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GENERALIZED TRANSPORTS AND DISPLACEMENT VECTORS

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1. INTRODUCTION

This work is devoted to the problem of construction of an invariant generalization in the case of an arbitrary differentiable manifold of the usual relative Euclidean radius-vector (the difference of two radius-vectors in an Euclidean space, i.e. the displacement vector between the end points of these radius-vectors).

In Sect.2, on the basis of /1,3/ we give a brief introduction to the theory of general linear transport (I-transport, generalized transport) (of tensors) along a given curve. Using the concept of an I-transport, in Sect.3 and Sect.4 we define and study some properties of the displacement and deviation vectors, respectively. We end the paper with some concluding remarks in Sect.5.

2. THE GENERALIZED TRANSPORT (I-TRANSPORT)

Let M be an n-dimensional ($n \geq 1$) real differentiable manifold of class C^k, $k \geq 1$, $T^p_{\cdot q}|_x(M)$ be the set of tensors of type $\binom{p}{q}$, p, q being non-negative integers, at $x \in M$, J be a nondegenerate real interval and $\gamma : J \to M$ be a given curve (map).

Following the works /1,3/ we give the following

Definition. The I-transport (=generalized (linear) transport) is the map

$$I^{\gamma}_{x\to y} : T^p_{\cdot q}|_x(M) \to T^p_{\cdot q}|_y(M), \qquad (2.1)$$

which corresponds to any curve $\gamma : J \to M$ and every points $x, y \in \gamma(J)$ and has the properties:

1) Linearity: if $\lambda, \mu \in \mathbf{R}$, $T', T'' \in T^p_{\cdot q}|_x(M)$ and $x, y \in \gamma(J)$, then

$$I^{\gamma}_{x\to y}(\lambda T' + \mu T'') := \lambda I^{\gamma}_{x\to y} T' + \mu I^{\gamma}_{x\to y} T''. \qquad (2.2)$$

2) For any $x \in \gamma(J)$

$$I^{\gamma}_{x\to x} := \mathrm{id}, \tag{2.3}$$

where id denotes the identity map.

3) For every $x, y, z \in \gamma(J)$

$$I^{\gamma}_{y\to z} \circ I^{\gamma}_{x\to y} := I^{\gamma}_{x\to z}. \tag{2.4}$$

4) If γ' is the restriction of γ on any subinterval $J' \subset J$ (i.e. $\gamma'(s) := \gamma(s)$ for $s \in J'$), then

$$I^{\gamma'}_{x\to y} := I^{\gamma}_{x\to y}, \quad x, y \in \gamma'(J') = \gamma(J). \tag{2.5}$$

Remark 1. In the following we shall use this definition especially for the case $\phi = 1$, $q = 0$ ($T^{1}_{\cdot 0}|_x(M) =: T_x(M)$).

Remark 2. For brevity instead of an I-transport along a curve γ we shall often speak about the map or the transport I^{γ}.

Remark 3. It can be proved (see [3]) that the transports such as parallel, Fermi-Walker, Jaumann and others used in the theoretical physics, satisfy the conditions (2.1)-(2.5), i.e., they are concrete examples for I-transports.

Remark 4. In the present work we shall not use the property (2.5). We shall only note that from (2.4) and (2.5) it follows that the I-transport corresponding to a composition of curves is equal to the composition of the I-transports corresponding to these curves [3].

Putting in (2.4) $z = y$ and using (2.3), we get

$$(I^{\gamma}_{x\to y})^{-1} = I^{\gamma}_{y\to x}, \tag{2.6}$$

and hence the maps I^{γ} are linear isomorphisms of the tensor spaces along γ.

