used later in discussion of the relation to the previous N = 4 model.

For the time being, our consideration will not be confined to any specific value of α . After covariant reduction of G_4/H_4 to $OSp^{(\alpha)}(4|2)/H_4$, the basic superfield

$$q^{ij} = (e^{-u \cdot I - 2i\,\varphi^{kl} t^{kl}})^{ij}$$

($(t^{kl})^{ij}$ are SO(4)-generators in the vector representation) satisfies the following system of equations

$$\begin{cases} D_+^i q^{jk} + D_+^k q^{ji} = \frac{1}{2}\,\delta^{ik} D_+^l q^{jl} \\ D_-^i q^{jk} + D_-^j q^{ik} = \frac{1}{2}\,\delta^{ij} D_-^l q^{lk} \end{cases} \tag{4.2}$$

$$\begin{aligned} D_-^i (q^{-1} D_+^k q)^{jl} = im\,(\delta^{kl} q^{ij} + \delta^{jl} q^{ik} - \delta^{jk} q^{il} + \\ + 2\alpha\, \varepsilon^{kjlr} q^{ir}). \end{aligned} \tag{4.3}$$

Comparing eqs. (4.2) and (4.3) with the general system (2.9), (2.10), one observes that the equation of motion has been slightly modified whereas the form of off-shell irreducibility conditions remained unchanged. Note that, as before, q^{ij} transforms under C_4 according to the general transformation law (2.4).

Let us turn to the component analysis of eqs.(4.2) and (4.3). Constraints (4.2) single out of q^{ij} an irreducible supermultiplet 16 + 16:

$$
\begin{aligned}
& q^{ij} = e^{-u}\tilde{q}^{ij} && \xi_+^i = \frac{i}{4}(q^{-1}D_+^m q)^{im} \\
& B^{ij} = \varepsilon^{ijkl} q^{mk} D_- \xi_+^l && \xi_-^i = \frac{i}{4}(D_-^m q \cdot q^{-1})^{mi} \\
& \left.\begin{aligned} A_+ &= D_+^i \zeta_+^i \\ A_- &= D_-^i \zeta_-^i \end{aligned}\right\} \partial_- A_+ - \partial_+ A_- = 0 && \zeta_+^i = \frac{i}{12}\varepsilon^{ijkl}(q^{-1}D_+^j q)^{kl} \\
& && \zeta_-^i = \frac{i}{12}\varepsilon^{ijkl}(q D_-^j q^{-1})^{kl} \\
& F = q^{mk} D_-^m \zeta_+^k \\
& G = q^{mk} D_-^m \xi_+^k
\end{aligned}
\tag{4.4}
$$

Like in the $N = 3$ case, the independent component fields are the $\theta = 0$ parts of these superfields. The eight fields B^{ij}, F and G are auxiliary; these enter the θ - expansion of q^{ij} as coefficients of the monomials $\sim \theta^+\theta^-$. Among the physical fields there is a vector field $A_\pm(x)$ which is subject to the differential constraint

$$\partial_- A_+(x) - \partial_+ A_-(x) = 0 \tag{4.5}$$

following (after some algebra) from the superfields constraint (4.2). So, $A_\pm(x)$ actually describes one degree of freedom off-shell (cf. notoph in four dimensions). The general solution of eq.(4.5) is via a SO(1,1) -scalar field $\varphi(x)$ of dimension cm^0:

$$A_\pm(x) = \partial_\pm \varphi(x) . \tag{4.6}$$

Thus, the irreducible manifold of bosonic fields consists of 8 physical fields (two real $SO_+(4)$ -singlets $u(x)$, $\varphi(x)$ and six real $SO_+(4)\times SO_-(4)/SO(4)$ -parameters $\varphi^{[ij]}(x)$) and 8 auxiliary fields (two real $SO_+(4)$ -singlets F, G and six real fields $B^{[ij]}(x)$ forming the two-rank skew symmetric tensor of $SO_+(4)$*)). The fermionic sector comprises 16 fields of physical dimension $cm^{-1/2}$ which fall

*)In the present model, any tensor representation of $SO_-(4)$ is equivalent to that of $SO_+(4)$ due to the existence of a "bridge" $\tilde{q}^{ij}$ relating vector indices of $SO_-(4)$ and $SO_+(4)$ to each other. Our convention on the representation content of fields ensures the most simple form of invariant action.

into two vectors of $SO_+(4)$ (ξ_+^i , ζ_+^i) and two vectors of $SO_-(4)$ (ξ_-^i , ζ_-^i).

The equation of motion (4.3), when rewritten in components, amounts to the system

$$
\begin{cases}
B^{ij} = 0 \\
F = 4\alpha m e^{-2u} \\
G = -4m e^{-2u}
\end{cases}
\tag{4.7}
$$

$$
\begin{cases}
\partial_-(\tilde{q}^{-1}\partial_+\tilde{q})^{ij} = -im\left[(\xi_-\tilde{q})^i\xi_+^j - (\xi_-\tilde{q})^j\xi_+^i - 2\alpha\varepsilon^{ijkl}(\xi_-\tilde{q})^k\xi_+^l\right]e^{-u} \\
\partial_+\partial_- u = m^2 e^{-2u} + im(\xi_-\tilde{q}\,\xi_+)e^{-u} \\
\partial_+\partial_- v = 0 \\
\partial_-\xi_+^i = m(\xi_-\tilde{q})^i e^{-u} \\
\partial_+\xi_-^i = -m(\tilde{q}\,\xi_+)^i e^{-u} \\
\partial_-\eta_+^i = 0 \\
\partial_+\eta_-^i = 0 ,
\end{cases}
\tag{4.8}
$$

where

$$
\begin{aligned}
\eta_\pm^i &\equiv 2(\zeta_\pm^i + \alpha\,\xi_\pm^i) \\
v &\equiv \frac{1}{2}(\varphi + 4\alpha u)
\end{aligned}
\tag{4.9}
$$

and eqs. (4.7) were used in the process of deriving the physical equations (4.8).

