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ON THE MEANING OF PERTURBATION EXPANSIONS IN QUANTUM FIELD THEORY

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1. INTRODUCTION

In this paper we construct a procedure how to get finite, nontrivial results for physical quantities which within the conventional perturbation theory in fixed renormalisation scheme (RS) are given by divergent expansions in appropriate coupling constant.

The method is based on the idea of Stevenson [1], who showed how the renormalisation group (RG) invariance of the theory can - under certain circumstances- lead to finite results even for highly divergent series. Contrary to him and other authors [2] we, however, do not think that this invariance, when applied to divergent series, implies a unique sum if such a sum can be defined at all. The rolé played by the renormalisation procedure in the construction of nontrivial quantum field theories [3] shows definitely that the renormalisation procedure **cannot be** regarded as purely perturbative in nature. It binds intimately together all aspects of the full theory and therefore its separation into "perturbative" and "nonperturbative" parts is bound to be ambiguous.

The paper is organised as follows. In the next Section the nature of the problem is recalled, necessary notation introduced and the main results of papers [1,2] briefly reviewed. The importance of the RG invariance for the attempts to sum perturbation expansions is discussed in Section 3, where also the main ingredients of our method are formulated and its close connection with the Borel summation technique [4] demonstrated. The implementation of this method by means of higher order RG parameters is covered in Section 4, followed in Section 5 by the comparison of their respective merits. In Section 6 the complications connected with the nonzero value of the coefficient c in eq.(2) below are sketched and numerical results presented. The relation of our results to conventional perturbation theory in fixed RS is clarified and the interpretation of the fundamental ambiguity in our procedure outlined in Section 7.

2. THE NATURE OF THE PROBLEM

In renormalised quantum field theory, such as QCD or QED, the physical quantities are conventionally expressed as perturbative expansions in powers of the renormalised couplant a (we adopt notation of [5]). This couplant depends in the massless case on a set of dimensionless paramaters c_i and a single dimensionfull scale parameter μ, introduced in the process of renormalisation. In the following we discuss in detail the case of massless QCD with n_f flavours of quarks.

For physical quantities each set of parameters μ, c_i defines certain RS (for Green functions additional parameters are needed for a unique specification of a given RS). Let us consider in such a fixed RS perturbation expansion of some physical quantity R, depending for simplicity on a single external variable Q, in the form

$$R(Q) = a^{d-1}\sum_{k=0}^{\infty} r_k a^{k+1}; \; r_0 = 1, \tag{1}$$

where the couplant $a(\mu, c_i)$ obeys the equation

$$\frac{da(\mu,c_i)}{d \ln\mu} \equiv b\hat{\beta}(a) = -ba^2(1+ca+c_2a^2+\dots). \tag{2}$$

We concentrate on the case d=1, for generalisation to $d \neq 1$ see Section 3.10. In massless QCD the coefficients b,c are fixed once the number of quark flavours is given: $b=(33-2n_f)/6$, $c=(153-19n_f)/(66-4n_f)$. The arbitrariness in the choice of the couplant $a(\mu, c_i)$ is then a direct consequence of the freedom in the choice of μ and c_i, $i \geq 2$.

Within the class of "finite" RS (i.e. those in which all but finite number of c_i's are zero and in fact in any RS in which the r.h.s. of (2) is well-defined convergent series) the equation (2) can be integrated with some consistent boundary condition like [5]

$$\tau \equiv b \ln\frac{\mu}{\Lambda} = \frac{1}{a} + c\ln\frac{ca}{1+ca} + \int_0^a \left(-\frac{1}{x^2B(x)} + \frac{1}{x^2(1+cx)}\right)dx, \tag{3}$$

where $B(x)=1+cx+c_2x^2+c_3x^3+\dots$. The dimensionfull parameter Λ appearing in (3) specifies unambiguously which of the solutions to (2) we have in mind. Conventional N-th order perturbation expansion for the quantity R is usually defined by truncating (1) and (2) to the same order [6]. This is, however, a rather arbitrary step. From conceptual point of view it would certainly be better to define once and for all orders of (1) our expansion parameter $a(\tau, c_i)$ taking it from the class of the well-defined "finite" RS and then to investigate the convergence of expansions like in (1), which is what we are really interested in. Unfortunately, there are numerous indications, reanalysed recently in [1], that the perturbation expansions in such a fixed RS are highly divergent. The next best choice is to allow for the variation of the RS, but in such a way that the corresponding couplant has a well-defined limit for $N \to \infty$.

There is little sense in complicating the situation by considering the expansion (1) in such RS where the expansion parameter itself is ill-defined in the limit $N \to \infty$. This attitude has originally been suggested in [7] and we stick to it in this paper too.

Although in some sense the parameter τ plays an exceptional role as it is connected with the regularisation procedure, mathematically all the parameters τ, c_i are on the same footing. We can for instance write down the analogue of eq.(2), looking this time for the derivative of the couplant with respect to c_i [5]

$$\frac{da(\tau,c_i)}{dc_i} \equiv \beta_i(a) = -\hat{\beta}(a)\int_0^a \frac{x^{i+2}}{[\hat{\beta}(x)]^2}dx = \sum_{k=i+1}^{\infty} h_k a^k. \tag{4}$$

The parameters τ, c_i can, within their definition region (i.e. so long as the couplant stays positive number) be chosen at will, but the RG invariance binds together the behaviour of the couplant a as a function of these variables with that of the coefficients r_k [5]

$$r_1(\tau) = \tau - \rho_1; \; r_2(\tau) = \rho_2 + r_1c + r_1^2 - c_2, \text{ etc.} \tag{5}$$

where all the ρ_i are RS invariants, depending merely on the external momentum Q. The relations (5) express the formal consistency of perturbative expansions in various RS, in the sense that the N-th order partial sum

$$R^N(\tau, c_i; i \leq N-1) \equiv \sum_{k=0}^{N-1} r_k(\tau,c_i)a^{k+1}(\tau,c_i) \tag{6}$$

varies by amount proportional to a^{N+1} when we change the RS, i.e., the values of τ, c_i, $i \in (2, N-1)$. Increasing the order N not only are further terms added in (6), but in general also the couplant may change as more of the coefficients c_i enter the game. Exploiting all the available parameters c_i, $i \leq N-1$ was essential for the Principal of Minimum Sensitivity [5] to work, but apart from this it has no special justification. We shall on the contrary take the number of c_i's used fixed for all orders and investigate the consequences of the RG invariance for each of these parameters separately.

Changing the value of τ or c_i we get by means of (5) another series corresponding to some other RS. Both the couplant and the coefficients r_k will be different, but should the original series be convergent, so would be the new one and moreover they would give the same result. This is the implication of the RG invariance for convergent series. For them the choice of a particular RS influences the results at each finite order, but the relation (5) guarantees that the full sum (1) is independent of it. There is a number of methods trying to resolve this ambiguity at finite order [5,8,9] each of them assuming that there is indeed a unique meaning of the full sum in (1) and that the problem is merely a question of how best and fast to approach it.