If $f, g : \gamma(J) \to \mathbf{R}$ are real functions, then from (2.2) we immediately find

$$I^{\gamma}_{x\to y}(f(x)T' + g(x)T'') = f(x)I^{\gamma}_{x\to y}T' + g(x)I^{\gamma}_{x\to y}T''. \tag{2.7}$$

Let $\{E_i(x)\}$ be a basis in $T^{1}_{\cdot 0}|_x(M) = T_x(M)$, $x \in M$, where here and from now on the latin indices run from 1 to $n = \dim M$ and are referred to an arbitrary (coordinate or noncoordinate as well as holonomic or anholonomic) basis and the Greek indices also run from 1 to n but are referred to a coordinate basis $\{\partial_\alpha = \partial/\partial x^\alpha\}$. So, we have $E_i(x) = A^{\alpha}_{i}(x)\partial_\alpha$, where A^{α}_{i} are some functions of $x \in M$ and $\det \|A^{\alpha}_{i}(x)\| \neq 0, \infty$.

Thus, due to (2.1) $I^{\gamma}_{x\to y}E_i(x) \in T_y(M)$, $x, y \in \gamma(J)$ and hence this vector can be expanded over the basis $\{E_i(y)\}$, i.e., if the map I^{γ} and the bases $\{E_i(y)\}$ and $\{E_i(y)\}$ are given, then there are uniquely defined functions $H^{i}_{\cdot j}(y, x; \gamma)$ such that

$$I^{\gamma}_{x\to y}E_j(x) = H^{i}_{\cdot j}(y, x; \gamma)E_i(x), \quad x, y \in \gamma(J), \tag{2.8}$$

or all the same

$$H^{i}_{\cdot j}(y, x; \gamma) = (I^{\gamma}_{x\to y}E_j(x))^{i}, \tag{2.8'}$$

are the components of the vector $I^{\gamma}_{x\to y}E_j(x)$ in the basis $\{E_i(y)\}$. From (2.8) it follow [3] that the functions (2.8') are components of a two-point tensor [4] from $T_y(M) \otimes T^{*}_{x}(M)$, where $\otimes$ is the tensor product sign and $T^{*}_{x}(M) := T^{0}_{\cdot 1}|_x(M)$.

If $V := V^{i}E_i(x) \in T_x(M)$, $x \in \gamma(J)$, then by virtue of (2.7) and (2.8), we get for any $y \in \gamma(J)$

$$I^{\gamma}_{x\to y}V = (H^{i}_{\cdot j}(y, x; \gamma)V^{j})E_i(x), \tag{2.9}$$

or

$$(I^{\gamma}_{x\to y}V)^{i} = H^{i}_{\cdot j}(y, x; \gamma)V^{j}. \tag{2.9'}$$

From the above discussion we observe that the definition of $I^{\gamma}_{x\to y}$ is equivalent to the definition, in some basis from $T_y(M) \otimes T^{*}_{x}(M)$, of the matrix $\|H^{i}_{\cdot j}(y, x; \gamma)\|$ consisting of the components of a two-point tensor from $T_y(M) \otimes T^{*}_{x}(M)$ in this basis.

Using (2.8) we easily see that (2.3) and (2.4) are equivalent respectively to

$$H^{i}_{\cdot j}(x, x; \gamma) = \delta^{i}_{j}, \tag{2.10}$$

$$H^{i}_{\cdot k}(z, y; \gamma)H^{k}_{\cdot j}(y, x; \gamma) = H^{i}_{\cdot j}(z, x; \gamma), \tag{2.11}$$

where $\delta^{i}_{j} = 1$ for $i = j$ and $\delta^{i}_{j} = 0$ for $i \neq j$ (Kronecker's delta symbol). In the same way we see that (2.6) is equivalent to

$$\|H^{i}_{\cdot j}(y, x; \gamma)\|^{-1} = \|H^{i}_{\cdot j}(x, y; \gamma)\|. \tag{2.12}$$

Remark. If we treat (2.10) and (2.11) as functional equations with respect to $\|H^{i}_{\cdot j}(y, x; \gamma)\|$, then their general solution in fixed bases $\{E_j(x)\}$ and $\{E_i(y)\}$ will be

$$\|H^{i}_{\cdot j}(y,x;\gamma)\| = L(y,\gamma)(L(x;\gamma))^{-1}, \tag{2.13}$$

where $L(z,\gamma)$ is an arbitrary nondegenerate $n\times n$ matrix function of $z \in \gamma(J)$ and $\gamma: J \to M (\det L(z,\gamma) \neq 0, \infty)$. We shall not prove (2.13) because it will not be used in the present work.