It is advantageous to rewrite both the system (4.8) and the original equations (4.2) and (4.3) in the isospinor two-component notation using the isomorphism $SO_+(4)\times SO_-(4) \sim [SU_+(2)]^2\times[SU_-(2)]^2$. Any vector index of superfield q^{ij}, spinor derivatives, spinor coordinates, etc. must be substituted by a pair of SU(2) -doublet indices according to the rules*)

*)Our conventions are as follows (both for ordinary and underlined indices):
$(\sigma^j)^{\alpha a}_{(j=4,1,2,3)} = (\varepsilon^{\alpha a}, i\vec{\sigma}^{\alpha a})$, $\varepsilon^{12} = -\varepsilon^{21} = -\varepsilon_{12} = \underline{\varepsilon_{21}} = -1$,
$\varepsilon^{\alpha a}\varepsilon_{a\beta} = \delta^\alpha_\beta$, $\varepsilon_{a\beta}\varepsilon^{\beta b} = \delta^b_a$, $\vec{\sigma}^{\alpha a} = \underline{\varepsilon^{\alpha\beta}\bar{\sigma}_\beta{}^a} = \varepsilon^{ab}\bar{\sigma}_b{}^\alpha = -(\vec{\sigma})_{\alpha a}$.
Here $(\vec{\sigma})^a{}_\beta = (\sigma^1,\sigma^2,\sigma^3)_\beta{}^a = \overline{(\vec{\sigma})_a{}^\beta}$ are ordinary hermitian Pauli matrices, symbol (–)means complex conjugation. Elsewhere SU(2)-indices are raised and lowered with the help of invariant tensors $\varepsilon^{\alpha\beta}$, $\varepsilon_{\alpha\beta}$, ε^{ab} and ε_{ab}.

$$A^i \to A^{\alpha a} = \frac{1}{\sqrt{2}} A^i (\sigma^i)^{\alpha a}; \quad A^{\alpha a} A_{\alpha a} \equiv \varepsilon_{\alpha\beta}\varepsilon_{ab} A^{\alpha a} A^{\beta b} = A^i A^i$$

$$B^{\underline{i}} \to B^{\underline{\alpha a}} = \frac{1}{\sqrt{2}} B^{\underline{i}} (\sigma^{\underline{i}})^{\underline{\alpha a}}; \quad B^{\underline{\alpha a}} B_{\underline{\alpha a}} \equiv \varepsilon_{\underline{\alpha\beta}}\varepsilon_{\underline{ab}} B^{\underline{\alpha a}} B^{\underline{\beta b}} = B^{\underline{i}} B^{\underline{i}}, \tag{4.10}$$

where ordinary and underlined indices refer, respectively, to $SO_+(4)$ and $SO_-(4)$.

Keeping in mind that the $SO_-(4) \times SO_+(4)$ matrix superfield $\tilde{q}^{\underline{\alpha a},\alpha a} = q^{\underline{\alpha a},\alpha a}\, e^{u}$ can be represented, without loss of generality, as

$$\tilde{q}^{\underline{\alpha a},\alpha a} \equiv \tilde{q}_1^{\underline{\alpha}\alpha} \cdot \tilde{q}_2^{\underline{a}a} \tag{4.11}$$

and passing further to components with doublet indices, one finally rewrites the system (4.8) as

$$\begin{aligned}
&\partial_-(\tilde{q}_1^{-1\alpha\underline{\alpha}}\partial_+\tilde{q}_{1\underline{\alpha}}{}^{\beta}) = -\frac{im}{2}(1+2\alpha)\,\xi_-^{\underline{\alpha a}}\tilde{q}_{2\underline{a}a}(\tilde{q}_{1\underline{\alpha}}{}^{\alpha}\xi_+^{\beta a} + q_{1\underline{\alpha}}{}^{\beta}\xi_+^{\alpha a})\,e^{-u}\\
&\partial_-(\tilde{q}_2^{-1a\underline{a}}\partial_+\tilde{q}_{2\underline{a}}{}^{b}) = -\frac{im}{2}(1-2\alpha)\,\xi_-^{\underline{\alpha a}}\tilde{q}_{1\underline{\alpha}\rho}(\tilde{q}_{2\underline{a}}{}^{a}\xi_+^{\rho b} + \tilde{q}_{2\underline{a}}{}^{b}\xi_+^{\rho a})\,e^{-u}\\
&\partial_+\partial_- u = m^2 e^{-2u} + im\,\xi_-^{\underline{\alpha a}}\tilde{q}_{1\underline{\alpha}\alpha}\tilde{q}_{2\underline{a}a}\xi_+^{\alpha a}\\
&\partial_+\partial_- v = 0\\
&\partial_+\xi_-^{\underline{\alpha a}} = -m\,\xi_+^{\alpha a}\tilde{q}_{1\alpha}{}^{\underline{\alpha}}\tilde{q}_{2a}{}^{\underline{a}}\,e^{-u}\\
&\partial_-\xi_+^{\alpha a} = m\,\xi_-^{\underline{\alpha a}}\tilde{q}_{1\underline{\alpha}}{}^{\alpha}\tilde{q}_{2\underline{a}}{}^{a}\,e^{-u}\\
&\partial_+\eta_-^{\underline{\alpha a}} = 0\\
&\partial_-\eta_+^{\alpha a} = 0 .
\end{aligned} \tag{4.12}$$

These equations follow from the action

$$\begin{aligned}
S = \frac{1}{\gamma^2}\int d^2x \Big\{ &\Big[\frac{1}{2}\partial_+ u\,\partial_- u - \frac{i}{2}\xi_+\partial_-\xi_+ - \frac{i}{2}\xi_-\partial_+\xi_- - \frac{m^2}{2}e^{-2u} -\\
&- im\,\xi_-^{\underline{\alpha a}}\tilde{q}_{2\underline{a}a}\tilde{q}_{1\underline{\alpha}\alpha}\xi_+^{\alpha a}e^{-u}\Big] + \frac{1}{1-4\alpha^2}\Big[\frac{1}{2}\partial_+ v\,\partial_- v - \frac{i}{2}\eta_+\partial_-\eta_+ -\\
&- \frac{i}{2}\eta_-\partial_+\eta_-\Big] + \frac{1}{2(1-2\alpha)}\mathcal{L}^{w.z.}(\tilde{q}_2) + \frac{1}{2(1+2\alpha)}\mathcal{L}^{w.z.}(\tilde{q}_1)\Big\}
\end{aligned} \tag{4.13}$$

$$\mathcal{L}^{w.z.}(\tilde{q}) = tr(\partial_+\tilde{q}^{-1}\partial_-\tilde{q}) + \int_0^1 dt\, tr\{\tilde{q}^{-1}\dot{\tilde{q}}(\tilde{q}^{-1}\partial_+\tilde{q})(\tilde{q}^{-1}\partial_-\tilde{q}) - (+\leftrightarrow-)\}$$

$$(\tilde{q}(t=1) \equiv \tilde{q}, \quad \tilde{q}(t=0) = I), \quad [\gamma] = cm^0 .$$

