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GAUGE FIELD GEOMETRY FROM COMPLEX AND HARMONIC ANALYTICITIES.

Hyper-Kähler Case

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In this paper we give a further application of the concept of harmonic analyticity in R^{4n} devopoled in [1]. We show that the constraints defining hyper-Kähler geometry in curved R^{4n} can be interpreted as the integrability conditions for the existence of harmonic analytic fields. These constraints can be solved by expressing all the elements of the differential geometry formalism in terms of two unconstrained analytic prepotentials (one of them is in fact a pure gauge). The geometric meaning of the prepotentials is brought out in an extended framework involving central charge coordinates. The prooedure of constructing the hyper-Kähler metric from the prepotentials involves solving a differential equation. In one approach it is a nonlinear equation for the bridge between the original basis with gauge parameters independent of the harmonic variables, and a new basis where analyticity becomes manifest. This approach is in many respects similar to the twistor one [2-5]. An alternative approaoh leads to a linear differential equation relating the vielbeins of the harmonic covariant derivatives in the new analytic basis. Examples illustrating both cases are given. Finally, we establish the one-to-one correspondence between hyper-Kähler geometry and N=2 supersymmetric off-shell sigma models. It turns out that the hyper-Kähler prepotentials determine the Lagrangian for the N=2 hypermultiplet (N=2 matter multiplet) superfields in just the same way as the Kähler potential prescribes the Lagrangian for the N=1 chiral (matter) superfields.

The reader is expected to be familiar with the ideas and notation of [1]. Like in [1], our consideration here is purely local, we do not concern any global aspects of hyper-Kähler geometry.

I. Harmonic analyticity and hyper-Kähler prepotentials

I.1. Hyper-Kähler constraints. The set-up for hyper-Kähler geometry is analogous to that for the self-dual Yang-Mills equations (see [1], sect. III). One considers the Euclidean space $R^{4n} = \{x^{\mu i}\}$, where $\mu = 1, \ldots 2n$ and $i = 1,2$. In it one defines fields $\varphi_{\alpha\beta\ldots ij\ldots}(x)$ which transform under the following gauge groups:

$$\delta\varphi_{\alpha\beta\ldots ij\ldots}(x)\equiv\varphi'_{\alpha\beta\ldots ij\ldots}(x+\delta x)-\varphi_{\alpha\beta\ldots ij\ldots}(x)=$$
$$=\tau_\alpha{}^{\alpha'}(x)\varphi_{\alpha'\beta\ldots ij\ldots}(x)+\tau_\beta{}^{\beta'}(x)\varphi_{\alpha\beta'\ldots ij\ldots}(x)+\ldots \qquad (I.1)$$
$$\delta x^{\mu i}=\tau^{\mu i}(x).$$

Here $\tau^{\mu i}(x)$ correspond to coordinate transformations; $\tau_\alpha{}^{\alpha'}(x)$ $(\alpha,\alpha'=1,\ldots 2n)$, to local $Sp(n)$ transformations; $i,j\ldots$ are rigid SU(2) indices. Thus the tangent space group is chosen to be a product of local $Sp(n)$ and rigid SU(2). Correspondingly, the covariant derivatives involve only $Sp(n)$-valued connections:

$$\mathcal{D}_{\alpha i}=e^{\mu k}_{\alpha i}\partial_{\mu k}+\omega_{\alpha i},\quad \omega_{\alpha i}=\omega_{\alpha i(\beta\gamma)}\Gamma^{(\beta\gamma)}, \qquad (I.2)$$

where Γ are the generators of $Sp(n)$. This framework allows us to impose the following constraints which define hyper-Kähler geometry:

$$[\mathcal{D}_{\alpha i},\mathcal{D}_{\beta j}]=\varepsilon_{ij}R_{(\alpha\beta)}\equiv\varepsilon_{ij}R_{(\alpha\beta)(\gamma\delta)}\Gamma^{(\gamma\delta)}, \qquad (I.3)$$

or, equivalently:

$$[\mathcal{D}_{\alpha(i},\mathcal{D}_{\beta j)}]\equiv R_{(ij)[\alpha,\beta]}=0. \qquad (I.3')$$

In order to establish a contact with the standard characterization of hyper-Kähler geometry (see Introduction to [1]) we take notice of the fact that the component $R_{\alpha[i\beta j]}\equiv\varepsilon_{ij}R_{(\alpha\beta)}$ of general Riemann tensor $R_{\alpha i\beta j}$ generates a subgroup $Sp(n)$ in the full holonomy group O(4n) generated by this tensor. So, eqs. (I.3) (or (I.3$'$)) amount to reducing the holonomy group to a subgroup of Sp(n), in accord with the general definition of hyper-Kähler manifolds.

From the Bianchi identities it follows that the nonvanishing part of the curvature $R_{(\alpha\beta)(\gamma\delta)}$ is totally symmetric. In the case of R^4 it is just the self-dual half of the Weyl tensor, so in four dimensions the hyper-Kähler constraints are equivalent to the condition of self-duality for the Riemann tensor.

The problem now is to find a way to present (I.3) as integrability conditions, which will lead to their solution. This can be achieved in the extended framework of harmonic space.

I.2. Harmonic space and analyticity. We begin by adding harmonic variables $u^{\pm i}\in SU(2)$ to the coordinates of R^{4n}, $\{x^{\mu i},u^{\pm i}\}$. Then the constraints (I.3), (I.3$'$) can be rewritten in the following equivalent form (see the discussion of the self-dual YM constraints in [1]):

$$[\mathcal{D}^+_\alpha,\mathcal{D}^+_\beta]=0\Leftrightarrow R^{++}_{[\alpha\beta]}\equiv u^{+i}u^{+j}R_{ij[\alpha,\beta]}=0 \qquad (a)$$
$$[\mathcal{D}^{++},\mathcal{D}^+_\alpha]=0\Leftrightarrow \mathcal{D}^+_\alpha=u^{+i}\mathcal{D}_{\alpha i}. \qquad (b) \qquad (I.4)$$

Here $\mathcal{D}^{++}=\partial^{++}\equiv u^{+i}\partial/\partial u^{-i}$ is the harmonic covariant derivative (in this case it coincides with the rigid one, since the gauge parameters (I.1) do not depend on $u^{\pm i}$). In addition we require that all harmonic functions have definite U(I) charge, i.e. they are eigenfunctions of the operator $\mathcal{D}^0=u^{+i}\partial/\partial u^{+i}-u^{-i}\partial/\partial u^{-i}$.