In QCD, however, the expansions in (1) are likely to be divergent in any fixed RS [1,7]. In such circumstances, the question of the uniqueness of the perturbation results cannot be answered prior to giving these formal expressions some good meaning. In this case the RG invariance gives us merely an infinite number of divergent series of the type (1), each of them associated with one particular RS, connected by relations (5). These relations express now only the formal consistency - in the sense mentioned above - of all these series, but do not by itself help us in summing them.

For divergent series we interpret the requirement of RG invariance, beside the relations (5), as the condition, for the moment rather vaguely defined, that all the RS should be treated on the same footing. In other words, when attempting to sum divergent series like (1) we should keep in mind that we are dealing not with one particular series, but rather with the whole infinite set of them. Starting from some initial series in RS $=\{\tau^0, c_i^0\}$, the RG invariance generates for us by means of (5) the coefficients r_k in any other RS$=\{\tau, c_i\}$. They, together with the new couplant given in (2), define another divergent series, which could equally well serve as the initial one. The sum we are looking for should not discriminate one RS with respect to others. This is the most we can get from RG invariance for divergent series.

In [1] Stevenson suggested a possible scenario of how to get finite and nontrivial results for the sum of divergent series, exploiting the above mentioned RG, restricted in his example to the subgroup associated with the change of the variable τ. Within the class of these "zero" schemes he discussed a toy example of the series

$$\sum_{k=0}^{\infty}(-1)^k k!\, a_0^{k+1} \tag{7}$$

which can be considered as (1) in some initial RS. For the above series his Principle of Minimum Sensitivity implies that for each finite sum of the first N terms in (1) the "optimal" value of τ is not constant behaving at large N as $\tau(N)=\chi_0 N$, $\chi_0 \doteq 0.278$ and consequently

$$\lim_{N\to\infty}\sum_{k=0}^{N-1} r_k(\tau(N))\, a^{k+1}(\tau(N)) = \int_0^{1/\chi_0}\frac{\exp(-u/a_0)}{1+u}\,du \tag{8}$$

is finite and closely related to the Borel sum of (7).

This example demonstrates that we can get a finite result for the limit $\lim_{N\to\infty} R^N(\tau(N))$ provided τ is not fixed as $N\to\infty$ but increases to infinity. The optimisation condition supplies just the right dependence $\tau(N)$ to yield the finite result. This conclusion has been generalised

in [2] to a wider class of series (7) and shown to depend on the assumption that the Borel transform of the series (1) has finite radius of convergence r^0. To get finite result in the case of series (7) we can, however, take any value $\chi \geq \chi_0$ of the factor χ in the relation $\tau(N)=\chi N$. In [2] the optimised result i.e. $\chi=\chi_0$ was shown to correspond to the maximal possible result of all the convergent limiting procedures (8). This fact was regarded in [2] as a strong argument in favour of considering (8) as a "correct representation" of the perturbative part of the physical quantity R(Q). We return to this claim in the next Section.

3. THE METHOD:

3.1 General remarks.

The physical question we want to have answered is the following: are the higher order terms in (1) really so overwhelmingly important as indicated by the divergence of these expansions in fixed RS, or do they in some way compensate each other between different orders, so that only a few lowest orders are of practical interest? We feel that the mere divergence of expansions (1) does not imply the dominance of high orders but to answer this question honestly we should from the beginning take them seriously.

Our aim is thus to construct a method that takes into account all orders of perturbation expansions, but yields finite results even in the case when (1) is divergent in fixed RS. The basic idea has already been mentioned in the previous Section. To make such a method of practical use, we furthermore require that it

a/ works order by order, using conventional calculations in fixed RS
b/ converges for $N\to\infty$ in the conventional sense
c/ contains no analytical extrapolations of any kind (as those employed in Borel summation technique and its variations)
d/ respects RG invariance in the sense mentioned earlier

3.2. Basic formulae

In this Section the results of ref. [1] are derived in a different manner, which, contrary to the original derivation in [1], is applicable to completely arbitrary coefficients $r_k(\tau)$ of (1) at some initial $\tau=\tau^0$

$$\sum_{k=0}^{\infty} r_k(\tau^0)\, a^{k+1}(\tau^0)\ ;\quad a_0 \equiv a(\tau^0) \tag{9}$$

and which makes transparent the close relation of the "sum" of (9) to the corresponding Borel sum. For the moment we stay in the zero RS, the case of $c_i \neq 0$ is subject of Sections 4 and 5. We furthermore assume c=0 in this Section. The technical complications connected with nonzero value of c in realistic QCD are considerable, but do not change the situation in any essential way (see Section 6). Defining now as the initial RS that corresponding to $\tau^0=0$, the RG invariance dictates the dependence of the coefficients $r_k(\tau)$ on τ through relation (5) which can be recast into a form of reccurence differential equations

$$\frac{dr_k(\tau)}{d\tau} = k\, r_{k-1}(\tau) \qquad (10)$$

with solutions $(r_k(\tau=0))\equiv r_k(0))$

$$r_k(\tau) = \sum_{l=0}^{k} \tau^l r_{k-l}(0)\binom{k}{l} = \tau^k \sum_{l=0}^{k}\frac{r_l(0)}{\tau^l}\binom{k}{l}.$$

From (10) we first form the N-th partial sum

$$R^N(\tau) \equiv \sum_{k=0}^{N-1} r_k(\tau)\, a^{k+1}(\tau) = \sum_{l=0}^{N-1}\frac{r_l(0)}{\tau^{l+1}}\sum_{k=l}^{N-1}\binom{k}{l} \qquad (11)$$

and then using the relation [10]

$$\sum_{k=l}^{N-1}\binom{k}{l} = \binom{N}{l+1}$$

arrive at

$$R^N(\tau) = \sum_{l=0}^{N-1}\frac{r_l(0)}{\tau^{l+1}}\binom{N}{l+1}. \qquad (12)$$

We now investigate the class of limitting procedures, defined by specifying the N-dependence of τ by means of two parameters

$$\tau(N) = \chi N^{\beta} \qquad (13)$$

and obtain

$$R(\chi,\beta) = \lim_{N\to\infty} R^N(\tau(N)) = \lim_{N\to\infty}\sum_{l=0}^{N-1}\frac{r_l(0)}{(l+1)!}\left(\frac{N^{1-\beta}}{\chi}\right)^{l+1}\alpha_l(N), \qquad (14)$$

where the factors $\alpha_l(N)=N!/((N-l-1)!N^{l+1})$ go to unity for fixed l when $N\to\infty$.