At the end of this section let us note that the tensor field T defined on $\gamma(J)$ is said[3] to undergo an I-transport (or to be I-transported) along $\gamma: J \to M$ if

$$T(y) = I^{\gamma}_{x\to y} T(x), \qquad x, y \in \gamma(J). \tag{2.14}$$

From (2.3) and (2.4) it follows that if T is I-transported along γ, then it is uniquely defined on $\gamma(J)$ by its value T_0 at an arbitrary fixed point $x_0 \in \gamma(J)$: $T(x) = I^{\gamma}_{x_0\to x} T_0$, $x \in \gamma(J)$.

3. THE DISPLACEMENT VECTOR

Definition. Let $x, y \in M$, $\gamma: [s', s''] \to M$ be a C^1-curve such that $\gamma(s') = x$ and $\gamma(s'') = y$, I^{γ} be an I-transport along γ and $\dot\gamma$ be the tangent to γ vector field:

$$\dot\gamma(s) := \left(\frac{\partial x^{\alpha}(\gamma(s))}{\partial s}\right)\frac{\partial}{\partial x^{\alpha}}\Big|_{\gamma(s)}, \tag{3.1}$$

where $s \in [s', s'']$ and $\{x^{\alpha}\}$ are coordinates in some neighbourhood of $\gamma(s)$. Then, the vector

$$h(x,y) := h(\gamma, I^{\gamma}; x, y) := \int_{s'}^{s''} I^{\gamma}_{\gamma(s)\to x}\dot\gamma(s)\,ds \in T_x(M) \tag{3.2}$$

is called the displacement vector of y with respect to x (defined by means of the curve γ and the map I^{γ}).

We shall not consider here the properties of the displacement vector because they have been presented in[1] from where the above definition is taken.

Let a covariant differentiation be defined on M, $\phi_x: T_x(M) \to \boldsymbol{R}^n$, $x \in M$ be an arbitrary fixed isomorphism and $\kappa_x: U_x \to T_x(M)$ be defined by $\kappa_x(y) := h(x,y)$, $y \in U_x$, where U_x is a normal neighbourhood of x and $h(x,y)$ is given by (3.2) in which γ is the unique geodesic connecting x and y in U_x and I^{γ} is a parallel transport along γ. Under these conditions in[2] it is proved that the map $\phi_x \circ \kappa_x: U_x \to \boldsymbol{R}^n$ is a coordinate diffeomorphism. In other words, within an isomorphism the

map $y \mapsto h(x,y)$ defines a coordinate system (with an origin at x in U_x). In another work the author will prove that there exist an infinite set of generalized transports I^{γ} and the relevant curves γ and neighbourhoods U_x such that in U_x the map $y \longmapsto h(x,y), y \in U_x$ (γ connects x with y) defines (within an isomorphism) a coordinate system on U_x. Begining with this place we shall suppose hereafter the generalized transports I^{γ} and the curves γ to be defined in such a way that for each $x \in M$ there exists a neighbourhood U_x of x in which the map $y \mapsto h(x,y)$, $y \in U_x$ (or more precisely the map $y \mapsto \phi_x(h(x,y))$, where $\phi_x: T_x(M) \to \boldsymbol{R}^n$ is a fixed isomorphism) to define (through (3.2)) a coordinate system on U_x.

In connection with this assumption let us note that (a) the arbitrariness in the choice of I^{γ} and γ in the definition (3.2) of the displacement vector can be considered as an arbitrariness in the definition of the coordinates of y with respect to x, and (b) as a consequence of this, the displacement vector (3.2) has a meaning of a relative coordinate of y with respect to x.