Obviously, (I.4a) now has the meaning of the integrability condition for the existence of harmonic analytic fields defined by

$$\mathcal{D}^+_\alpha\varphi_{\beta\gamma\ldots}(x,u)=0. \qquad (I.5)$$

As we have seen before [1], the way to the solution of such constraints goes through the introduction of an analytic basis in harmonic space, where the analyticity (I.5) becomes manifest. It is defined with the help of bridges $v^{\mu\pm}(x,u)$

$$x^{\mu\pm}_A=x^{\mu k}u^\pm_k+v^{\mu\pm}(x,u)$$
$$\delta x^{\mu+}_A=\lambda^{\mu+}(x^+_A,u) \qquad (I.6)$$
$$\delta x^{\mu-}_A=\lambda^{\mu-}(x^+_A,x^-_A,u),\quad \delta u^{\pm i}=0.$$

In this basis the analyticity of scalar fields simply means that they do not depend on $x^{\mu-}_A$. This statement is covariant under the group (I.6). In other words, the covariant derivative $\mathcal{D}^+_\alpha$ can be chosen in the form

$$\mathcal{D}^+_\alpha=E^\mu_\alpha\partial^+_{A\mu}+\omega^+_\alpha. \qquad (I.7)$$

However, the tangent space group is still the original one (I.1), and it violates the explicit analyticity of fields with $Sp(n)$ indices. However, just as in Kähler case (see discussion in [1] after eq. (II.26)) the basic constraint $R^{++}_{[\alpha,\beta]}=0$ (I.4a) guarantees the existence of analytic tangent space. Indeed, it follows from this constraint that the τ covariant connection ω^+_α can be presented in a "pure gauge" form:

$$\omega^+_{\alpha\,\beta}{}^{\gamma} = (M^{-1})_\beta{}^\delta\, \mathbb{E}_\alpha^\mu\, \partial^+_{A\mu} M_\delta{}^\gamma. \tag{I.8}$$

Symplectic matrix $M_\alpha{}^\beta(x, u)$ provides a bridge to the tangent space with manifest analyticity:

$$\Phi_\alpha(x,u) = M_\alpha^\beta(x,u)\,\varphi_\beta(x,u)\,,\quad M\Omega M^T = \Omega$$
$$\delta M_\alpha{}^\beta = \lambda_\alpha^\gamma M_\gamma^\beta - M_\alpha^\gamma \tau_\gamma{}^\beta\,,\quad \delta\Phi_\alpha = \lambda_\alpha^\beta \Phi_\beta\,. \tag{I.9}$$

The new $Sp(n)$ parameters $\lambda_\alpha{}^\gamma$ are analytic, $\lambda_\alpha{}^\gamma = \lambda_\alpha{}^\gamma(x^+_A, u)$. Correspondingly, in this analytic tangent λ frame there will be no connection ω^+_α

$$(\mathcal{D}^+_\alpha{}_\beta^\gamma)_\lambda = M_\alpha^\rho M_\beta^\nu (\mathcal{D}^+_\rho{}_\nu^\varphi)(M^{-1})_\varphi^\gamma = \delta_\beta^\gamma M_\alpha^\sigma \mathbb{E}_\sigma^\mu \partial^+_{A\mu} \equiv \delta_\beta^\gamma E_\alpha^\mu \partial^+_\mu \equiv \delta_\beta^\gamma \Delta^+_\alpha. \tag{I.10}$$

Note an important difference with the Kähler case. There, the analytic tangent frame group was induced by the analytic world transformations and was essentially complex. In the present case, this group is again $Sp(n)$ with real parameters $\lambda_\alpha{}^\beta(x,u)$ (reality is defined now with respect to generalized conjugation $\sim$ [1] which has no analog in Kähler case).

In what follows we shall work in the analytic basis (I.6) and frame (I.9), so we shall often drop the identifiers λ, A. The new vielbeins E^μ_α transform as follows

$$\delta E_\alpha{}^\mu = \lambda_\alpha{}^\beta E_\beta{}^\mu + E_\alpha{}^\nu \partial^+_\nu \lambda^{\mu-}. \tag{I.11}$$

They are still subject to the constraint (I.4a):

$$\Delta^+_{[\alpha} E^\mu_{\beta]} = 0\,. \tag{I.12}$$

It is not hard to find the general solution of (I.12), but we postpone this until we introduce some other useful objects.

I.3. <u>Harmonic covariant derivative $\mathcal{D}^{++}$</u>. Going to the analytic basis (I.6) and frame (I.9) we have made the derivative $\mathcal{D}^+_\alpha$ almost simple. Instead, the harmonic one acquires vielbeins and connection:

$$\mathcal{D}^{++} = \partial^{++} + H^{++\mu+}\partial^-_\mu + H^{++\mu-}\partial^+_\mu + \omega^{++} \equiv \Delta^{++} + \omega^{++}. \tag{I.13}$$

Here

$$H^{++\mu+} = \Delta^{++} v^{\mu+}\,,\quad H^{++\mu-} = \Delta^{++} v^{\mu-} - v^{\mu+} + x^{\mu+}_A$$
$$\omega^{++}{}_\alpha{}^\beta = M_\alpha{}^\gamma \Delta^{++} M^{-1}_\gamma{}^\beta \tag{I.14}$$
$$\delta H^{++\mu\pm} = \Delta^{++}\lambda^{\mu\pm}\,,\quad \delta\omega^{++}{}_\alpha{}^\beta = -\Delta^{++}\lambda_\alpha{}^\beta.$$

The parameter $\lambda^{\mu-}(x^\pm_A, u)$ is a general function; so one is able to gauge the vielbein $H^{++\mu-}$ into its flat space limit $x^{\mu+}_A$:

$$H^{++\mu-} = x^{\mu+}_A \rightarrow \Delta^{++} v^{\mu-} = v^{\mu+} \rightarrow \Delta^{++}\lambda^{\mu-} = \lambda^{\mu+}. \tag{I.15}$$

This gauge will be always implied henceforth.

Recall that in the YM theory [1] the bridges connecting the λ and τ frames were secondary objects, they could be expressed in terms of the analytic connection V^{++} of $\mathcal{D}^{++}$. The latter was the unconstrained prepotential of the self-dual YM theory. The situation here is different in that it involves one more step. The coordinate and frame bridges $v^{\mu\pm}$ and $M_\alpha{}^\beta$ can be found as solutions of (I.14) given the λ world (= frame and basis) vielbeins and connection of $\mathcal{D}^{++}$ (I.13). However, the latter are not unconstrained this time. As we shall see in sect. I.6, they will be expressed in terms of some unconstrained analytic prepotentials.

We remark that the equations (I.14) relating the bridges to the vielbeins and connection of $\mathcal{D}^{++}$ are highly nonlinear because v and M are defined in the τ basis, while H^{++} and ω^{++} are analytic (see (I.16)) fields defined in the λ basis. Under certain simplifying assumptions they can be solved (see an example in sect. III.2), but it is hard to deal with them in general. There is an alternative approach which requires living only in the λ world, thus avoiding the use of the bridges to the τ world. In this approach one has to deal with linear differential equations (see sect. I.4).