Transformations of $N = 4$ superconformal symmetry which leave (4.13) invariant can be found by the general recipe, starting with the realization of C_4 as left shifts in C_4/H_4 and substituting the expressions (4.7) for auxiliary fields. We restrict our presentation here, like in the $N = 3$ case, to giving supersymmetric transformations from the right branch of C_4 :

$$\begin{aligned}
&(\tilde{q}_1^{-1}\delta\tilde{q}_1)^{\alpha\beta} = \frac{i}{2}\Big[(1+2\alpha)(\mu^{+\alpha a}\xi_{+a}^{\beta} + \mu^{+\beta a}\xi_{+a}^{\alpha}) - \mu^{+\alpha a}\eta_{+a}^{\beta} - \mu^{+\beta a}\eta_{+a}^{\alpha}\Big]\\
&(\tilde{q}_2^{-1}\delta\tilde{q}_2)^{ab} = \frac{i}{2}\Big[(1-2\alpha)(\mu^{+\alpha a}\xi_{+\alpha}^{b} + \mu^{+\alpha b}\xi_{+\alpha}^{a}) + \mu^{+\alpha a}\eta_{+\alpha}^{b} + \mu^{+\alpha b}\eta_{+\alpha}^{a}\Big]\\
&\delta u = -i\mu^{+\alpha a}\xi_{+\alpha a}\\
&\delta v = -i\mu^{+\alpha a}\eta_{+\alpha a}\\
&\delta\xi_+^{\alpha a} = -\mu^{+\alpha b}(\tilde{q}_2^{-1}\partial_+\tilde{q}_2)_b{}^{a} - \mu^{+\beta a}(\tilde{q}_1^{-1}\partial_+\tilde{q}_1)_\beta{}^{\alpha} - \mu^{+\alpha a}\partial_+ u + \partial_+\mu^{+\alpha a} +\\
&\qquad + 2i\alpha(\mu^{+\beta a}\xi_{+\beta}^{b}\xi_{+b}^{\alpha} - \mu^{+\alpha b}\xi_{+b}^{\beta}\xi_{+\beta}^{a}) - i(\mu^{+\beta a}\eta_{+\beta}^{b}\xi_{+b}^{\alpha} - \mu^{+\alpha b}\eta_{+b}^{\beta}\xi_{+\beta}^{a})\\
&\delta\eta_+^{\alpha a} = -(1+2\alpha)\mu^{+\alpha b}(\tilde{q}_2^{-1}\partial_+\tilde{q}_2)_b{}^{a} + (1-2\alpha)\mu^{+\beta a}(\tilde{q}_1^{-1}\partial_+\tilde{q}_1)_\beta{}^{\alpha} - \mu^{+\alpha a}\partial_+ v -\\
&\qquad - \frac{i}{2}(1-4\alpha^2)\Big[\mu^{+\beta a}\xi_{+\beta}^{b}\xi_{+b}^{\alpha} - \mu^{+\alpha b}\xi_{+b}^{\beta}\xi_{+\beta}^{a}\Big] + 2\alpha\,\partial_+\mu^{+\alpha a} -\\
&\qquad - \frac{i}{2}\Big[\mu^{+\beta a}\eta_{+\beta}^{b}\eta_{+b}^{\alpha} - \mu^{+\alpha b}\eta_{+b}^{\beta}\eta_{+\beta}^{a}\Big] + 2\nu_+^{\alpha a}\\
&\delta\xi_-^{\underline{\alpha a}} = -m\,\tilde{q}_1^{\underline{\alpha}\beta}\tilde{q}_2^{\underline{a}b}\mu^{+}_{\beta b}\,e^{-u}\\
&\delta\eta_-^{\underline{\alpha a}} = 0,\\
&\mu^{+\beta b} = \mu^{+\beta b}(x^+), \quad \nu_+^{\alpha a} = \nu_+^{\alpha a}(x^+) .
\end{aligned} \tag{4.14}$$

Let us outline the main features of the model specified by eqs. (4.12) and (4.13). Its bosonic sector involves two WZ sigma models defined on two independent spaces $SU_+(2) \times SU_-(2)/SU(2)$. These models are interrelated solely via the fermionic sector due to Yukawa couplings with the common set of spinor fields $\xi_+^{\alpha a}$ and $\xi_-^{\underline{\alpha a}}$. In the bosonic sector one finds also an additional scalar field $v(x)$. Though the latter possesses the free action for any m, its presence is necessary for N=4 superconformal invariance of the whole action.

(4.13). Again, all the fields involved are of the Nambu-Goldstone nature. In particular, $A_{\pm}(x) = \partial_{\pm}\varphi(x)$ are C_4/H_4 coset parameters associated with the noncanonical vector generators present in superalgebra $\mathcal{G}_4$ (Appendix A, eq. (A.8)).

Just as in the N = 3 case, at m = 0 all the nonderivative couplings vanish, and the action is reduced to a sum of two bosonic WZ actions, free actions of scalars $u(x)$, $v(x)$ and free fermionic actions.

Formula (3.8) for quantization of the overall coupling constant is now modified due to the presence of two independent WZ terms in the action (4.13) with an additional free parameter α. The quantization conditions involve two independent integers k_1, k_2

$$f^2(1-2\alpha) = \frac{4\pi}{k_1}, \quad f^2(1+2\alpha) = \frac{4\pi}{k_2}. \tag{4.15}$$

For $\alpha \neq \pm 1/2$ these conditions amount to

$$\frac{f^2}{2\pi} = \frac{1}{k_1} + \frac{1}{k_2}, \quad \alpha = \frac{1}{2}\,\frac{k_1-k_2}{k_1+k_2} \tag{4.16}$$

while at the singularity points $\alpha = \pm 1/2$, respectively, to

$$\text{a) } k_1 \to \infty, \ \frac{f^2}{2\pi} = \frac{1}{k_2}; \qquad \text{b) } k_2 \to \infty, \ \frac{f^2}{2\pi} = \frac{1}{k_1}. \tag{4.17}$$

In the rest of this Section we discuss the relation to the N = 4 model constructed earlier/14,15/ .