Using (2.9) we get from (3.2) that in a coordinate basis the displacement vector has the components

$$h^{\alpha}(x,y) = \int_{s'}^{s''} H^{\alpha}_{\cdot\beta}(x,\gamma(s);\gamma)\dot\gamma^{\beta}(s)\,ds. \tag{3.3}$$

Integrating by parts the right-hand side of this equality and using (2.10), we find

$$\begin{aligned} h^{\alpha}(x,y) &= H^{\alpha}_{\cdot\beta}(x,\gamma(s);\gamma)\gamma^{\beta}(s)\Big|_{s'}^{s''} - \int_{s'}^{s''}\frac{\partial H^{\alpha}_{\cdot\beta}(x,\gamma(s);\gamma)}{\partial s}\gamma^{\beta}(s)\,ds = \\ &= H^{\alpha}_{\cdot\beta}(x,y;\gamma)y^{\beta} - x^{\alpha} - \int_{s'}^{s''}\frac{\partial H^{\alpha}_{\cdot\beta}(x,\gamma(s);\gamma)}{\partial s}\gamma^{\beta}(s)\,ds = (y^{\alpha} - x^{\alpha}) + \\ &+ \Big[(H^{\alpha}_{\cdot\beta}(x,y;\gamma) - \delta^{\alpha}_{\beta})y^{\beta} - \int_{s'}^{s''}\frac{\partial H^{\alpha}_{\cdot\beta}(x,\gamma(s);\gamma)}{\partial s}\gamma^{\beta}(s)\,ds\Big], \end{aligned} \tag{3.4}$$

where for brevity we have put $\gamma^{\beta}(s) := (x^{\beta}\circ\gamma)(s)$, $s \in [s', s'']$.

In the Euclidean case (see[2], Sect.III.3.1), when $M = E_n$ (E_n is an n-dimensional standard Euclidean space) and I^{γ} is a parallel transport along γ, we have $H^{\alpha}_{\cdot\beta}(x,y;\gamma) \equiv \delta^{\alpha}_{\beta}$ for every x, y and γ; so, from (3.4) we get

$$h^{\alpha}(x,y)\big|_{E_n} = y^{\alpha} - x^{\alpha}. \tag{3.5}$$

From here we see that the expression in the square brackets in (3.4) yields the deviation of the general displacement vector (3.2) from the usual Euclidean displacement vector (3.5). From (3.4) we see that in the general case this deviation depends on the choice of the curve γ and generalized transport I^{γ} and also on the topology of the manifold M. Generally speaking this deviation has an order of $O((s''-s')^2)$. In fact, on the one hand we have

$$y^{\alpha} - x^{\alpha} = \gamma^{\alpha}(s'') - \gamma^{\alpha}(s') = \dot{\gamma}^{\alpha}(s')(s''-s') + O((s''-s')^2), \quad (3.6)$$

and one the other hand in [1], Sect.II.3, eq.(3.3) we showed that

$$h^{\alpha}(x,y) = \dot{\gamma}^{\alpha}(s')(s''-s') + O((s''-s')^2), \quad (3.7)$$

hence the above result follows from (3.4)-(3.7).

If $s''-s'$ is an infinitesimal constant, i.e., if $s''=s'+ds$, then according to (3.7) the infinitesimal displacement vector[1]

$$\xi(x,y) = \dot{\gamma}(s')ds, \quad (3.8)$$

defines the general displacement vector (3.2) up to second order in ds quantities. Let us note that the vector (3.8) depends on γ only through its tangent vector $\dot{\gamma}$ at the point $x = \gamma(s')$.

4. THE DEVIATION VECTOR

Let us consider the following general construction: Let be given curves $x_a:[s_a', s_a''] \to M$, $a = 1,2$ and $x:[s',s''] \to M$ and one-to-one maps $\tau_a:[s',s''] \to [s_a', s_a'']$ which map the parameter $s \in [s',s'']$ into the parameters $s_a = \tau_a(s) \in [s_a', s_a'']$, $a = 1,2$. Let there be given two one-parameter families of curves $\gamma_s:[r_s', r_s''] \to M$ and $\eta_s:[\rho_s', \rho_s''] \to M$, such that $\gamma_s(r_s') := x_1(\tau_1(s))$, $\gamma_s(r_s'') := x_2(\tau_2(s))$, $\eta_s(\rho_s') := x_1(\tau_1(s))$ and $\eta_s(\rho_s'') := x(s)$, $s \in [s',s'']$. Finally, let there be given generalized transports I^{γ_s} and I^{η_s} (along γ_s and η_s, respectively) having the properties pointed out in Sect.3.