We return to the discussion of the constraints (I.4b). We have to insert the expressions (I.10) for $\mathcal{D}^+_\alpha$ and (I.13) for $\mathcal{D}^{++}$ into (I.4b) and examine the consequences:

$$\mathcal{D}^{++}E_\alpha^{\ \mu} = \Delta^{++}E_\alpha^{\ \mu} + \omega^{++\ \beta}_{\alpha} E_\beta^{\ \mu} = 0 \qquad \text{(a)}$$

$$\Delta^+_\alpha \, H^{++\mu+} = 0 \qquad \text{(b)} \qquad \text{(I.16)}$$

$$\Delta^+_\alpha \, \omega^{++\ \gamma}_{\beta} = 0 . \qquad \text{(c)}$$

First we consider (I.16a). Note that it is covariant because the parameter $\partial^+_\nu \lambda^{\mu-}$ in (I.11) has the property $\Delta^{++}(\partial^+_\nu \lambda^{\mu-}) = 0$ in the gauge (I.15) (taking into account (I.16b)). From (I.16a) one can express the connection ω^{++} in terms of the vielbein $E_\alpha^{\ \mu}$:

$$\omega^{++}_{\alpha\beta} = E_{(\alpha}^{\ \ \mu} \Delta^{++} E^{-1}_{\mu\beta)} . \qquad \text{(I.17)}$$

Here the Sp(n) indices of ω^{++} and E^{-1} are lowered with the help of the symplectic tensor $\Omega_{\alpha\beta}$ ($\omega^{++}_{\alpha\beta} = \Omega_{\beta\gamma}\,\omega^{++\ \gamma}_{\alpha}$, etc); $\omega^{++}_{\alpha\beta}$ is symmetric in α, β because it takes its values in the algebra of Sp(n). The antisymmetric part of (I.16a) remains as a constraint which will be solved later on:

$$E_{[\alpha}^{\ \ \mu} \Delta^{++} E^{-1}_{\mu\beta]} = 0 . \qquad \text{(I.18)}$$

Secondly, (I.16b) means that the vielbein $H^{++\mu+}$ must be analytic, $H^{++\mu+} = H^{++\mu+}(x^+_A, u)$. Finally, the analyticity of $\omega^{++}_{\alpha\beta}$ (I.17) (following from (I.16c)) will be a consequence of the explicit expression for $E_\alpha^{\ \mu}$ in terms of independent prepotentials (see eq. (I.27) below)

I.4. <u>Harmonic covariant derivative $\mathcal{D}^{--}$</u> . As in the case of Y.M. self-duality the vielbeins of $\mathcal{D}^{--}$ in the λ world will prove essential in the construction of the differential geometry formalism. In the τ world $\mathcal{D}^{--} = \partial^{--} \equiv u^{-i}\partial/\partial u^{+i}$. Passing to the λ world, it becomes

$$\mathcal{D}^{--} = \partial^{--} + H^{--\mu+}\partial^-_\mu + H^{--\mu-}\partial^+_\mu + \omega^{--} \equiv \Delta^{--} + \omega^{--}$$

$$\delta H^{--\mu\pm} = \Delta^{--}\lambda^{\mu\pm} , \quad \delta\omega^{--\ \beta}_{\alpha} = -\Delta^{--}\lambda_\alpha^{\ \beta} . \qquad \text{(I.19)}$$

As in (I.14) the new vielbeins and connection can be expressed in terms of the bridges v and M . However, following B.M.Zupnik [6]

we prefer to find them directly from the relation

$$[\mathcal{D}^{++}, \mathcal{D}^{--}] = \mathcal{D}^0 \equiv \partial^0 + x_A^{\mu+}\partial^-_\mu - x_A^{\mu-}\partial^+_\mu . \qquad \text{(I.20)}$$

This relation is obviously true in the τ world (there $\mathcal{D}^0 = \partial^0 \equiv \equiv u^{+i}\partial/\partial u^{+i} - u^{-i}\partial/\partial u^{-i}$). Note that $\mathcal{D}^0$ is not covariantized in the λ world. The reason is that every object in our framework is an eigenfunction of $\mathcal{D}^0$, as we have stated above.

In the λ world, from (I.13), (I.19) and (I.20) one gets

$$\Delta^{++}H^{--\mu-} - \Delta^{--}H^{++\mu-} = -x_A^{\mu-}$$

$$\Delta^{++}H^{--\mu+} - \Delta^{--}H^{++\mu+} = x_A^{\mu+} . \qquad \text{(I.21)}$$

In addition, the curvature term in (I.20) must also vanish. This becomes true after deriving from $\mathcal{D}^{++}E_\alpha^{\ \mu} = 0$ (I.16a) the relation $\mathcal{D}^{--}E_\alpha^{\ \mu} = 0$ (it can be proved by going back to the τ world). Then ω^{--} is expressed in terms of $E_\alpha^{\ \mu}$:

$$\omega^{--}_{(\alpha\beta)} = E_{(\alpha}^{\ \ \mu}\Delta^{--}E^{-1}_{\mu\beta)} \qquad \text{(I.22)}$$

which together with (I.17) implies $R^{++--} = 0$.

The equations (I.21) are <u>linear differential equations for $H^{--\mu\pm}$</u> . Their solution always exists and is unique. One way to see this is to construct the solution perturbatively [6] . To this end one introduces an auxiliary basis: $x_0^{\mu i} = x_A^{\mu+}u^{-i} - x_A^{\mu-}u^{+i}$. There the operators Δ^{++} and Δ^{--} become

$$\Delta^{++} = \partial_0^{++} + h^{++} , \quad \Delta^{--} = \partial_0^{--} + h^{--} ; \quad h^{\pm\pm} = h^{\pm\pm\mu i}\partial_{\mu i} ;$$

$$h^{++\mu i} = H^{++\mu+}u^{-i} , \quad h^{--\mu i} = (H^{--\mu+} - x_A^{\mu-})u^{-i} - H^{--\mu-}u^{+i} .$$

Then (I.21) takes the same form

$$\partial_0^{++}h^{--} - \partial_0^{--}h^{++} + [h^{++}, h^{--}] = 0$$

as the YM equation (III.26) from [1] and is solved in the same way. Of course, one should try to solve (I.21) nonperturbatively. In sect. III.1 we give an example.

I.5. <u>Construction of the vielbein $E_\alpha^{\ \mu}$</u> . Now it is easy to find an expression for $E_\alpha^{\ \mu}$ which satisfies (I.12):

$$E_\alpha^{\ \mu} = e_\alpha^{\ \nu} (\partial H)^{-1\ \mu}_{\nu}\,; \quad (\partial H)_\nu^{\ \mu} \equiv \partial_\nu^+ H^{--\mu+}, \quad \partial_\mu^+ e_\alpha^{\ \nu} = 0. \qquad (I.23)$$

In fact, the matrix $(\partial H)^{-1}$ alone does satisfy (I.12). However, it transforms as follows (see (I.6),(I.19))

$$\delta (\partial H)^{-1\ \nu}_{\mu} = -\partial_\mu^- \lambda^{\rho+} (\partial H)^{-1\ \nu}_{\rho} + (\partial H)^{-1\ \rho}_{\mu} \partial_\rho^+ \lambda^{\nu-}. \qquad (I.24)$$

The second term here corresponds to the second term in (I.11), but the first one is different. Therefore one introduces the new object $e_\alpha^{\ \mu}$ which is analytic (so it does not spoil (I.12)) and supplies the correct tangent group transformation:

$$\delta e_\alpha^{\ \mu} = \lambda_\alpha^{\ \beta} e_\beta^{\ \mu} + e_\alpha^{\ \nu} \partial_\nu^- \lambda^{\mu+}. \qquad (I.25)$$

The solution (I.23) of (I.12) is unique. Indeed, if there were another solution $E_\alpha^{\prime\ \mu}$, the combination $E_\alpha^{\prime\ \mu} E^{-1\ \beta}_{\mu}$ would be a dimensionless tensor; the only such tensor in the theory is δ_α^β.