Our starting points in/14,15/ were minimal N = 4 superalgebra $\tilde{\mathcal{G}}_4$ and its covariant reduction subalgebra $su(2|1,1)$. As has been already remarked, this $su(2|1,1)$ corresponds to the choice of α = 1/2 or α = -1/2 in generators (4.1) (depending on the way one singles out $\tilde{\mathcal{G}}_4$ from $\mathcal{G}_4$). Thus, to descend to the model of refs./14,15/ from the one given here we should put, e.g., α = 1/2 in the equations of the latter and simultaneously reduce initial C_4-symmetry to symmetry under supergroup $\tilde{C}_4$ with the algebra $\tilde{\mathcal{G}}_4$. This is achieved with the following ansätze for $q^{\pm\underline{a},\alpha a}$:

$$q^{\pm\underline{a},\alpha a} = \varepsilon^{\underline{a}a}\, q^{\pm\alpha}. \tag{4.18}$$

Employing general q^{ij}-transformation law (2.4) one may check that the ansätze (4.18) breaks C_4 down to semi-direct product of $\tilde{C}_4$ and diagonal SU(2)-subgroup of those two SU(2)'s from $SO_-(4)\times SO_+(4)$ which act on indices $\underline{a}$, a . This diagonal SU(2) gives rise to purely global rotations of Grassmann coordinates and the related component fields in $q^{\pm\alpha}$. Two other SU(2)'s fall into supergroup $\tilde{C}_4$ and extend as before to Kac-Moody gauge symmetries.

Substitution of (4.18) into eqs.(4.2), (4.3) yields the self-consistent system of equations for $q^{\pm\alpha}$:

$$\begin{cases} D_+^{\alpha\beta} q^{\pm\beta} + D_+^{\beta\beta} q^{\pm\alpha} = 0 \\ D_-^{\pm\underline{a}} q^{\beta\alpha} + D_-^{\beta\underline{a}} q^{\pm\alpha} = 0 \end{cases} \tag{4.19}$$

$$D_-^{\pm\underline{a}}\left[(q^{-1})^{\nu}_{\underline{\beta}}\, D_+^{\beta\beta}(q)^{\beta\gamma}\right] = 2im\, \varepsilon^{\underline{a}\beta}\varepsilon^{\nu\beta} q^{\pm\gamma}, \tag{4.20}$$

where

$$D_+^{\alpha\beta} \equiv \frac{1}{\sqrt{2}} D_+^i (\sigma^i)^{\alpha\beta}, \quad D_-^{\pm\underline{a}} \equiv \frac{1}{\sqrt{2}} D_-^i (\sigma^i)^{\pm\underline{a}}.$$

Let us set

$$D_+^{\alpha 1} \equiv D_+^{\alpha}, \quad D_-^{\pm 1} \equiv \mathcal{D}_-^{\pm}$$
$$D_+^{\alpha 2} \equiv \bar{D}_+^{\alpha}, \quad D_-^{\pm 2} \equiv \bar{D}_-^{\pm}$$

which amounts to regarding real coordinates $\theta^{+\alpha a}$ and $\theta^{-\pm\underline{a}}$ as complex doublets of $SU_{\pm}(2) \subset \tilde{C}_4$. Then eqs. (4.19) and (4.20) take the same form as those in/14,15/. In particular, the irreducibility constraints (4.19) become

$$D_+^{(\alpha} q^{\pm\beta)} = 0, \quad D_-^{(\pm} q^{\beta)\alpha} = 0 \tag{4.21}$$

(by the reality property $\bar{q}_{\pm\beta} = -\varepsilon_{\pm\beta}\varepsilon_{\beta\rho} q^{\beta\rho}$, the same conditions hold with $\bar{D}_{\pm}$). It is worth mentioning that the supermultiplet subjected to the constraints (4.21) has been independently invented by Gates, Hall and Roček/8/ in searching for new D = 2 sigma models with extended supersymmetry (it was called "twisted chiral N = 4 multiplet"). These authors used a bit different definition of spinor derivatives leading to a more intricate form of the constraint. The explicitly $SO_-(4)\times SO_+(4)$ -invariant form (4.19) has been given for the first time by Siegel/19/.

The superfield condition (4.18) in application to physical components of $q^{\underline{\alpha}\alpha,\underline{a}a}$ amounts to

$$\tilde{q}_2^{\underline{a}a}(x) = \varepsilon^{\underline{a}a}\ , \quad v(x) = 0\ , \quad \eta_+^{\alpha a} = \eta_-^{\underline{\alpha}\underline{a}} = 0\ . \tag{4.22}$$

Relabelling the fermionic components as

$$\xi_+^{\alpha 1} \equiv \chi_+^{\alpha}\ , \qquad \xi_-^{\underline{\alpha}1} \equiv \psi_-^{\underline{\alpha}}$$

$$\xi_+^{\alpha 2} \equiv \bar{\chi}_+^{\alpha} = \varepsilon^{\alpha\beta}\overline{(\chi_+^{\beta})}\ , \quad \xi_-^{\underline{\alpha}2} \equiv \bar{\psi}_-^{\underline{\alpha}} = \varepsilon^{\underline{\alpha}\underline{\beta}}\overline{(\psi_-^{\underline{\beta}})}$$

and choosing $\alpha = 1/2$ in the action (4.13) (this choice is implementable only when followed by eqs.(4.22))one finds the reduced action

$$S = \frac{1}{f^2}\int d^2x \Big[\frac{1}{2}\partial_+ u\,\partial_- u + i\bar{\chi}_+\partial_-\chi_+ + i\bar{\psi}_-\partial_+\psi_- - \frac{1}{2}m^2 e^{-2u} -$$
$$- im\left(\psi_-^{\underline{\alpha}}\tilde{q}_{1\underline{\alpha}}^{\ \alpha}\bar{\chi}_{+\alpha} - \bar{\psi}_-^{\underline{\alpha}}\tilde{q}_{1\underline{\alpha}}^{\ \alpha}\chi_{+\alpha}\right) + \frac{1}{4}\mathcal{L}^{W.Z.}(\tilde{q}_1)\Big] \tag{4.23}$$

which reproduces the component equations obtained in /14,15/. Quantization of the coupling constant f^2 is given by formula (4.17a).