3.3 The case $\beta=1$

For $\beta=1$ and provided that the Borel transform of the initial series (9) at $\tau^0=0$ has a nonzero radius of convergence r^0, we can set $\alpha_l(N)=1$ in (14) and thus get

$$R(\chi,\beta=1) = \sum_{l=0}^{\infty}\frac{r_l(0)}{(l+1)!}\left(\frac{1}{\chi}\right)^{l+1}. \qquad (15)$$

This expression points immediately to the close relation of our procedure to the Borel summation as it can equivalently be rewritten as

$$R(\chi,\beta=1) = \int_0^{1/\chi}\sum_{l=0}^{\infty}\frac{r_l(0)}{l!}u^l\,du \qquad (16)$$

which **differs** from the Borel sum of (9) at $\tau^0=0$ merely by the finite

upper integration bound (at $\tau^0=0$, $a(\tau^0)=\infty$ and thus the weight factor $\exp(-u/a(\tau^0))$ equals unity). The intimate connection of (15) with the Borel sum, so immediate above is in the case of general r_k far from obvious if one follows the derivation in [1,2]. We stress that the factor multiplying $r_k(0)$ in (15) is a nontrivial limit of the sum of contributions coming from increasing number of terms $r_k(\tau(N))\,a^{k+1}(\tau(N))$ as N and consequently $\tau(N)$ go to infinity.

The series (15) provides therefore well-defined representation of $R(\chi,\beta=1)$ for $1/\chi < r^0$. For $1/\chi$ beyond r^0 (15) is of no direct use, but in certain cases (14) can still converge if we carefully take into account the factors $\alpha_l(N)$. This had been demonstrated in [2] for the toy example $r_k(0)=(-1)^k k!$, where $r^0=1$, but (14) converges up to $1/\chi_0 \doteq 3.55$. A word of caution is, however, in order here. Although in the above example we can go beyond r^0, this possibility is of little practical use. We obtained finite limit (14) because we knew exactly all the coefficients r_k. In practice we can derive the asymptotic behaviour of r as $k\to\infty$ and calculate **explicitly** a few of the lowest ones. This would be sufficient to determine r^0 but would not allow us to go beyond it. There the eventual finite limit of (14) is a consequence of subtle cancellations between large numbers of opposite signs, which necessitate exact knowledge of all r_k. Were such an information available, as in the toy example of [1,2], we could evaluate $R(\chi,\beta=1)$ beyond r^0 also through analytical continuation from the region $1/\chi < r^0$. Both procedures require exactly the same kind of information, but the latter allows us to calculate $R(\chi,\beta=1)$ even further, up to $1/\chi=\infty$ where we recover the Borel sum (if it exists)! The point $\chi=\chi_0$ plays no exceptional role, contrary to claims in [2].

3.4 General β

If the original series, given by $r_k(0)$, has $r^0>0$, the formula (14) with $\beta=1$ gives nontrivial, finite result (15). What happens if we take $\beta\neq 1$? To find out, we rewrite (14), setting $\alpha_l(N)=1$:

$$R(\chi,\beta) = \lim_{N\to\infty}\int_0^{N^{1-\beta}/\chi}\sum_{l=0}^{N-1}\frac{r_l(0)}{l!}u^l\,du. \qquad (17)$$

For $\beta>1$ and $N\to\infty$ the upper integration bound $(N^{1-\beta})/\chi$ goes to zero and as the integrand is finite within the nonzero $r^0>0$ the limit **necessarily** vanishes. For $\beta<1$ we have two possibilities:

a/ the series (15) has finite $r^0>0$. Then the upper integration limit goes to infinity crossing eventually r^0 and causing divergence of (17). The same happens if $r^0=\infty$ but $\lim_{\chi\to 0}R(\chi,\beta=1)=\infty$.

b/ $r^0=\infty$ and $\lim_{\chi\to 0}|R(\chi,\beta=1)|<\infty$ as for instance for the series of the type $(-1)^k(qk)!$ with $q<1$. For such series the difference

$$-\int_0^{\left(\frac{1}{\chi}N^{1-\beta}\right)}\sum_{l=N}^{\infty}\frac{r_l(0)}{l!}u^l du-\int_{\left(\frac{1}{\chi}N^{1-\beta}\right)}^{\infty}\sum_{l=0}^{\infty}\frac{r_l(0)}{l!}u^l du$$

of (17) and the Borel sum behaves in the limit $N\to\infty$ as follows. The second term evidently vanishes and the first one approaches the series

$$-\sqrt{q}\,\frac{e^{q-1}}{q^q}\sum_{l=N}^{\infty}(-1)^{l+1}\left[z\left(\frac{N}{l}\right)^{1-q}N^{q-\beta}\right]^{l+1}\frac{1}{l^q};\quad z=\frac{q^q e^{1-q}}{\chi}.$$

This vanishes in the limit $N\to\infty$ for $\beta>q$ and for $\beta=q, z\leq 1$ and oscillates with increasing amplitude for $\beta<q$ and $\beta=q, z>1$. Thus we get in the limit $N\to\infty$ the Borel sum if $q<\beta<1$ as well as for $\beta=q, z<1$. For $\beta<q$ or $\beta=q, z>1$ the sequence $R^N(\chi,\beta)$ diverges. The Borel sum is obtained also for $\beta=1$ in the limit $\chi\to 0$.

3.5 Conditions for finite, nontrivial limit of $R^N(\chi,\beta)$

The above discussion suggests a somewhat special role of $\beta=1$ and the importance of the finite r^0. For series with $r^0=0$, $\beta=1$ does not yield finite results. But could we not choose $\beta>1$ in this case and still get what we want: finite, nontrivial limit of the sequence $R^N(\chi,\beta)$? Unfortunately, we **cannot**. To investigate convergence properties of this sequence we first write down the expression for the difference $\Delta R^N\equiv R^N(\tau(N))-R^{N-1}(\tau(N-1))$ using $Q_N\equiv r_{N-1}(\tau(N))a^N(\tau(N))$

$$\Delta R^N=Q_N+\sum_{k=0}^{N-2}\left[r_k(\tau(N))-r_k(\tau(N-1))\right]a^{k+1}(\tau(N))+\sum_{k=0}^{N-2}r_k(\tau(N))\left[a^{k+1}(\tau(N))-a^{k+1}(\tau(N-1))\right]. \quad (18)$$

The second and third terms in (18) can be reduced by means of (2) and (10) so that we obtain in the limit $N\to\infty$

$$\Delta R^N\approx Q_N-Q_{N-1}+(1-\beta)Q_N.$$

We shall discuss two distinct possibilities for the sequence R^N to converge to a finite limit:

i/ Q_N is a smooth function of N and does not change sign with N (in the asymptotic region $N\to\infty$). Then (18) approximates well the derivative dR^N/dN and the finite limit requires this derivative to vanish for $N\to\infty$ faster than $1/N$. For power behaviour $Q_N\approx N^{\gamma}$ it implies $\gamma>1$, except in the case $\beta=1$, when it suffices $\gamma>0$. This fact shall be crucial in the following.

ii/ Q_N does change sign with $N\to\infty$. Then the criterion with the derivative **cannot** be used, but $\gamma>0$ is still sufficient, because now **the successive terms conspire in such a manner that finite limit** $\lim_{N\to\infty}R^N(\tau(N))$ results, as exemplified by the series $\sum_{k=0}^{\infty}(-1)^k/k^{\alpha}$ which has the finite limit as $N\to\infty$ for any $\alpha>0$.