After constructing $E_\alpha^{\ \mu}$ we can find a new expression for $\omega^{++\ \beta}_{\ \alpha}$ (I.17):

$$\omega^{++}_{\alpha\beta} = e_{(\alpha}^{\ \ \mu} \Delta^{++} e_{\mu\beta)} + e_{(\alpha}^{\ \ \mu} (\partial H)^{-1\nu}_{\mu} \Delta^{++} \partial_\nu^+ H^{--\rho+} e_{\rho\beta)}. \qquad (I.26)$$

Further, using (I.16b), (I.21), (I.19) one finds

$$\Delta^{++}\partial_\nu^+ H^{--\rho+} = \partial_\nu^+ (x_A^{\rho+} + \Delta^{--} H^{++\rho+}) = \partial_\nu^+ H^{--\delta+} \partial_\delta^- H^{++\rho+}\,;$$

so (I.26) simplifies to

$$\omega^{++}_{\alpha\beta} = e_{(\alpha}^{\ \ \mu}\Delta^{++} e_{\mu\beta)} + e_{(\alpha}^{\ \ \mu} \partial_\mu^- H^{++\nu+} e_{\nu\beta)}. \qquad (I.27)$$

In this form the analyticity (I.16b) of ω^{++} is obvious.

I.6. Hyper-Kähler prepotentials. So far we have been able to solve all the constraints following from (I.4) except for (I.18). Here we shall solve this constraint by expressing the two basic analytic quantities $e_\alpha^{\ \mu}$ and $H^{++\mu+}$ in terms of unconstrained prepotentials.

Repeating the steps (I.26,27) one can rewrite (I.18) in the following form:

$$e_{[\alpha}^{\ \ \rho}\Delta^{++} e_{\rho\beta]} + e_{[\alpha}^{\ \ \rho}\partial_\rho^- H^{++\lambda+} e_{\lambda\beta]} = 0. \qquad (I.28)$$

Next one multiplies (I.28) by $e_\mu^{\ \alpha} e_\nu^{\ \beta}$ and introduces the notation [1)]

$$H_{\mu\nu} = e_{\mu\alpha} e^{\alpha}_{\ \nu} = -H_{\nu\mu}. \qquad (I.29)$$

after which (I.28) becomes

$$\Delta^{++} H_{\mu\nu} - 2\,\partial^-_{[\mu} H^{++\rho+} H_{\nu]\rho} = 0. \qquad (I.30)$$

This equation implies

$$H^-_{\mu\nu\lambda} \equiv \partial^-_{[\mu} H_{\nu\lambda]} = 0 \Rightarrow H_{\mu\nu} = \partial^-_{[\mu} \mathcal{L}^+_{\nu]}. \qquad (I.31)$$

To see this one differentiates (I.30) with ∂^-_λ and uses the relation (recall definition of Δ^{++} (I.13) and property $\partial^+_\rho H_{\mu\nu} = 0$):

$$\partial^-_\lambda \Delta^{++} H_{\mu\nu} = \Delta^{++}\partial^-_\lambda H_{\mu\nu} + \partial^-_\lambda H^{++\rho+}\partial^-_\rho H_{\mu\nu}.$$

The result is

$$\Delta^{++} H^-_{\mu\nu\lambda} + 3\,\partial^-_{[\mu} H^{++\rho+} H^-_{\nu\lambda]\rho} = 0. \qquad (I.32)$$

This is nothing but the covariant (under world transformations with parameter $\partial^-_\mu \lambda^{\nu+}$) derivative, $\mathcal{D}^{++} H^-_{\mu\nu\lambda} = 0$. Returning to the τ world this equation reads $\partial^{++} H^-_{\mu\nu\lambda} = 0$ which implies (I.31) (the reason is that in the harmonic expansion of $H^-_{\mu\nu\lambda}$ there are no terms vanishing under ∂^{++}).

The new object $\mathcal{L}^+_\mu$ in (I.31) should transform under the world ($\partial^-_\mu \lambda^{\nu+}$) transformations as a vector so that $H_{\mu\nu}$ will be a proper world tensor (see (I.29)). In addition, $\mathcal{L}^+_\mu$ is defined up to terms of the type $\partial^-_\mu \lambda^{++}$ (a pregauge freedom of (I.31)). So, the transformation law for $\mathcal{L}^+_\mu$ is

$$\delta \mathcal{L}^+_\mu = -\partial^-_\mu \lambda^{\nu+} \mathcal{L}^+_\nu - \partial^-_\mu \lambda^{++}. \qquad (I.33)$$

Without loss of generality both $\mathcal{L}^+_\mu$ and λ^{++} can be taken to be analytic

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1) Note that the "vielbeins" $e_\mu^{\ \alpha}$ can be obtained from (1.29) as "square roots" of the "metric" $H_{\mu\nu}$, up to the tangent space Sp(n) freedom λ_α^β (much like the relationship between vierbeins and metric in general relativity).

$$\partial^+_\mu \mathcal{L}^+_\nu = \partial^+_\mu \lambda^{++} = 0 . \tag{I.34}$$

(the nonanalytic , i.e. $x^{\mu-}$ dependent part of $\mathcal{L}^+_\mu$ must be of pregauge form, because $H_{\mu\nu}$ is analytic; then it can be gauged away by the corresponding part of λ^{++}).

The next step is to plug (I.31) in (I.30) and rewrite it in the following form:

$$\partial^-_{[\mu}(\Delta^{++}\mathcal{L}^+_{\nu]} + \partial^-_{\nu]} H^{++\rho+}\mathcal{L}^+_\rho) = 0 .$$

This equation implies

$$\Delta^{++}\mathcal{L}^+_\mu + \partial^-_\mu H^{++\nu+}\mathcal{L}^+_\nu = -\partial^-_\mu H^{++(++)} . \tag{I.35}$$

Once again, one can choose the new object $H^{++(++)}$ to be analytic. Taking into account (I.33), (I.14) it can be checked that

$$\delta H^{++(++)} = \Delta^{++}\lambda^{++} . \tag{I.36}$$

Finally, one introduces yet another analytic object:

$$\mathcal{L}^{(+4)} = H^{++(++)} + H^{++\mu+}\mathcal{L}^+_\mu , \quad \partial^+_\mu \mathcal{L}^{(+4)} = 0 \tag{I.37}$$

$$\delta\mathcal{L}^{(+4)} = \partial^{++}\lambda^{++} + \partial^{++}\lambda^{\mu+}\cdot\mathcal{L}^+_\mu .$$

This makes it possible to solve (I.35) for $H^{++\mu+}$

$$H^{++\mu+} = \frac{1}{2} H^{\mu\nu}(\partial^-_\nu \mathcal{L}^{(+4)} + \partial^{++}\mathcal{L}^+_\nu) , \quad H^{\mu\nu}H_{\nu\lambda} = \delta^\mu_\lambda . \tag{I.38}$$

Summarizing the above discussion we can say that all the constraints of hyper-Kähler geometry have been solved in terms of <u>two unconstrained analytic prepotentials $\mathcal{L}^{\mu+}$ and $\mathcal{L}^{(+4)}$</u> . Like the Kähler prepotential K (see [1]), they have their own pregauge transformations with the analytic parameter λ^{++} . The dimension of these prepotentials and parameter are peculiar ([$\mathcal{L}^{\mu+}$]= cm^1, [$\mathcal{L}^{(+4)}$] = [λ^{++}] = cm^2), their geometric meaning is obscure in the present scheme. The origin of those objects will become clear in an extended framework involving central charge coordinates (see sect. II).