In accordance with [2], Sect.IV.1, we define the deviation vector of x_2 with respect to x_1 relatively to x at $x(s)$ as

$$h_{12}(s,x) := I^{\eta_s}_{x_1(\tau_1(s))\to x(s)} \int_{r_s'}^{r_s''} I^{\gamma_s}_{\gamma_s(r)\to x_1(\tau_1(s))} \dot{\gamma}_s(r)\,dr \in T_{x(s)}(M), \quad (4.1)$$

where

$$\dot{\gamma}_s^{\alpha}(r) := \frac{\partial \gamma_s^{\alpha}(r)}{\partial r}, \qquad r \in [r_s', r_s'']. \quad (4.2)$$

Evidently (cf. (3.2)), the integral in (4.1) is exactly the displacement vector of the point $x_2(\tau_2(s))$ with respect to the point $x_1(\tau_1(s))$. Thus, transporting the displacement vector of $x_2(\tau_2(s))$ with respect to $x_1(\tau_1(s))$ from the point $x_1(\tau_1(s))$ to the point $x(s)$ along the curve η_s by means of the generalized transport I^{η_s}, we get the corresponding deviation vector of x_2 with respect to x_1 relative to x at the point $x(s)$. If the curves x and x_1 coincide geometrically ($x(s) \equiv x_1(\tau_1(s))$ for each $s \in [s',s'']$), then (see (2.3)) the deviation vector is identical with the corresponding displacement vector.

The described above construction physically may be interpreted in the following way (for example, in the 4-dimensional Riemannian space V_4 of the general relativity). We can treat the curves x_1 and x_2 as the trajectories (world lines) of two observed particles and the curve x as a trajectory of an observer who studies their movement. The parameters s_1, s_2 and s can be interpreted as "proper times" of the corresponding particles. The maps τ_1 and τ_2 give the connection between these proper times and define the "process of observation" in this case and in a certain sense they define some simultaneity between the three particles: the maps τ_1 and τ_2 define the simultaneity between the observer and the observed particles; and the map $\tau_2 \circ \tau_1^{-1}:[s_1', s_1''] \to [s_2', s_2'']$, between the observed particles themselves. For a fixed $s \in [s',s'']$ the curves γ_s and η_s can be considered as trajectories (world lines) of "signals" which physically realize the maps $\tau_2 \circ \tau_1^{-1}$ and τ_1. (For instance, if γ_s and η_s are zero (isotrope) geodesic (e.g., in V_4), then this corresponds to defining the simultaneity by means of light signals - see [4]).

Before going on we want to note that constructions like the one described above arise every time when one considers questions connected somehow with the deviation equation (see the references quoted in [1]) and also in the investigations of a relative kinematics (and some time dynamics) of particles in the general relativity and other alike theories. In connection with this, as good examples of concrete applications of some special cases of our general construction (under the condition $x(s) = x_1(\tau_1(s))$), it is enough to point the following places in the book [4] where one can also find suitable illustrations: ch.I, § 6 (geodesic deviation equation); ch.II: § 3 (the world function), § 6 (geodesic triangles), § 11 (metric in Fermi and optical coordinates) and § 12 (geodesics in Fermi and optical coordinates); ch.III: § 5 (Born rigidity), § 6 (measurement directions), § 7 (relative velocity and Doppler effect) and § 12. (physical meaning of the first normal and the curvative of space-like curves); ch.VII: § 1 (de Sitter universe) and § 9 (spectral shift); ch.VIII, § 3 (cosmological red-shift); ch.XI: § 5 (astronomical observations), § 6 (stellar aberration) and § 9 (spectral shift in a continuum). The same techique is used in other places in [4] as well but we shall not poiht term out here.