To determine when we get nontrivial results we calculate the derivative

$$\frac{dR^N(\chi,\beta)}{d(1/\chi)}=\frac{d\tau(N)}{d(1/\chi)}\sum_{k=0}^{N-1}\frac{r_k(0)}{\tau^k}\binom{N}{k}\equiv\chi N Q_N.$$

For finite limit of $R^N(\chi,\beta)$, depending nontrivially on χ, Q_N must not change sign and must behave as $1/N$ for $N\to\infty$. Then, however, only $\beta=1$ is acceptable. So only $\beta=1$ can lead to finite result depending nontrivially on χ.

For series with $r^0=0$, $\beta>1$ is clearly necessary to guarantee finite results. These, however, can not depend on χ and must in fact be equal to zero as $R^N(\chi=\infty,\beta)=0$ for each N and $R(\chi,\beta)$ is continuous function at $1/\chi=0$. For $\beta>1$ we therefore get either zero, ∞ or oscillating behaviour of R^N as $N\to\infty$.

Summarising, we see that $\beta=1$ is the largest value of β, leading to finite, nontrivial results for $R(\chi,\beta)$. For $\beta>1$ only zero, ∞, or oscillating behaviour of R^N is possible, while for $\beta<1$ either ∞, oscillating behaviour or finite, χ-independent result, equal to the Borel sum comes out.

3.6 The case of general τ^0

The derivation of our basic result i.e. the formula (15) is extremely simple due to the choice $\tau^0=0$. If the initial $\tau^0>0$, we can, however, proceed quite analogously with only minor modifications. Defining now $\tau=\bar{\tau}+\tau^0$ so that $a(\bar{\tau})=1/(\bar{\tau}+\tau^0)$ and repeating all previous steps setting this time $\bar{\tau}=\chi N$ we get

$$R(\chi,\beta=1,\tau^0)\equiv\lim_{N\to\infty}\sum_{l=0}^{N-1}r_l(\tau^0)Z_l^N(\chi,\tau^0)=\sum_{l=0}^{\infty}r_l(\tau^0)Z_l(\chi,\tau^0), \quad (19)$$

where

$$Z_l(\chi,\tau^0)=\lim_{N\to\infty}Z_l^N(\chi,\tau^0)=\lim_{N\to\infty}\frac{1}{\bar{\tau}^{l+1}}\sum_{k=l}^{N-1}\left(\bar{\tau}a(\bar{\tau})\right)^{k+1}\binom{k}{l} \quad (20)$$

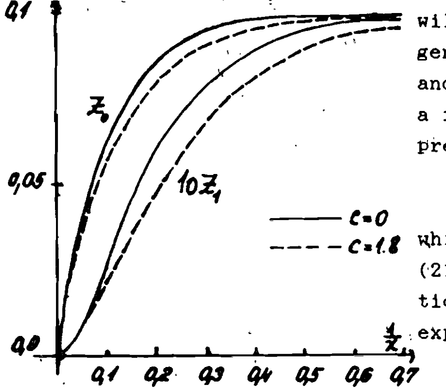
obey

$$\frac{dZ_l(\chi,\tau^0)}{d\tau^0}=-(l+1)Z_{l+1}(\chi,\tau^0). \quad (21)$$

Evaluating **explicitly** $Z_0=(1-\exp(-\tau^0/\chi))/\tau^0$ from (20) we find

$$Z_l(\chi,\tau^0)=\frac{(-1)^l}{l!}\left(\frac{1}{\chi}\right)^{l+1}\frac{d^l}{dy^l}z_0(y),$$

where $z_0(y)=(1-\exp(-y))/y$ and $y=\tau^0/\chi$. Obviously, $Z_l(\chi,\tau^0)$ reduces to $(1/\chi)^{l+1}/(l+1)!$ for fixed χ and $\tau^0\to 0$ as well as for τ^0 fixed and $\chi\to\infty$. For $\tau^0>0$ and $\chi\to 0$ (i.e. $y\to\infty$) only the first term in $z_0(y)$ contributes and we get $Z_l(0,\tau^0)\to(1/\tau^0)^{l+1}=a_0^{l+1}$, recovering, as we must, the original series (9) in fixed RS={ τ^0 }. The shape of the functions $z_\ell(\chi,\tau^0)$ for $\ell=0,1$ and a typical value $a_0=0.1$ **is displayed** in the Figure. If (15), corresponding to $\tau^0=0$, has $r^0>0$, then series (19) will have the same radius of convergence. The reason is that for fixed χ and τ^0 the behaviour of $z_\ell(\chi,\tau^0)$ as a function of ℓ is given by the expression



$$z_\ell(\chi,\tau^0) \underset{\ell\to\infty}{\sim} \left(\frac{1}{\chi}\right)^{\ell+1} \frac{1}{(\ell+1)!} \exp\left(-\frac{\tau^0}{\chi}\right)$$

which is an immediate consequence of (21) and the analyticity of the funtion $z_0(y)$, guaranteeing the Taylor expansion around $y=0$.

We now come to the important point of the potential dependence of our results (19) on τ^0. We began Section 3 with the rather loosely formulated requirement that our procedure must respect RG invariance. We can now be more specific: our results (19) must not depend on the choice of the initial RS, specified in this case by τ^0! That this is indeed the case can be verified quite straightforwardly by differentiating (19) with respect to τ^0, employing in the process (21) and the fact that according to (10) $dr_k(\tau^0)/d\tau^0 = kr_{k-1}(\tau^0)$. We proceed in fact very much in the same way as in the conventional framework where the knowledge of (10) and (2) leads to the formal independence of (9) of τ^0. For $1/\chi < r^0$ our expansions are, on the other hand, convergent series so that all mathematical operations with them make good sense and the results, i.e. (19), are really independent of τ^0!

3.7 The influence of higher RG parameters

So far we have varied τ with N according to (13) but stayed in the zero schemes, where all $c_i=0$. Allowing these parameters to assume arbitrary, but fixed values complicates the derivation, but doesn't influence the final result (19). Taking into account merely c_2, eq. (10) is replaced by

$$\frac{dr_k(\tau,c_2)}{d\tau} = kr_{k-1}(\tau,c_2) - (k-2)c_2 r_{k-3}(\tau,c_2) \tag{22}$$

and also the couplant has a bit more complicated form. Nevertheless, as we shall demonstrate in Section 4.3 the final result for $R(\chi,1)$ is independent of c_2. The same holds for all c_i, $i\geq 2$.