We would like to point out that the approach to hyper-Kähler

geometry developed here is closely related to the twistor approach to the self-dual Einstein equations (in R^4) [2,3,5]. This relationship is analogous to the one we discussed in [1] in the context of self-dual YM theory. Here we shall only recall the fact that the twistor approach makes use of the τ basis and the central problem there is to solve the nonlinear bridge equation (the "splitting procedure" in the twistor language). In sect. III.2 we present the Ansatz due to Ward [5] which allows to solve this problem in a certain class of cases. In the harmonic framework we suggest an alternative way which is confined to the analytic λ world. The problem in this case is to solve the linear differential equation (I.21). We hope that this task may prove easier. An example is worked out in sect. III.1.

I.7. <u>The hyper-Kähler metric in the analytic basis.</u> To comlete the differential geometry formalism we need the covariant derivative $\mathcal{D}^-_\alpha$. It is defined by the conventional constraint (which obviously takes place in the τ world):

$$\mathcal{D}^-_\alpha = [\mathcal{D}^{--}, \mathcal{D}^+_\alpha] \equiv \Delta^-_\alpha + \omega^-_\alpha = \tag{I.39}$$

$$= -e_\alpha{}^\mu\partial^-_\mu - \Delta^+_\alpha H^{--\mu-}\partial^+_\mu - \Delta^+_\alpha\omega^{--} \equiv E^{-\mu\pm}_\alpha \partial^\mp_\mu - \Delta^+_\alpha\omega^{--} .$$

Taking into account that $\mathcal{D}^+_\alpha = e^\mu_\alpha(\partial H)^{-1}_\mu{}^\nu\partial^+_\nu = E^{+\nu-}_\alpha\partial^+_\nu$ one derives the contravariant components of the hyper-Kähler metric in the λ basis:

$$\begin{aligned} & g^{\mu+\nu+} = 0 \\ & g^{\mu+\nu-} = g^{\nu-\mu+} = -E^{\alpha-\mu+}E^{+\nu-}_\alpha = -H^{\mu\lambda}(\partial H)^{-1}_\lambda{}^\nu \\ & g^{\mu-\nu-} = E^{\alpha+\mu-}E^{-\nu-}_\alpha + (\mu\leftrightarrow\nu) = \\ & \quad = 2H^{\rho\sigma}(\partial H)^{-1}_\sigma{}^\lambda(\partial H)^{-1}_\rho{}^{(\mu}\partial^+_\lambda H^{--\nu)-} . \end{aligned} \tag{I.40}$$

The transformation law for the metric (which is a world tensor) can be read off from the law for a world vector $A^{\mu+}, A^{\mu-}$. The λ basis is asymmetric ($\lambda^{\mu+}$ is analytic, but $\lambda^{\mu-}$ is not), therefore $A^{\mu+}$ transforms homogeneously

$$\delta A^{\mu+} = A^{\nu+}\partial^-_\nu\lambda^{\mu+}$$

while $\delta A^{\mu-}$ involves both $A^{\mu-}$ and $A^{\mu+}$:

$$\delta A^{\mu-} = A^{\nu-}\partial^+_\nu\lambda^{\mu-} + A^{\nu+}\partial^-_\nu\lambda^{\mu-}$$

A peculiarity of the metric (I.40) is its dependence on the harmonic variables, $g = g(x^+, x^-, u^\pm)$. This is quite natural in the λ basis, where we now live. However, it is covariantly independent of $u^\pm$ in the sense of the following relations:

$$\mathcal{D}^{++} g^{\mu+\nu-} = \mathcal{D}^{++} g^{\mu-\nu+} = 0$$
$$\mathcal{D}^{++} g^{\mu-\nu-} = g^{\mu+\nu-} + g^{\mu-\nu+} .$$

Here $\mathcal{D}^{++}$ contains suitable Christoffel terms.

I.8. Gauge choices and normal coordinates. In addition to the gauge (I.15) which fixes the parameter $\lambda^{\mu-}$ one can impose three further gauges on the parameters $\lambda^{\mu+}$, $\lambda_\alpha{}^\beta$ and λ^{++} .

From (I.33) and the fact that $\mathcal{L}^+_\mu$ is analytic and has the flat space limit $\Omega_{\mu\nu} x^{\nu+}_A$ one concludes that the following gauge is possible:

$$\mathcal{L}^+_\mu = x^+_{A\mu} \rightarrow H_{\mu\nu} = \Omega_{\mu\nu} . \tag{I.41}$$

This implies $\lambda^+_\mu + \partial^-_\mu \lambda^{\nu+} x^+_{A\nu} + \partial^-_\mu \lambda^{++} = 0$. Introducing the new analytic parameter

$$\hat{\lambda}^{++} = \lambda^{++} + \lambda^{\nu+} x^+_{A\nu} \tag{I.42}$$

one can express λ^+_μ in terms of $\hat{\lambda}^{++}$

$$\lambda^+_\mu = \partial^-_\mu \hat{\lambda}^{++} \Rightarrow \partial^-_{[\nu} \lambda^+_{\mu]} = 0 . \tag{I.43}$$

Under these circumstances the remaining prepotential $\mathcal{L}^{(+4)}$ transforms as follows:

$$\delta \mathcal{L}^{(+4)} = \partial^{++} \hat{\lambda}^{++} . \tag{I.44}$$

The second gauge fixing concerns the Sp(n) parameter λ_α^β . Using (I.41), (I.29), (I.25) one can demand:

$$e^\mu_\alpha = \delta^\mu_\alpha \rightarrow \lambda_{\alpha\beta} = -\partial^-_{(\alpha} \lambda^+_{\beta)} . \tag{I.45}$$

The third gauge corresponds to finding a normal set of coordinates in the analytic basis. This means that one uses the full remaining gauge freedom (in our case (I.44)) to gauge away whatever possible from the prepotential $\mathcal{L}^{(+4)}$. The remainder is a coordinate

expansion of $\mathcal{L}^{(+4)}$ at a given point, where the coefficients coincide with the values of the nonvanishing tensors at this point. To achieve this one considers the expansion of a function of charge $+s$

$$F^{(+s)}(x^+, u) = \sum_{\substack{n,m=0 \\ m+n \geq s}}^{\infty} x^+_{\mu_1} \cdots x^+_{\mu_n} u^+_{i_1} \cdots u^+_{i_m} u^-_{i_{m+1}} \cdots u^-_{i_{n+m-s}} \times f^{\mu_1 \cdots \mu_n i_1 \cdots i_{n+m-s}} . \tag{I.46}$$