Applying the expansion (2.9) to (4.1), we see that in a coordinate basis the components of the deviation vector (4.1) are

$$h^{\alpha}_{12}(s,x) = H^{\alpha}_{\cdot\beta}(x(s), x_1(\tau_1(s)); \eta_s) \times \times \int_{r'_s}^{r''_s} H^{\beta}_{\cdot\sigma}(x_1(\tau_1(s)), \gamma_s(r); \gamma_s)\dot{\gamma}^{\sigma}_s(r)dr. \qquad (4.3)$$

Integrating here by parts we can put (4.3) in the form (cf. (3.4))

$$h^{\alpha}_{12}(s,x) = [x^{\alpha}_2(\tau_2(s)) - x^{\alpha}_1(\tau_1(s))] + + \{(H^{\alpha}_{\cdot\beta}(x,(s), x_1(\tau_1(s)); \eta_s) - \delta^{\alpha}_{\beta})(x^{\beta}_2(\tau_2(s)) - x^{\beta}_1(\tau_1(s))) + + H^{\alpha}_{\cdot\beta}(x(s), x_1(\tau_1(s)); \eta_s)[(H^{\beta}_{\cdot\sigma}(x_1(\tau_1(s)), x_2(\tau_2(s)); \gamma_s) - - \delta^{\beta}_{\sigma})x^{\sigma}_2(\tau_2(s)) - \int_{r'_s}^{r''_s} \frac{\partial H^{\beta}_{\cdot\sigma}(x_1(\tau_1(s)), \gamma_s(r); \gamma_s)}{\partial r} \gamma^{\sigma}_s(r)dr]\}. \qquad (4.4)$$

As we have already said (see Sect.3), in the Euclidean case $H^{\alpha}_{\cdot\beta}(\dots) \equiv \delta^{\alpha}_{\beta}$; hence, the Euclidean deviation vector has the components

$$h^{\alpha}_{12}(s,x)|_{E_n} = x^{\alpha}_2(\tau_2(s)) - x^{\alpha}_1(\tau_1(s)). \qquad (4.5)$$

From (3.5) and (4.5) we conclude that the Euclidean deviation vector is exactly equal to the Euclidean displacement vector (in E_n) between the corresponding points of the curves x_1 and x_2.

So, we see that the expression in the curly brackets in (4.4) describes the deviation of the general deviation vector from the usual Euclidean deviation (displacement) vector. Using the same method as at the end of Sect.3, one can easily prove that this deviation has an order of $O(\rho''_s - \rho'_s)O(r''_s - r'_s) + O((r''_s - r'_s)^2)$.

If the curves x, x_1 and x_2 are infinitely close to each other in the sense that $\rho''_s - \rho'_s$ and $r''_s - r'_s$ are infinitesimal constants, i.e., $\rho''_s = \rho'_s + d\rho_s$ and $r''_s = r'_s + dr_s$, then due to $x^{\alpha}_2(\tau_2(s)) - x^{\alpha}_1(\tau_1(s)) = \gamma_s(r''_s) - \gamma_s(r'_s) = \dot{\gamma}_s(r'_s)(r''_s - r'_s) + O((r''_s - r'_s))$ (see also (4.4)) the infinitesimal deviation vector

$$\xi(s,x) := \dot{\gamma}_s(r'_s)dr_s. \qquad (4.6)$$

defines the general deviation vector (4.1) up to the second order quantities.

5. CONCLUSION

In this paper, on the basis of the concept of a generalized transport we have defined in the general case the displacement and deviation vectors. We have given a physical interpretation of these vectors and examined their deviation from the corresponding vectors in the usual Euclidean case.

In a forthcoming work we are going to show how on the basis of the here developed theory one can define (in the "nonlocal" case of a curved space) some kinematical and dynamical quantities such as relative velocity and relative momentum and find some important connections between them.

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