Finally, we note that an alternative reduction to the N = 3 model of Sect.3 is effected, on the component level, by choosing $\alpha = 0$ in (4.13) and setting (with unessential numeric coefficients ignored)

$$\tilde{q}_1^{\underline{\alpha}\beta} = q_2^{\underline{a}b}\ , \quad \eta_+^{\alpha a} = \varepsilon^{\alpha a}\eta_+\ , \quad \eta_-^{\underline{\alpha}\underline{a}} = \varepsilon^{\underline{\alpha}\underline{a}}\eta_-\ , \quad v = 0\ . \tag{4.24}$$

5. DISCUSSION

For reader's convenience, we list now the most important peculiarities of the proposed new class of superconformal WZ sigma models with focusing on its distinctions from the type considered previously /5,6/:

(a) These nonlinear sigma models are defined on coset manifolds of N-extended D = 2 conformal supergroups. Though the number of coset parameters is originally infinite one succeeds in imposing co-

variant constraints on them so as to leave in game a finite set of parameters which are identified then with the physical bosonic and fermionic fields. The internal symmetry group of underlying bosonic WZ sigma model for a given N is $SO_-(N) \times SO_+(N)$. It enters the whole supergroup in a nontrivial way and does not commute with D = 2 supersymmetry being the automorphism group of the latter.

(b) The manifold of physical bosons for a given N is the product $M_{D_N} \times \frac{SO_-(N)\times SO_+(N)}{SO(N)}$, M_{D_N} being a D_N-dimensional Euclidean space with

$$D_N = 2^{N-1} - \frac{N(N-1)}{2} \tag{5.1}$$

(the bosonic manifold dimension is determined by counting the pairs of independent bosonic left and right movers, for details see Appendix B). Among the fields with values in M_{D_N} one always finds the dilaton $u(x)$. The remainder transforms according to real representations $\left.\begin{array}{c}\square\\ \vdots\\ \square\end{array}\right\}k$ ($k = 4,6,\ldots,2\left[\frac{N}{2}\right]$, $N \geq 4$) of $SO_-(N)$ (or, equivalently, of $SO_+(N)$, see footnote on p. 14).

(c) On the classical level there arises the relation between the parameters of bosonic WZ action needed to maintain conformal invariance in the quantum case. This is owing to the presence of two commuting $SO_\pm(N)$ - Kac-Moody symmetries in the underlying N-extended superconformal symmetry.

(d) The complete sigma model action may include potential terms which are superconformally invariant on their own and necessarily incorporate the Liouville term for dilaton. Thus, the models in question can be equally viewed as higher N superextensions of the Liouville theory. Note that in the standard D = 2 sigma models with extended supersymmetry an invariant introduction of potential terms becomes possible only after extending the original superalgebra by central charges (see, e.g. /20/).

(e) For any N these models are classically integrable. Their equations of motion are equivalent to vanishing of curvature of certain 1-superform (with values in $osp(N|2)$ or $su(2|1,1)$).

(f) The N=3 and N=4 models (3.6) and (4.13) provide us with the first examples of Lagrangian field theories which possess D = 2 superconformal symmetry having among its generators the noncanonivariant constraints on them so as to leave in game a finite set of parameters which are identified then with the physical bosonic and fermionic fields. The internal symmetry group of underlying bosonic WZ sigma model for a given N is $SO_-(N) \times SO_+(N)$. It enters the whole supergroup in a nontrivial way and does not commute with D = 2 supersymmetry being the automorphism group of the latter.

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cal ones. These models are self-contained (at least, classically), all the involved fields have the true physical dimension and SO(1,1) -weights and enter the action with correct kinetic terms. It should be pointed out that noncanonical symmetries in the present case are spontaneously broken, in contrast, e.g., with the standpoint of[9] where those assumed to be unbroken. It would be interesting to inquire the quantum structure of these models, at least in the limit $m = 0$ where the standard techniques of refs.[8,5,21] are applicable. This would allow to test them for quantum self-consistency and, in particular, to evaluate the central terms (if exist) in corresponding quantum commutators of superconformal charges.

There remain many other things still to be completed, such as solving of general constraints (2.9), (2.11) off-shell via proper unconstrained prepotentials and finding the relevant superfield actions. This can hopefully be done in the framework of the harmonic superspace approach[22] that proved to be most adequate for handling extended supersymmetries. However, as mentioned in the end of Sect.2, the construction of invariant actions for $N > 4$ may encounter troubles related to the appearance of the notoph-type differential constraints off-shell. These are necessarily required in order to ensure a correct number of off-shell bosonic degrees of freedom but in general may have no local solution via potentials because of nonlinearities in WZ fields (as distinct from the simplest N = 4 condtion (4.5)). A possible way out is, of course, to implement them in the action with the help of Lagrange multipliers.

An interesting development would be the generalization to the case of D = 2 heterotic superconformal symmetries of the type (N,M), $N \neq M$. An immediate heterotic analog of our square N x N -matrix superfield $q^{ij}(z)$ is a rectangular N x M -matrix superfield $q^{ij}(z)$ ($i = 1,\ldots N$; $j = 1,\ldots M$) depending on coordinates $(x^\pm, \theta^{-i}, \theta^{+j})$ and subjected, by each of its indices, to the constraints of the type (2.9). The dynamical equation (2.10) is also generalized in an obvious way. The WZ fields will parametrize the nonsymmetric coset $SO_-(N) \times SO_+(M)/SO(P)$, $P = \min(N,M)$.