Comparing the expansions of $\mathcal{L}^{(+4)}$ and $\hat{\lambda}^{++}$, from (I.44) one derives the following normal gauge form of $\mathcal{L}^{(+4)}$.

$$\mathcal{L}^{(+4)} = \sum_{n=0}^{\infty} x^+_{\mu_1} \cdots x^+_{\mu_{n+4}} u^-_{i_1} \cdots u^-_{i_n} C^{\mu_1 \cdots \mu_{n+4} i_1 \cdots i_n} . \tag{I.47}$$

Note the absence of u^+_i in (I.47). In the case of R^4 the coefficient $C^{\mu_1 \cdots \mu_4}$ is the value of the self-dual Weyl tensor at the point $x = 0$, and the higher rank coefficients correspond to the totally symmetrized covariant derivatives of the Weyl tensor at that point. Still in R^4 , $\mathcal{L}^{(+4)}$ is a function of three complex variables (2 from x^+ + 2 from u^- − 1 from U(I) charge) 2) , as predicted in [2] . From the reality of the analytic space ($x^+, u^\pm$) under the conjugation $\sim$ (see [1]) it follows that the coefficients C are pseudo-real (since $\mathcal{L}^{(+4)}$ is real). The generalization of the above interpretation to the case of R^{4n} is straightforward.

In the gauge (I.47) a few constant parameters survive in the expansion of $\hat{\lambda}^{++}$;

$$\hat{\lambda}^{++} = x^+_\mu x^+_\nu \lambda^{\mu\nu} + x^+_\mu u^+_i a^{\mu i} + u^+_i u^+_j \omega^{ij} . \tag{I.48}$$

where $\lambda^{\mu\nu}$ are rigid Sp(n) rotations, $a^{\mu i}$ are rigid translations. The meaning of ω^{ij} ($[\omega]$= cm^2) will become clear in the next section.

Note an intersting correspondence with the self-dual YM problem considered in [1] . As follows from eq. (I.38) $\mathcal{L}^{(+4)}$ enters $H^{++\mu+}$, and hence all the other geometric objects including the metric, via a derivative $\partial^-_\nu \mathcal{L}^{(+4)}$. Clearly, at least in the normal gauge

one may write

2) Recall that x^+, u^+, u^- are real in the sense of operation $\sim$ but are complex in the conventional sense [1] . The same regards also the U(I)-parameter. In fact, one has no need to require u^+ and u^- to be mutually conjugated, it suffices to keep them real under $\sim$.

$$\partial_{\nu}^{-}\mathcal{L}^{(+4)} = (V^{++})_{\nu}^{\ \mu} x_{\mu}^{+}, \tag{I.49}$$

where $(V^{++})_{\nu}^{\ \mu}$ is defined by $\mathcal{L}^{(+4)}$ (up to a gauge freedom). As a matter of fact, $(V^{++})_{\nu}^{\ \mu}$ can always be regarded as some particular case of unconstrained self-dual YM prepotential in R^{4n} for gauge group $Sp(n)$, with the internal and space $Sp(n)$-indices identified. Thus, we observe a surprising isomorphism between the complete set of hyper-Kähler metrics in R^{4n} (given by $\mathcal{L}^{(+4)}$) and a subclass of self-dual solutions of $Sp(n)$ YM equations in R^{4n} (given by V^{++}). This isomorphism deserves a further study.

II. Central charges and the geometric meaning of the hyper-Kähler prepotentials

In sect. I we have presented a solution of the hyper-Kähler constraints. We saw that all the objects of differential geometry (vielbeins, connections, etc) could be expressed in terms of two analytic prepotentials $\mathcal{L}_{\mu}^{+}$ and $\mathcal{L}^{(+4)}$ with their own pregauge freedom with analytic parameter λ^{++}. However, neither the prepotentials nor the pregauge group naturally fit in the existing geometric framework. A similar situation was observed in Kähler geometry [1]. There we found an extension of the space R^{2n} with a new central charge coordinate z, $[z] = \text{cm}^2$. Then it became possible to interpret the Kähler pregauge parameter as a local translation of z, and the Kähler prepotential as the bridge between the τ and λ coordinates z and z_A. In this section we shall develop an analogous interpretation of hyper-Kähler geometry in an extended harmonic space with an $SU(2)$ triplet of central charge coordinates.

II.I. <u>Central charge extension of flat harmonic space.</u> We begin by considering the following extension of the Poincaré algebra (with $Sp(n) \times SU(2)$ as an automorphism group):

$$[P_{\mu i}, P_{\nu j}] = 2i\,\Omega_{\mu\nu} Z_{ij},$$

$$[P_{\mu i}, Z_{jk}] = [Z_{ij}, Z_{kl}] = 0. \tag{II.1}$$

Here $Z_{ij} = Z_{ji}$ is a real $SU(2)$ triplet of central charge generators. This algebra can be realized in the space $R^{4n,3} = \{x^{\mu i}, z^{ij}\}$,

where $[x] = \text{cm}$, $[z] = \text{cm}^2$. The derivatives covariant with respect to (II.1) are

$$D_{\mu i} = \frac{\partial}{\partial x^{\mu i}} + \Omega_{\mu\nu} x^{\nu j}\frac{\partial}{\partial z^{ij}}, \quad \partial^{z}_{ij} = \frac{\partial}{\partial z^{ij}}. \tag{II.2}$$

Further, adding harmonic variables $u^{\pm i}$ to $R^{4n,3}$ we find an analytic subspace

$$\{x^{\mu +} = x^{\mu i}u_i^{+},\ z^{++} = z^{ij}u_i^{+}u_j^{+},\ u^{\pm i}\} \tag{II.3}$$

which is closed under the action of the full algebra (II.1). There we can introduce analytic fields defined by the conditions

$$D_{\mu}^{+}\varphi = \left(\frac{\partial}{\partial x^{\mu -}} + \frac{1}{2}x_{\mu}^{+}\frac{\partial}{\partial z^{+-}} + x_{\mu}^{-}\frac{\partial}{\partial z^{--}}\right)\varphi = 0 \tag{II.4}$$

$$\partial_z^{++}\varphi = \partial_z^{+-}\varphi = 0.$$

The solution of (II.4) is

$$\varphi = \varphi(x^{\mu +}, z^{++}, u^{\pm i}). \tag{II.5}$$

II.2. <u>Curved $R^{4n,3}$ and hyper-Kähler geometry.</u> Next we shall discuss the curved version of the space introduced above and locate the places where the hyper-Kähler prepotentials occur as geometric objects. Our discussion will be brief, we shall only point out the modifications to the scheme developed in sect. I.