3.8 Application to convergent series.

Let us take $\tau^0=0$ and consider simple convergent series $\sum_{k=0}^{\infty}(-1)^k a_0^{k+1}/k!$, which sums to $a_0\exp(-a_0)$ and thus for $a_0 \to \infty$ (i.e. for $\tau^0=0$) yields conventional result equal to zero. From (14-15) we get for this series

$$R(\chi,\beta=1) = \sum_{k=0}^{\infty}\left(\frac{1}{\chi}\right)^{k+1}\frac{1}{k!(k+1)!} = \frac{1}{\sqrt{\chi}}J_1\left(\frac{2}{\sqrt{\chi}}\right);\quad R(\chi,\beta<1)=0$$

which vanishes for $\chi=0$ (and $\chi=\infty$ as any series (15) with $r^0>0$). This is an example of a general situation: conventional results are for convergent series recovered only for $\beta<1$ or for $\beta=1$, but $\chi\to 0$. For $\beta=1$ and $\chi\neq 0$ the procedure embodied in (14-15) is therefore not regular. For divergent series there is no reason to reject these values and all pairs χ,β yielding finite results are in principle equally acceptable.

The fact that our procedure is **not** regular for the values of χ,β which are required to get finite results in the case of divergent series is nothing absurd, but indicates that it makes little sense to try to split divergent series into "convergent" and "divergent" parts Adding the conventional sum of a convergent series to a divergent one "summed" by means of (15) differs from the result of applying (15) to the formal sum of these series. Similarly for other operations.

3.9 The case of nonoscillatig series

Traditionally the oscillating character of the divergent series has been considered vital for obtaining finite generalised sums. For instance the series with coefficients $r_k(0)=k!$ has in the conventional sense an improper limit ∞ for any a_0. Not so according to (15). The basic reason for this unexpected situation is again the fact that we deal not with one particular series of the mentioned type, but rather with the whole infinite set of them. The formula (15) yields finite results - $R(\chi,1)=-\ln(1-1/\chi)$ - even in the mentioned case.

Contrary to the oscillating series we **cannot**, however, go beyond the point $1/\chi=1$ (our result becomes complex and thus looses physical sense there). For $1/\chi<1$, on the other hand, our result $-\ln(1-1/\chi)$ is no less sensible **than** the one for the oscillating series (7) i.e. $\ln(1+1/\chi)$. The point $1/\chi=1$ corresponds to the conventional result.

3.10 Generalisation to the case $d\neq 1$

For $d\neq 1$ the invariants ρ_i appearing in (5) acquire simple dependence on d [5]. If d is a natural number (1) can be rewritten as

$$\sum_{k=0}^{\infty} r_k(\tau)a^{k+d}(\tau) = \sum_{k=0}^{\infty}\bar{r}_k(\tau)a^{k+1}(\tau), \tag{23}$$

where $\bar{r}_k(\tau)=r_{k+1-d}(\tau)$. We can then apply (15) and get

$$R(\chi,\beta=1)=\sum_{k=0}^{\infty}\frac{r_k(0)}{(k+d)!}\left(\frac{1}{\chi}\right)^{k+d}. \tag{24}$$

We stress that for us (23) is a definition of the formal expression (1) as our procedure does not commute with multiplication of the series. Should we first apply (15) to (1) with d=1 and then multiply the result with $\lim_{N\to\infty} a^{d-1}(\tau(N))=0$ we would get zero instead of (24).

4. The method : c_2

4.1 Basic formulae

In this Section we show that beside τ also any other RG parameters c_i can serve to define a procedure similar to that of the previous Section. We describe in some detail the modifications connected with the use of c_2. For $c_2<0$ the couplant $a(\tau,c_2)$ is defined as a solution of the equation (assuming as above c=0)

$$\tau=\frac{1}{a}+\frac{c_2}{\Delta}\ln\left|\frac{2c_2a-\Delta}{2c_2a+\Delta}\right|, \tag{25}$$

where $\Delta=2\sqrt{|c_2|}$. The introduction of c_2 complicates the dependence of the couplant $a(\tau,c_2)$ on τ but as we shall hold τ fixed and vary c_2 we assume for the moment $\tau=0$ and only later in Section 4.3 do show that the results are in fact independent of this assumption.

For $\tau=0$ the solution of (25) is again simple : $a(0,c_2)=\alpha/\sqrt{-c_2}$ where $\alpha \doteq 0.84$ is the solution of the equation

$$\frac{1}{\alpha}-\frac{1}{2}\ln\left|\frac{1+\alpha}{1-\alpha}\right|=0\,. \tag{26}$$

For $\tau=0$ we also have simply: $d a(0,c_2)/dc_2=h_3 a^3(0,c_2)$, where $h_3=1/2\alpha^2$.
In [1,2] the asymptotic freedom of QCD has been regarded as crucial property for the construction of the finite limit as $N\to\infty$. From the point of view of mathematics involved, the only essential condition for the procedure to work is, however, that as $N\to\infty$ the couplant vanishes sufficiently fast. In the case of c_2 the same situation arises if we send $c_2\to-\infty$. In the following all the steps of the preceding Section will be straightforwardly reformulated for c_2.

The RG invariance with respect to c_2 determines the dependence of the coefficients $r_k(c_2)\equiv r_k(\tau=0,c_2)$ on c_2 through the relations

$$\frac{dr_k(c_2)}{dc_2}=-(k-1)h_3 r_{k-2}(c_2) \tag{27}$$

analogous to (10). They have the solutions

$$r_{2k+1}(c_2)=\sum_{j=0}^{k}\frac{(2k)!!}{(2j)!!}\frac{r_{2j+1}(0)}{(k-j)!}(|c_2|h_3)^{k-j}$$

$$r_{2k}(c_2)=\sum_{j=0}^{k}\frac{(2k-1)!!}{(2j-1)!!}\frac{r_{2j}(0)}{(k-j)!}(|c_2|h_3)^{k-j},$$

where $(2k-1)!!=(2k-1)(2k-3)\ldots1=2^k\Gamma(k+1/2)/\sqrt{\pi}$. Forming the N-th partial sum, separately for odd and even orders, we get $(N=2M+1)$

$$R^N_{odd}(c_2)=\sum_{j=0}^{M}\frac{r_{2j+1}(0)}{(2|c_2|h_3)^{j+1}}\binom{M+1}{j+1}$$

$$R^N_{even}(c_2)=\sum_{j=0}^{M}\frac{r_{2j}(0)}{(2|c_2|h_3)^{j+1/2}}\frac{\Gamma(M+3/2)}{\Gamma(j+1/2)(M-j)!}$$

closely reminiscent of (12). Assuming now

$$c_2(M)=-\chi(2M)^\beta \tag{28}$$

we obtain

$$R^N_{odd}(\chi,\beta)=\sum_{j=0}^{M}\frac{r_{2j+1}(0)}{(j+1)!}\left(\frac{\alpha}{\sqrt{\chi}}\frac{1}{2^{\beta/2}}M^{\frac{1-\beta}{2}}\right)^{2j+2}\alpha_j(M) \tag{29}$$