We end with several comments concerning a possible relation to string theories. As is known, any conformally invariant D = 2 sigma model can be regarded as a candidate for string theory, bosonic or fermionic (with proper Virasoro-type constraints added). What sort of fermionic strings could be associated then with the models constructed here? These strings should possess extended D = 2 superconformal cal ones. These models are self-contained (at least, classically), all the involved fields have the true physical dimension and SO(1,1) -weights and enter the action with correct kinetic terms. It should be pointed out that noncanonical symmetries in the present case are spontaneously broken, in contrast, e.g., with the standpoint of [9] where those assumed to be unbroken. It would be interesting to inquire the quantum structure of these models, at least in the limit $m = 0$ where the standard techniques of refs. [8,5,21] are applicable. This would allow to test them for quantum self-consistency and, in particular, to evaluate the central terms (if exist) in corresponding quantum commutators of superconformal charges.

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An interesting development would be the generalization to the case of $D = 2$ heterotic superconformal symmetries of the type (N,M), $N \neq M$. An immediate heterotic analog of our square $N \times N$ -matrix superfield $q^{ij}(z)$ is a rectangular $N \times M$ -matrix superfield $q^{ij}(z)$ ($i = 1,\ldots N$; $j = 1,\ldots M$) depending on coordinates $(x^{\pm}, \theta^{-i}, \theta^{+j})$ and subjected, by each of its indices, to the constraints of the type (2.9). The dynamical equation (2.10) is also generalized in an obvious way. The WZ fields will parametrize the nonsymmetric coset $SO_-(N) \times SO_+(M)/SO(P)$, $P = \min(N,M)$.

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Clearly, the question of how consistent is the string interpretation of the proposed models will be possible to answer in a full generality only within the complete quantum framework. Keeping in mind the above peculiarities, one may expect serious modifications as compared with the current schemes. This regards, e.g., the flat-space critical dimension formulas. It may happen that the present models exist quantum-mechanically only for special values of N, namely those which ensure the quantum critical dimension to coincide with the classical one.

An important step towards a string interpretation of models in question is their reformulation in a locally supersymmetric way by coupling to conformal $D = 2$ supergravities. This would help to reveal what are the relevant generalized Virasoro conditions. As a matter of fact, it is the whole infinite-dimensional superconformal symmetry that has to be gauged and we expect a close contact at this point with the recent construction by van Holten [11].

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APPENDIX A. General D = 2 Superconformal Algebras

N-extended superconformal algebra in two dimensions consists of two independent light-cone components, each being generated by an infinite set of generators $A_n^{[i_1\dots i_R]}$. Here $R = 0,1,\dots \mathcal{N}$, every index i runs over vector representation of SO(N), [ ] means antisymmetrization and $n \in \mathbb{Z}$ or $\mathbb{Z} + 1/2$, depending on whether R is even or odd. (We assume the Neveu-Schwarz type grading). Discarding possible central terms, these generators obey the following (anti) commutation relations/10,11/

$$A_n^{[i_1\dots i_R]} A_m^{[j_1\dots j_S]} + (-1)^{RS+1} A_m^{[j_1\dots j_S]} A_n^{[i_1\dots i_R]} =$$
$$= (i)^{-RS}\Big\{[n(2-S)-m(2-R)]\, A_{n+m}^{[i_1\dots i_R j_1\dots j_S]} -$$
$$-i\sum_{h=1}^{R}\sum_{k=1}^{S}(-1)^{h+k+S}\,\delta^{i_h,j_k}\,A_{n+m}^{[i_1\dots \hat{i}_h\dots i_R j_1\dots \hat{j}_k\dots j_S]}\Big\} \tag{A.1}$$

(a hat means that the corresponding index should be missed).

We are interested in a contact subalgebra $\mathbb{K}(1|\mathcal{N})$ of (A.1)/10/ generated by $A_n^{[i_1\dots i_R]}$ $(R = 0,1,\dots \mathcal{N};\ \frac{R-2}{2} \le n < +\infty)$.

Note that one may impose the reality condition

$$\bar{A}_n^{[i_1\dots i_R]} = (-i)^{R+2} A_n^{[i_1\dots i_R]} \tag{A.2}$$

consistently with the relations (A.1). It is convenient to redefine generators so that the reality property takes the familiar form:

$$\overline{\tilde{A}_n^{[i_1\dots i_R]}} = \tilde{A}_n^{[i_1\dots i_R]}$$

$$\tilde{A}_n^{[i_1\dots i_R]} = e^{-\frac{\pi i}{4}[2+(-1)^{R+1}R]}\, 2^{\frac{2R+3-(-1)^R}{4}} A_n^{[i_1\dots i_R]}, \tag{A.3}$$

where the numeric coefficient has been included in order to simplify subsequent formulas. Besides we introduce a condensed notation

$$L_n^{2A} \equiv \tilde{A}_n^{[i_1\dots i_{2A}]}, \qquad (A-1 \le n < +\infty)$$

$$G_m^{2A+1} \equiv \tilde{A}_m^{[i_1\dots i_{2A+1}]}, \quad (A-\tfrac{1}{2} \le m < +\infty),\ A = 0,1,\dots,[\tfrac{\mathcal{N}}{2}]. \tag{A.4}$$

Then, the structure relations of superalgebra K(1|N) read

$$i[L_n^{2A}, L_m^{2B}] = [n(1-B)-m(1-A)]\,L_{n+m}^{2(A+B)} + \sum_{h=1}^{2A}\sum_{k=1}^{2B}(-1)^{h+k}\,\delta^{i_h j_k}\,L_{n+m}^{2(A+B-1)},$$

$$i[L_n^{2A}, G_m^{2B+1}] = [n(\tfrac{1}{2}-B)-m(1-A)]\,G_{m+n}^{2(A+B)+1} + $$
$$+\sum_{h=1}^{2A}\sum_{k=1}^{2B+1}(-1)^{k+h}\,\delta^{i_h j_k}\,G_{m+n}^{2(A+B)-1}, \tag{A.5}$$

$$\{G_n^{2A+1}, G_m^{2B+1}\} = 2[n(\tfrac{1}{2}-B)-m(\tfrac{1}{2}-A)]\,L_{n+m}^{2(A+B+1)} -$$
$$-2\sum_{h=1}^{2A+1}\sum_{k=1}^{2B+1}(-1)^{h+k}\,\delta^{i_h j_k}\,L_{n+m}^{2(A+B)}, \quad 0 \le A,B \le [\tfrac{\mathcal{N}}{2}].$$

Finally, we give a detailed exposition of (A.5)for the cases N=3 and N=4 treated in the paper. The generators are specialized so as to keep closer to the notation of/15/. We denote the corresponding superalgebras by the same letter $\mathcal{G}$ which was used throughout the text to represent sums of two isomorphic light-cone branches.