First we extend the τ group by adding local translations of z^{ij}, $\delta z^{ij} = \tau^{ij}(x)$. As in the case of Kähler geometry [1], neither the gauge group nor the gauge fields will depend on z^{ij}. Only matter fields are allowed to do so. The requirement that matter z-dependent analytic fields (II.5) should exist in the curved case has important consequences for the geometry. In order to make the analyticity (II.5) manifest we introduce an analytic basis and λ group with the following new (in addition to (I.6)) elements:

$$z_A^{\pm\pm} = z^{ij}u_i^{\pm}u_j^{\pm} + v^{\pm\pm}(x,u)$$

$$\delta z_A^{++} = \lambda^{++}(x^{+},u),\ \delta z_A^{+-} = \lambda^{+-}(x^{\pm},u),\ \delta z_A^{--} = \lambda^{--}(x^{\pm},u). \tag{II.6}$$

This gives rise to several new vielbeins in the λ world covariant harmonic derivative $\mathcal{D}^{++}$ (c f. (I.13)):

$$\mathcal{D}^{++} = \Delta^{++} + H^{++(++)}\partial_z^{--} + z_A^{++}\partial_z^{+-} + 2z_A^{+-}\partial_z^{++} + \omega^{++} . \tag{II.7}$$

For simplicity the vielbeins of ∂_z^{+-} and ∂_z^{++} have been gauged into their flat space values and the corresponding parameters have been fixed:

$$\Delta^{++}\lambda^{+-} = \lambda^{++} , \quad \Delta^{++}\lambda^{--} = 2\lambda^{+-} . \tag{II.8}$$

The vielbein $H^{++(++)}$ transforms as follows

$$\delta H^{++(++)} = \Delta^{++}\lambda^{++} \tag{II.9}$$

which coincides with (I.36). This reveals the meaning of the prepotential $\mathcal{L}^{(+4)}$ which is related to $H^{++(++)}$ by the field redefinition (I.37). We stress that the vielbein $H^{++(++)}$ as well as all the other new vielbeins appearing below have no need to depend on the central charge coordinates.

The next step is to express the new vielbeins in $\mathcal{D}^{--}$:

$$\mathcal{D}^{--} = \Delta^{--} + (H^{--(++)} + 2z_A^{+-})\partial_z^{--} + (H^{--(+-)} + z_A^{--})\partial_z^{+-} + H^{--(--)}\partial_z^{++} + \omega^{--} \tag{I.10}$$

in terms of H^{++} as solutions to the differential equations following from (I.20). Further, the covariant derivative $\mathcal{D}_\alpha^+$ becomes

$$\mathcal{D}_\alpha^+ = \Delta_\alpha^+ + E_\alpha^{+(+-)}\partial_z^{+-} + E_\alpha^{+(--)}\partial_z^{--} . \tag{I.11}$$

Note the absence of a ∂_z^{++} term in (II.11), which is due to the analyticity of the parameter $\lambda^{++}(x^+, u)$ (II.6). Finally, the derivative $\mathcal{D}_\alpha^-$ is defined as before (I.39) and has the following new terms:

$$\mathcal{D}_\alpha^- = \Delta_\alpha^- + \mathcal{L}_\alpha^+\partial_z^{--} + (\Delta^{--}E_\alpha^{+(+-)} - \Delta_\alpha^+ H^{--(+-)})\partial_z^{+-}$$
$$+ (\Delta^{--}E_\alpha^{+(--)} - \Delta_\alpha^+ H^{--(--)})\partial_z^{++} + \omega_\alpha^- . \tag{I.12}$$

Here

$$\mathcal{L}_\alpha^+ = e_\alpha^\mu \mathcal{L}_\mu^+ = \frac{1}{2}\Delta_\alpha^+(\Delta^{++})^2 H^{--(--)} + 2E_\alpha^{+(+-)} \tag{I.13}$$

and one obtains (see (I.25),(II.6))

$$\delta\mathcal{L}_\mu^+ = -\partial_\mu^-\lambda^{\nu+}\mathcal{L}_\nu^+ - \partial_\mu^-\lambda^{++} \tag{II.14}$$

which coincides with (I.33). Thus we find a natural place for the prepotential $\mathcal{L}_\mu^+$ also.

The set of covariant derivatives shown above must satisfy a number of constraints. They look the same as the ones considered in sect.I with the exception of

$$[\mathcal{D}_\alpha^+, \mathcal{D}_\beta^+] = -2\Omega_{\alpha\beta}\partial_z^{++}$$
$$[\mathcal{D}_\alpha^+, \mathcal{D}_\beta^-] = -2\Omega_{\alpha\beta}\partial_z^{+-} + R_{\alpha\beta}^{+-} \tag{II.15}$$
$$[\mathcal{D}_\alpha^-, \mathcal{D}_\beta^-] = -2\Omega_{\alpha\beta}\partial_z^{--} .$$

These reproduce the flat space commutation relations of the derivatives (II.2). The essential new point is the presence of new torsion terms in the constraints, related to the central charge coordinates. They naturally give rise to a number of relations which were derived in sect. I as consequences of higher-order differential constraints. Thus, the vanishing of the torsion

$$T_\alpha^+|_\beta^-|^{(++)} = 0$$

yields the analyticity of $\mathcal{L}_\mu^+$ (I.34). The relation

$$T_\alpha^+|_\beta^+|^{(--)} = 0$$

explains why $H_{\mu\nu}$ (I.29) is expressed in terms of $\mathcal{L}_\mu^+$ (I.31). Further,

$$T^{++}|_\alpha^+|^{(++)} = 0$$

implies the analyticity of $H^{++(++)}$ (and thus of $\mathcal{L}^{(+4)}$). Finally, the relation (I.35) which allows us to express $H^{++\mu+}$ in terms of the prepotentials is now equivalent to the torsion constraint

$$T^{++}|_\alpha^+|^{(+-)} = 0 .$$

So, we have seen that the introduction of the auxiliary central charge coordinates proved very useful in the geometric interpretation of the hyper-Kähler prepotentials and their pregauge group. However, none of these objects depend on the new coordinates. At the same time, the geometry permits the existence of analytic matter fields with non-trivial central charge dependence. It is intriguing to find out the meaning of such fields.

III. Examples of explicit construction of hyper-Kähler metrics

In this section we shall give two rather simple R^4 examples in which we shall explicitly carry out the procedure for construction of hyper-Kähler metrics. They illustrate the two alternative approaches to the problem. The first is to work in the λ world and solve the linear differential equations for the vielbeins H^{--} (I.21). The second is to find the bridge to the τ world by solving the nonlinear equation (I.14). This second approach is analogous to the one of Newman, Penrose, Ward and others [2-5].

III.1. <u>The Taub-NUT metric in the analytic</u> basis. We consider the harmonic extension of R^4 with coordinates

$$x^{\mu\pm} = (x^{\pm}, -\bar{x}^{\pm}), \quad u^{\pm i} . \tag{III.1}$$

All the gauges discussed in Sect.1.8 are assumed to be fulfilled. Our choice of the prepotentials is prompted by the results in [7] where the Taub-NUT metric [8] was obtained from an N=2 supersymmetric sigma model (the precise relationship between hyper-Kähler prepotentials and N=2 supersymmetric Lagrangians will be discussed in Sect.IV):

$$\mathcal{L}^{+}_{\mu} = x^{+}_{\mu}, \quad \mathcal{L}^{(+4)} = -\frac{1}{2}(x^{+})^{2}(\bar{x}^{+})^{2} . \tag{III.2}$$

This choice corresponds to an SU(2) x U(I) isometry of the manifold (SU(2) follows from the absence of explicit harmonic dependence in (III.2), U(I) rotates $x \to e^{i\alpha}x$, $\bar{x} \to e^{-i\alpha}\bar{x}$). It is worth remarking that the form (III.2) is the simplest example of prepotentials in the normal coordinates of sect.I.8, where all the symmetrized covariant derivatives of the Weyl tensor vanish at the point x = 0, except for the Weyl tensor itself.