$$R^N_{even}(\chi,\beta)=\sum_{j=0}^{M}\frac{r_{2j}(0)}{\Gamma(j+1/2)}\left(\frac{\alpha}{\sqrt{\chi}}\frac{1}{2^{\beta/2}}M^{\frac{1-\beta}{2}}\right)^{2j+1}\alpha_j(M), \tag{30}$$

where $\alpha_j(M)$ are the same as in (14). Combining (29-30) we arrive at

$$R(\chi,\beta)=\lim_{N\to\infty}R^N(\chi,\beta)=\lim_{N\to\infty}\sum_{j=0}^{N}\frac{r_j(0)}{\Gamma(\frac{j+1}{2}+1)}\left(\frac{\alpha}{\sqrt{\chi}}\frac{1}{2^{\beta/2}}M^{\frac{1-\beta}{2}}\right)^{j+1}\alpha_{[j/2]}(M)$$

which for $\beta=1$ yields finally

$$R(\chi,\beta=1)=\sum_{j=0}^{\infty}\frac{r_j(0)}{\Gamma(\frac{j+1}{2}+1)}\left(\frac{\alpha}{\sqrt{2\chi}}\right)^{j+1} \tag{31}$$

provided the r.h.s. of it converges. The only essential difference of (31) from (15) lies in the presence of mere $\Gamma((j+1)/2+1)$ compared to $\Gamma(j+2)$ there. The above defined procedure based on the use of c_2 and embodied in (31) is again closely (although not as much as for τ) related to the generalised Borel sum of the series $\sum_{k=0}^{\infty} f_k z^{k+1}$

$$F(z)\equiv\int_0^\infty e^{-t}B(t^\nu z)dt;\quad B(x)=\sum_{k=0}^{\infty}\frac{f_k}{\Gamma(\nu(k+1)+1)}x^{k+1} \tag{32}$$

for $\nu=1/2$. Due to lower value of ν the use of c_2 is less powerful and leads to finite results only for series behaving at most like (k/2)!. For such series, however, both τ- and c_2-based procedures are equally good, although they lead in general to different results.

As far as the conditions for finite, nontrivial results of the limiting procedure specified by (28) are concerned the situation turns out to be the same as for τ: $\beta=1$ is the only value allowing nontrivial, χ-dependent results for $R(\chi,\beta)$. The arguments parallel closely those of Section 3.4 and we shall not repeat them here.

4.2 The independence of $R(\chi, 1)$ of the fixed τ

For technical reasons we took at the beginning of this Section $\tau=0$. To prove that the results (31) do not in fact depend on the value of fixed τ, we first calculate the derivative

$$\frac{dR^N(\chi,\beta,\tau)}{d\tau}=\frac{d}{d\tau}\sum_{k=0}^{N-1} r_k(\tau,c_2(N))\,a^{k+1}(\tau,c_2(N)).$$

Using (2) and (22) this can be written as

$$-(Na)Q_N-(c_2a^2)\left[NaQ_N+(N-1)aQ_{N-1}+(N-2)aQ_{N-2}\right].$$

For $\tau=0$ and c_2 negative we have $-c_2a^2=\alpha^2$, $Na\doteq(\alpha/\sqrt{\chi})N^{1-\beta/2}$ and thus

$$\left.\frac{dR^N(\chi,\beta,\tau)}{d\tau}\right|_{\tau=0}\xrightarrow[N\to\infty]{}-\frac{\alpha}{\sqrt{\chi}}N^{1-\beta/2}Q_N(1-3\alpha^2). \tag{33}$$

If $\beta=1$ then Q_N must behave as $1/N$ to yield finite, χ-dependent results and so in this case $dR^N(0,c_2)/d\tau\sim N^{-1/2}$ as $N\to\infty$. Repeating the above procedure to calculate second and higher derivatives we find that they vanish even faster than (33). As all the derivatives $dR^N/d\tau$ at $\tau=0$ vanish in the limit $N\to\infty$ and the functions $R(0,c_2(N))$ have finite limit too, $R(\chi,\beta=1)=\lim_{N\to\infty}R^N(\chi,\tau,\beta=1)$ must be independent of τ.

Similar steps can be taken to prove the claim made in Section 3.7. In this case the roles of τ and c_2 are reversed and we must use (4) instead of (2) and (27) instead of (10). At $c_2=0$, $da(\tau,c_2)/dc_2=a^3$, $dr_k(\tau,c_2)/dc_2=-(k-1)r_{k-2}$ and thus

$$\left.\frac{dR^N(\tau(N),c_2)}{dc_2}\right|_{c_2=0}\xrightarrow[N\to\infty]{}Na^2Q_N+(N-1)a^2Q_{N-1} \tag{34}$$

which behaves for $N\to\infty$ as $(2/\chi^2)Q_N N^{1-2\beta}$. For finite limit we need again $\beta=1$, $Q_N\approx1/N$ or $\beta\neq1$, $Q\approx1/N^{\gamma}$, $\gamma>1$. In both cases (34) vanishes as $N\to\infty$. Higher derivatives are again vanishing even faster than (34) and therefore the **limiting** $R(\chi,\beta)$ is independent of fixed c_2.

4.3 The case of general c_2^0

Instead of (28), which means we take for the initial RS that specified by $\tau=0, c_2^0=0$, we can build our procedure starting from general c_2^0: $c_2(M)=-2\chi M+c_2^0=-2\chi M(1-y/M)$, where $y=c_2^0/2\chi$. Instead of (31) we get

$$R(\chi,\beta=1)=\sum_{k=0}^{\infty}r_k(c_2^0)\,\tilde{Z}_k(\chi,c_2^0) \tag{35}$$

which is analogous to (19) and where $\tilde{Z}_k(\chi,c_2)$ obey the relations

$$\frac{d\tilde{Z}_k(\chi,c_2^0)}{dc_2^0}=k_3(k+1)Z_{k+2}(\chi,c_2^0)$$

and are therefore given through $\tilde{z}_0(y)=(\sqrt{2\chi}/\alpha)\tilde{Z}_0$, $\tilde{z}_1(y)=(2\chi/\alpha^2)\tilde{Z}_1$:

$$\tilde{Z}_{2k+1}(\chi,c_2^0)=\frac{1}{\Gamma(k+1)}\left(\frac{\alpha}{\sqrt{2\chi}}\right)^{2k+2}\frac{d^k\tilde{z}_1(y)}{dy^k}\ ;\quad y=\frac{c_2^0}{2\chi}$$

$$\tilde{Z}_{2k}(\chi,c_2^0)=\frac{2}{\Gamma(k+\frac{1}{2})}\left(\frac{\alpha}{\sqrt{2\chi}}\right)^{2k+1}\frac{d^k\tilde{z}_0(y)}{dy^k}.$$

The functions $\tilde{z}_0(y)$, $\tilde{z}_1(y)$ have qualitatively the same behaviour as the function $z_0(y)$ in Section 3.6 (with the substitution $y\to-y$, corresponding to the fact that while $\tau^0>0$, $c_2^0<0$). For $\chi\to0$, ($y\to-\infty$) we again have $\tilde{Z}_k(c_2^0,\chi)\to a^{k+1}(c_2^0)$.