N = 3.

$$\mathcal{G}_3 = \{L_n, G_k^i, L_s^{[ij]} \equiv T_s^k\varepsilon^{ijk}, G_\rho^{[ijk]} \equiv \varepsilon^{ijk}\Gamma_\rho\} \tag{A.6}$$

$$\begin{aligned}
&i[L_n, L_m] = (n-m)L_{n+m}\\
&i[L_n, G_k^i] = (\tfrac{n}{2}-k)G_{k+n}^i\\
&\{G_k^i, G_r^j\} = -2\delta^{ij}L_{k+r} + (k-r)\varepsilon^{ijk}T_{k+r}^k\\
&i[L_n, \Gamma_p] = -(\tfrac{n}{2}+p)\Gamma_{p+n}\\
&i[\Gamma_p, T_s^i] = \{\Gamma_p, \Gamma_q\} = 0\\
&i\{\Gamma_p, G_r^i\} = -T_{p+r}^i\\
&i[T_s^i, T_e^j] = \varepsilon^{ijk}T_{s+e}^k\\
&i[T_s^i, G_r^j] = \varepsilon^{ijk}G_{r+s}^k + s\,\delta^{ij}\Gamma_{r+s}\\
&i[T_s^i, L_m] = s\,T_{s+m}^i
\end{aligned} \tag{A.7}$$

N = 4.

$$\mathcal{G}_4 = \{ L_n ,\ G_r^i ,\ L_s^{[ij]} \equiv T_s^{ij} ,\ G_p^{[ijk]} \equiv \varepsilon^{ijkl} \Gamma_p^l ,\ L_s^{[ijkl]} \equiv \varepsilon^{ijkl} \Delta_s \} \tag{A.8}$$

$$\begin{aligned}
& i[L_n, L_m] = (n-m) L_{n+m} \\
& i[L_n, G_r^i] = \left(\tfrac{n}{2} - r\right) G_{r+n}^i \\
& i[L_n, \Gamma_p^i] = -\left(\tfrac{n}{2} + p\right) \Gamma_{p+n}^i \\
& i[L_n, T_m^{ij}] = -m T_{m+n}^{ij} \\
& i[L_n, \Delta_s] = -(n+s) \Delta_{n+s} \\
& i[T_n^{ij}, T_m^{kl}] = \delta^{ik} T_{n+m}^{jl} - \delta^{il} T_{n+m}^{jk} + \delta^{jl} T_{n+m}^{ik} - \delta^{jk} T_{n+m}^{il} \\
& i[T_n^{ij}, G_r^k] = \tfrac{1}{2} n \varepsilon^{ijkl} \Gamma_{n+r}^l + \delta^{ik} G_{r+n}^j - \delta^{jk} G_{n+r}^i \\
& i[T_n^{ij}, \Gamma_p^k] = \delta^{ik} \Gamma_{p+n}^j - \delta^{jk} \Gamma_{p+n}^i \\
& i[T_n^{ij}, \Delta_p] = 0 \\
& \{ G_p^i , G_r^j \} = -2 \delta^{ij} L_{p+r} + (p-r) T_{p+r}^{ij} \\
& \{ G_r^i , \Gamma_s^j \} = (r+s) \delta^{ij} \Delta_{r+s} - \varepsilon^{ijkl} T_{r+s}^{kl} \\
& i[G_r^i, \Delta_p] = \Gamma_{p+r}^i \\
& \{ \Gamma_p , \Gamma_s \} = 0 \\
& i[\Gamma_p, \Delta_s] = 0 \\
& i[\Delta_p, \Delta_s] = 0
\end{aligned} \tag{A.9}$$

Superalgebra $\mathcal{G}_4$ contains two minimal N = 4 superconformal subalgebras $\tilde{\mathcal{G}}_4^{(\pm)}$ arranged as

$$\tilde{\mathcal{G}}_4^{(\pm)} = \begin{cases} \mathcal{L}_n^{(\pm)} = L_n \pm \dfrac{n(n+1)}{4} \Delta_n , & n = -1, 0, 1, \ldots \\ \mathcal{G}_r^{(\pm)i} = G_r^i \mp \tfrac{1}{2}\left(r + \tfrac{1}{2}\right) \Gamma_r^i , & r = -\tfrac{1}{2}, \tfrac{1}{2}, \ldots \\ \mathcal{T}_m^{(\pm)ij} = \tfrac{1}{2}\left(T_m^{ij} \pm \tfrac{1}{2} \varepsilon^{ijkl} T_m^{kl}\right) , & m = 0, 1, 2, \ldots \end{cases} \tag{A.10}$$

Here, $\mathcal{T}_n^{(\pm)ij}$ generate two independent SU(2) Kac-Moody algebras whose sum is SO(4) Kac-Moody algebra with generators T_n^{ij}. The superalgebra $SU(1,1|2)$ corresponding to the choice of $\alpha = 1/2\ (-1/2)$ in eq.(4.1) is formed by sums of generators $\mathcal{L}_{-1}^{(+)}$, $\mathcal{L}_1^{(+)}$, $\mathcal{G}_{-\frac{1}{2}}^{(+)i}$, $\mathcal{G}_{\frac{1}{2}}^{(+)i}$, $\mathcal{T}_0^{(+)ij}$ ($\mathcal{L}_{-1}^{(-)}$, $\mathcal{L}_1^{(+)}$, $\mathcal{G}_{-\frac{1}{2}}^{(-)i}$, $\mathcal{G}_{\frac{1}{2}}^{(-)i}$, $\mathcal{T}_0^{(-)ij}$) and of the analogous generators coming from the complementary light-cone branch of full N = 4 contact superalgebra. Superalgebra $\tilde{\mathcal{G}}_4^{(+)}$ together with its copy contained in the second branch constitute the symmetry algebra of minimal N = 4 sigma model (4.23).

APPENDIX B. On-Shell Structure of WZ Supermultiplet for any N

Here we examine the component content of the left-moving superfield $q_L^{il}(Z^-)$ defined by eqs.(2.12) and (2.13b). The analysis for right movers proceeds similarly.