5. GENERALISATIONS AND COMPARISON OF τ AND c_i

The procedure described in detail for τ and c_2 can straightforwardly be generalised to any RG parameter c_i. Technical complications rapidly increase, but one feature of the results persists: the higher the value of i, the less powerful the procedure based on the corresponding c_i. So from both principal and practical points of view it is crucial to know the asymptotic behaviour of the coefficients r_k.

Although there are arguments [11], indicating that in QCD perturbation expansions, when considered in fixed RS, are asymptotically factorially divergent and even of constant sign, they may contain flaws [1] to be taken at the face value. Nevertheless, as stressed earlier, our procedure, when τ is employed, works even for this situation. From all the singularities discussed in [6] those associated with infrared renormalons are most likely to find their reflection also in our perturbation expansions. Fortunately, these singularities would lie in the $1/\chi$ plane far from the origin, the first pole being expected [6] at $1/\chi\doteq30$ and therefore should not thwart our procedure.

In previous Sections we demostrated close connection of our formulae and the results of generalised Borel summation techniques. Specifically we saw that the use of τ corresponds to $\nu=1$ in (32) while c_2 is similar in effect to $\nu=1/2$ and generally c_i to $\nu=1/i$. But what about $\nu>1$? From the point of view of mathematics all c_i's are equally good, so could we not define a procedure, analogous to those of previous Sections, which would correspond to $\nu>1$? We can but it suffers a serious drawback.

The whole philosophy of [1] and our paper is based on RG invariance. It is this invariance which generates for us from perturbation expansions in one particular RS an infinite set of other series, associated with other RS. And it was precisely the existence of these series which enabled us to construct our procedure. Without them we would be left with only the "initial" series, such as in (7) and the Borel sum would probably seem as the only plausible result.

For series with $r^0=0$, the formula (15) defines another divergent series, the degree of divergence of which is lowered by a factor $k!$, compared with that of the original series. One might then be tempted to repeat the whole procedure this time starting with (15). In such a way we could handle series up to $(k!)^2$. Repeating it ν-times, series up to $(k!)^{\nu}$ would lead to finite results, which, moreover, would be closely related to the generalised Borel sums [32] for $\nu > 1$.

Although formally conceivable, this second and further steps lack the fundamental ingredient, namely some analogue of the RG invariance which indeed provides the only justification for the whole construction in Sections 3,4.

6. THE EFFECTS OF $c \neq 0$

In the previous Sections we have assumed, for technical reasons, c=0, although in realistic QCD the value of c=1.5-2 is nonnegligible. The complications due to c=0 would, however, otherwise obscure the essence of the exposition. In this Section we indicate the main steps in the derivation of the generalisation of formula (19) to the case $c \neq 0$. Besides the obvious and harmless dependence of the coefficients r_k on c, the only change therein lies in the fact that now also the functions $\mathcal{Z}_\ell(\chi, \tau^0, c)$ do depend on it. Their calculation had been done numerically and the results for $\ell=0$ and 1, $a_0=0.1$ and c=1.8 is displayed **in the Figure,** together with the curve corresponding to c=0. Clearly, the effect of $c \neq 0$ is nonnegligible, but the shapes remain qualitatively the same as for c=0. Specifically it still holds that

a/ $\mathcal{Z}_\ell(\chi, \tau^0, c) \to (1/(\ell+1)!)(1/\chi)^{\ell+1}$ for a_0, c fixed and $1/\chi \to 0$

b/ $\mathcal{Z}_\ell(\chi, \tau^0, c) \to a_0^{\ell+1}$ for a_0, c fixed and $\chi \to 0$.

The essential steps in the construction of our procedure for the case of the τ-variable (i.e. $c_i=0, i\geq 2$) are the same as in Section 3.

1/ We start in some initial RS specified by τ^0 or equivalently the corresponding couplant a_0 as given in (3). We define $\bar{\tau}$ with respect to this τ^0 as $\bar{\tau}=\tau-\tau^0$. The couplant $a(\tau)=a(\bar{\tau}, a_0, c)$ is then given as a solution of the equation

$$\bar{\tau} = \frac{1}{a} - \frac{1}{a_0} + c \ln\left[\frac{a(1+ca_0)}{a_0(1+ca)}\right]. \tag{36}$$

Note that for given a_0, $\bar{\tau}$ the couplant is an analytical function of c in the neighbourhood of c=0. In the course of the derivation we use for instance, the fact that $da/dc = -a^2 \ln(a/a_0)$ at c=0. The equation

(36) is for $c \neq 0$ transcendental and so its solution must be found by numerical methods.

2/ The coefficients r are now determined by a generalisation of (10)

$$\frac{d r_k(\bar{\tau}, c)}{d\bar{\tau}} = k\, r_{k-1}(\bar{\tau}, c) + c(k-1)\, r_{k-2}(\bar{\tau}, c) \tag{37}$$

which suggest the general form of its solution

$$r_k(\bar{\tau}, c) = \sum_{\ell=0}^{k} \bar{\tau}^{\ell} \sum_{j=0}^{k-\ell} c^j r_{k-\ell-j}(\tau_0)\, \alpha(k, \ell, j), \tag{38}$$

where the coefficients $\alpha(k,l,j)$ obey, as the consequence of (37), the reccurence relations ($\alpha(k,l,-1)\equiv 0$)

$$(\ell+1)\alpha(k, \ell+1, j) = k\,\alpha(k-1, \ell, j) + (k-1)\alpha(k-2, \ell, j-1) \;;\quad 0 \leq j \leq k-\ell-1 \tag{39}$$

with the boundary condition $\alpha(k,0,j) = \delta_{0j}$.

3/ Combining now (36) with (38) and assuming $\bar{\tau}(N)=\chi N$ we have

$$R(\chi, \beta=1) = \lim_{N\to\infty} \sum_{k=0}^{N-1} r_k(\bar{\tau}(N))\, a^{k+1}(\bar{\tau}(N), a_0, c) = \sum_{\ell=0}^{\infty} r_\ell(\tau_0)\, \mathcal{Z}_\ell(\chi, \tau^0, c) \tag{40}$$

where

$$\mathcal{Z}_\ell(\chi, \tau^0, c) = \lim_{N\to\infty} \mathcal{Z}_\ell^N(\bar{\tau}(N), \tau^0, c);\; \mathcal{Z}_\ell^N(\bar{\tau}, \tau^0, c) = \left(\frac{1}{\bar{\tau}}\right)^{\ell+1} \sum_{j=0}^{N-\ell-1} \left(\frac{c}{\bar{\tau}}\right)^j \sum_{k=\ell+j}^{N-1} (\bar{\tau} a(\bar{\tau}))^{k+1} \alpha(k, k-\ell-j, j). \tag{41}$$

4/ To prove that the above limit as $N\to\infty$ does indeed exist is complicated by the nontrivial and implicit dependence of $\bar{\tau} a(\tau(N))$ on N and a_0 (for c=0 we had simply $\bar{\tau} a(\tau(N))=1/(1+ 1/(a_0\tau(N)))$). One way how to proceed is to expand $\mathcal{Z}_\ell^N$ around c=0 and then to investigate the limit of each term in the Taylor expansion separately. The problem **boils down** to the proof that for fixed ℓ all the derivatives $d^i \mathcal{Z}_\ell^N / dc^i |_{c=0}$, i=1,2,... have finite limits as $N\to\infty$.

5/ In the simplest case i=1 we first prove by explicit evaluation of (41) that the limit $\lim d\mathcal{Z}_0^N/dc \equiv d\mathcal{Z}_0/dc$ at c=0 does exist and is a differentiable function of τ^0. Then we derive **recurrence** relations

$$\frac{d}{d\tau^0} \frac{\partial \mathcal{Z}_\ell(\chi, \tau^0, c)}{\partial c}\Big|_{c=0} = -(\ell+1)\left[\frac{\partial \mathcal{Z}_{\ell+1}(\chi, \tau^0, c)}{\partial c}\Big|_{c=0} - a_0 \mathcal{Z}_{\ell+1}(\chi, \tau^0, c=0) + \mathcal{Z}_{\ell+2}(\chi, \tau^0, c=0)\right].$$

They prove that also the derivatives of higher $\mathcal{Z}_\ell$ do exist at c=0. Moreover, these relations can be used to show that the total derivatives with respect to c (taking into account also the dependence of a_0 on c as given in (3)) do obey, at c=0, reccurence relations

$$\frac{d}{d\tau^0} \frac{d\mathcal{Z}_\ell(\chi, \tau^0, c)}{dc}\Big|_{c=0} = -(\ell+1)\frac{d}{dc}\left[\mathcal{Z}_{\ell+1}(\chi, \tau^0, c) + c\,\mathcal{Z}_{\ell+2}(\chi, \tau^0, c)\right]_{c=0}.$$

They suggest the following relation at general c :

$$\frac{d\mathcal{Z}_\ell(\chi, \tau^0, c)}{d\tau^0} = -(\ell+1)\left[\mathcal{Z}_{\ell+1}(\chi, \tau^0, c) + c\,\mathcal{Z}_\ell(\chi, \tau^0, c)\right] \tag{42}$$

which for c=0 reduces to (21). It is easy to show that (42) is exactly what is needed to prove that the full sum (40) is independent of τ^0 also in the case $c\neq 0$!

6/ To close the construction we should repeat, with appropriate modifications, step 5 for all higher derivatives. This can be done but is tedious and will be omitted here.

7. INTERPRETATION AND APPLICATIONS - AN OUTLINE

In the previous Sections we have described an algorithm, showing how a judicious use of the RG invariance can lead to finite, nontrivial results for series divergent when considered in fixed RS. These results can, for the τ-variable, be written in two alternative ways

$$R(\chi,\beta=1)=\lim_{N\to\infty}\sum_{k=0}^{N-1} r_k(\tau(N))\,a^{k+1}(\tau(N)) = \sum_{k=0}^{\infty} r_k(\tau^0)\,\mathcal{L}_k(\chi,\tau^0,c), \tag{43}$$

where $\tau(N)=\tau^0+\chi N$. Provided the coefficients r_k diverge at most like k! this series has nonzero radius of convergence r^0, inside which $R(\chi,1)$ can be approached order by order by making use in (43) of conventional perturbation calculations which **supply** the coefficients $r_k(\tau^0)$. To a finite order the sum (43) depends, besides χ, also on the choice of τ^0, exactly in the same way as in conventional approach which, as we know, corresponds to $\chi=0$. We face now three questions.

The first concerns the meaning of the χ-dependence in eq.(43). We consider it a manifestation of the inherent ambiguity in the separation of the full theory into its "perturbative" and "nonperturbative" parts. Accoding to our understanding this ambiguity **cannot** be resolved within the perturbation theory itself. In the forthcoming paper [12] we shall give a number of arguments in favor of this conjecture employing, among other facts, the triviality of the full $\lambda\varphi^4$ theory and draw analogy between the role of our parameter χ and Λ from (3).

Secondly, if the perturbation theory should have any predictive power, then there must be a unique value of χ - to be determined together with Λ from comparison with data - describing with reasonable accuracy all the physical quantities of "perturbative" nature. This last notion is, however, only very loosely defined concept without giving first the formal expressions (1) some good mathematical and physical sense, which is just what we are trying to do. Even without fixing the value of χ we can use our formula (43) to derive mathematically well-defined, χ-independent relations between any pair of physical quantities R_1, R_2

$$R_1=\sum_{k=0}^{\infty} v_k^{12}\,(R_2)^{k+1}, \tag{44}$$

where the coefficients v_k^{12} (RS invariants, formed from r_j, $j\leq k$) are <u>different</u> from those we would obtain by formal manipulations with (1).

We consider relations like (44) as basic results of perturbation theory. Also this point is thoroughly covered in [12].

Third, in practical situations (43) must always be truncated to low order and therefore we must also choose carefully, as in conventional framework, the "optimal" τ^0. Recall that the coefficients r_k

$$r_k(\tau)=r_k(\tau-t) = b_0^k\left(\ln\frac{\mu}{Q}\right)^k + b_1^k\left(\ln\frac{\mu}{Q}\right)^{k-1}+\dots\; r_k(\mu=Q)$$

are in fact functions of the difference $\tau-t$, $t=b\ln(Q/\Lambda)$, and the **last** term above is μ and thus also Q-independent. For fixed k and Q going to infinity the first term dominates and requires $\mu=\varkappa Q$, $\varkappa\cong 1$, to avoid the large logarithms. This is what is done in the conventional framework, where the only problem seems to be the question what exactly $\varkappa$ should be. However, it is **also** the "constant" terms $r_k(\mu=Q)$ usually neglected, which are factorially divergent! It is these terms, which then require the introduction of nonzero χ in the relation (13). By choosing $\tau^0=b\ln(Q/\Lambda)$, as usually, and assuming $\chi>0$, we therefore do not in the least violate the spirit of the RG improved perturbation theory, but merely take into account also the presence of the constant terms, which are usually disregarded due to their small **values for k=1.** If τ^0 is very big, due to big Q, and χ modest, it takes a large N before $\tau(N)$ differs in any significant way from τ^0 and thus for small N the sum $\sum_{k=0}^{N-1} r_k(\tau(N))a^{k+1}(\tau(N))$ shall be practically the same as in fixed RS=$\{\tau^0\}$.

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