It will be convenient for us to define the superfield projections of $q_L^{il}(Z^-)$ in the following way

$$\begin{aligned} & q_L^{il}(z^-),\ F^{i_1 i_2 i_3}(z^-) \equiv q_L^{i_1 l}(z^-) D_-^{i_2} q_L^{i_3 l}(z^-), \ldots , \\ & \qquad F^{i_1 \ldots i_{m+1}}(z^-) \equiv q_L^{i_1 l}(z^-) D_-^{[i_2} \ldots D_-^{i_m]} q_L^{i_{m+1} l}(z^-) , \ldots \end{aligned} \tag{B.1}$$

Without loss of generality indices of spinor derivatives are assumed to be totally antisymmetrized because any symmetric pair of them would yield x^--derivative by the anticommutation relations (2.5).

We wish to show that the constraint (2.13b) together with the nonlinear orthogonality conditions

$$q_L^{il}(z^-) q_L^{jl}(z^-) = \frac{\delta^{ij}}{N} (q_L^{sm} q_L^{sm}),\quad q_L^{il}(z^-) q_L^{im}(z^-) = \frac{\delta^{lm}}{N} (q_L^{pn} q_L^{pn}) \tag{B.2}$$

leave among the superfields (B.1) an irreducible set of the type pictured in eq. (2.14)

$$q_L^{il}(z^-),\ F^i(z^-) = q_L^{kl}(z^-) D_-^i q_L^{kl}(z^-),\ F^{[i_1 i_2 i_3]}(z^-), \ldots , F^{[i_1 i_2 \ldots i_N]}(z^-) . \tag{B.3}$$

The proof amounts to demonstrating that all the remaining superfield projections are expressed in terms of the basis ones (B.3) and x^--derivatives of the latter.

To begin with, the singlet and two-rank skew-symmetric superfields from the set (2.14) are already contained in the N x N -matrix $q_L^{i\ell}(Z^-)$ as its $1 + \frac{N(N-1)}{2}$ independent superfield parameters. Next, we consider the three-rank tensor $F^{i_1 i_2 i_3}(z^-)$. Using repeatedly eqs.(2.13b) and (B.2) we are able to show

$$F^{i_1 i_2 i_3} = F^{[i_1 i_2 i_3]}(z^-) + \frac{1}{N}\left(\delta^{i_1 i_3} F^{i_2}(z^-) + \delta^{i_2 i_3} F^{i_1}(z^-) - \delta^{i_1 i_2} F^{i_3}(z^-)\right). \qquad (B.4)$$

Further, any subsequent tensor from the set (B.1) can be generated by applying a spinor derivative to the preceding tensor (modulo products of lower rank objects and their x^--derivatives). Thus, the independent components of $F^{i_1 i_2 i_3 i_4}(Z^-)$ can be specified by inspecting $D^{i_1} F^{i_2}$ and $D^{i_1} F^{[i_2 i_3 i_4]}(z^-)$. Taking advantage of anticommutation relations (2.5) together with the general identities following from eqs. (2.13b) and (B.2)

$$D_-^{i_1} \ldots D_-^{i_m} D_-^{\ell} q_L^{\ell n} = iN D_-^{i_1} \ldots D_-^{i_{m-1}} \partial_- q_L^{i_m n} \qquad (B.5)$$

$$D_-^{i_1} \ldots D_-^{i_m} q_L^{js} \cdot q_L^{ks} = -q_L^{js} D_-^{i_1} \ldots D_-^{i_m} q_L^{ks} + \frac{2}{N}\delta^{jk} D_-^{i_1} \ldots D_-^{i_m} q_L^{is} \cdot q_L^{is} + \qquad (B.6)$$

(+ products of lower rank tensors and their x^- derivatives), one easily establishes that $D_-^{i_1} F^{i_2}$ produces no new structures. So, the latter can arise only from $D_-^{i_1} F^{[i_2 i_3 i_4]}(z^-)$ and one has now to show the absence of independent components of the mixed type in this object.

In fact, from this step we may proceed by induction. Assume that the conjecture (B.3) has been proven up to tensors of rank $m \geqslant 3$. Then, we should prove it for the tensor of rank $m + 1$ i.e. to show that all the components of the mixed type $D_-^{(i_{m+1}} F^{[i_1) i_2 \ldots i_m]}$ are expressed through the lower rank tensors and, x-derivatives of the latter.

Note first that $D_-^{i_{m+1}}$ can be carried through $q_L^{\kappa\ell}$ in $F^{[i_1 \ldots i_m]}$ (again discarding certain products of tensors of ranks 3 and $m-2$). Further, when index i_{m+1} is symmetrized with an index from the set $[i_1, \ldots i_m]$, one meets three typical situations. First, another index may hit a spinor derivative. No new structures arise in this case because any symmetrized pair of spinor derivatives reduces to x^--derivative. Alternatively, i_{m+1} may pair with a free index of the second superfield $q_L^{\kappa\ell}(Z^-)$. Up to x^--derivatives, all such terms are reduced to the expressions of the type

$$q_L^{i_1 n}(z^-) D_-^{i_2} \ldots D_-^{(i_{m+1}} q_L^{i_m) n}(z^-)$$

which are in turn expressed via x^--derivatives of the lower rank tensors by the constraint (2.13b) and the identity (B.5) (following from (2.13b)). Finally, the situation when i_{m+1} joins an index of the first superfield $q_L(x^-)$ is reduced to the previous ones by exploiting the general identity (B.6). Thus, the only new independent piece of $F^{i_1 \ldots i_{m+1}}$ is $F^{[i_1 \ldots i_{m+1}]}$. The induction procedure clearly terminates at $m = N$ due to the nonexistence of totally skew-symmetric tensors of rank $> N$. This completes our analysis of the irreducible field content of the (N,0) -superfield $q_L^{i\ell}(Z^-)$ subject to the constraints (2.13b), (B.2). Finally, recall that passing to the fields with correct dimensions involves taking a suitable number of derivatives ∂_- off the fields (B.3), starting with

$$F^{[i_1 i_2 i_3 i_4]} \equiv \partial_- \tilde{F}^{[i_1 i_2 i_3 i_4]} \quad ([F] = cm^{-1}, \ [\tilde{F}] = cm^0) \quad \text{, etc.}$$

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