The next step is to calculate $H^{++\mu+}$ according to (I.38) and write down the differential equations (I.21) for $H^{-\mu+} = (H^{-}, -\bar{H}^{-})$ and $H^{--\mu-} = (H^{(-3)}, -\bar{H}^{(-3)})$:

$$[x^{+}\partial^{+} + \bar{x}^{+}\bar{\partial}^{+} + x^{+}\bar{x}^{+}(x^{+}\partial^{-} - \bar{x}^{+}\bar{\partial}^{-})]H^{-} =$$
$$= x^{+} + 2x^{+}\bar{x}^{+}H^{-} + (x^{+})^{2}\bar{H}^{-}$$

$$[x^{+}\partial^{+} + \bar{x}^{+}\bar{\partial}^{+} + x^{+}\bar{x}^{+}(x^{+}\partial^{-} - \bar{x}^{+}\bar{\partial}^{-})]H^{(-3)} = \tag{III.3}$$
$$= -x^{-} + H^{-} .$$

It is not hard to check that (III.3) has the following unique solution:

$$H^{-} = -\frac{x^{-}}{a}\ln(1-a)\left[1+\ln(1+b)\right]$$

$$H^{(-3)} = -\frac{(x^{-})^{2}\bar{x}^{-}}{a^{2}b}\ln(1+b)\left[a+\ln(1-a)\right], \tag{III.4}$$

$$a = x^{-}\bar{x}^{+}, \quad b = x^{+}\bar{x}^{-} = -\tilde{a} .$$

Finally, a straightforward calculation using (I.40) and (I.23) produces the metric in the λ basis. For instance:

$$g^{\mu+\nu-} = -\varepsilon^{\mu\lambda}(\partial H)^{-1\,\nu}_{\ \lambda} = \begin{pmatrix} x^{+}x^{-}\dfrac{(1-a)\ln(1-a)}{a[1+\ln(1-a)(1+b)]} & \dfrac{(1+b)[1+\ln(1+b)]}{1+\ln(1-a)(1+b)} \\ \dfrac{(1-a)[1+\ln(1-a)]}{1+\ln(1-a)(1+b)} & \tilde{x}^{+}\tilde{x}^{-}\dfrac{(1+b)\ln(1+b)}{b[1+\ln(1-a)(1+b)]} \end{pmatrix} .$$

As follows from the computation in ref.[7], the τ basis form of the metric indeed coincides with the standard Taub-NUT one[8].

This example, although rather simple, illustrates the main advantage of the analytic basis approach, the linearity of the differential equations which determine the metric.

III.2. <u>The Ansatz of Ward and the bridges to the τ world</u>. As we explained, the main obstacle on the way to constructing a τ basis hyper-Kähler metric from the prepotentials is the nonlinear differential equation (I.14) for the bridges. The nonlinearity is due to the fact that the prepotential $\mathcal{L}^{(+4)}(x^{+}_{A}, u)$ is naturally defined in the λ basis, whereas the bridge $v^{\mu+}(x^{\mu i}, u)$ has natural τ basis coordinates (for the purpose of calculating the τ metric one does not need the symplectic bridge M (I.8) which affects only the vielbeins). Nevertheless, a clever trick proposed by Ward [5] makes it possible to avoid this nonlinearity in a class of examples (in four dimensions). The idea (in our notation) is to choose a special dependence of $\mathcal{L}^{(+4)}$ on its argument x^{+}_{A} :

$$\mathcal{L}^{(+4)} = \mathcal{L}^{(+4)}(p^{\mu+}x^{+}_{A\mu}, u) . \tag{III.5}$$

Here $p^{\mu+} = p^{\mu i}u^{+}_{i}$ where $p^{\mu i}$ ($\mu=1,2$; $i=1,2$) is a constant real <u>vector</u> [3]. Then from (I.38) one finds

[3] This choice is simplest though not unique. Original Ward's Ansatz [5] corresponds to using three-rank spinor $p^{\mu++} = p^{\mu ij}u^{+}_{i}u^{+}_{j}$ rather than $p^{\mu+}$.

$$H^{++\mu+} = p^{\mu+} g^{++}(p^+ x^+_A, u), \quad g^{++} = \frac{\partial \mathcal{L}^{(+4)}}{\partial (p^+ x^+_A)} . \tag{III.6}$$

Further, in τ basis the equation (I.14) for the bridge $v^{\mu+}$ now becomes:

$$\partial^{++} v^{\mu+}(x,u) = p^{\mu+} g^{++}(p^+ x^+_A, u) . \tag{III.7}$$

By inspection of (III.7), one concludes that $v^{\mu+}$ can be sought in the form

$$v^{\mu+}(x,u) = p^{\mu+} v(x,u) . \tag{III.8}$$

The advantage of this Ansatz is now clear. With a bridge like (III.8) one has

$$p^{+\mu} x^+_{A\mu} = p^{+\mu}(x^i_\mu u^+_i), \quad (p^{+\mu} p^+_\mu = 0) \tag{III.9}$$

so the equations (I.14) for the bridges become linear. In particular, eq. (III.7)

$$\partial^{++} v(x,u) = g^{++}(p^+ x^+, u)$$

has the solution

$$v^{\mu+} = p^{\mu+} \cdot \int dw \, \frac{u^+ w^-}{u^+ w^+} \, g^{++}(p^+_w x^+, w) . \tag{III.10}$$

Here u and w are two sets of harmonic variables, $u^+ w^- = u^{+i} w^-_i$, the harmonic distribution $(u^+ w^+)^{-1}$ has the property [9]

$$\partial^{++}_u (u^+ w^+)^{-1} = \delta^{(1,-1)}(u,w) .$$

Analogously, in the gauge (I.15) one finds

$$v^{\mu-} = p^{\mu+} \int dw \, \frac{u^- w^-}{u^+ w^+} \, g^{++}(p^+ x^+, w) . \tag{III.II}$$

Having obtained the bridges, it is a matter of tedious calculations to find the expression for $H^{--\mu+} = \partial^{--} v^{\mu+} + x^{\mu i} u^-_i$ and then to evaluate $\partial^+_{A\nu} x^{\rho k}$ and $(\partial H)^\mu_\nu = \partial^+_{A\nu} H^{--\mu+} =$ $= \partial^+_{A\nu} x^{\rho k} \cdot \partial_{\rho k} H^{--\mu+}$:

$$\partial^+_{A\nu} x^{\rho k} = A^\mu_\nu(x) \left(\delta^\rho_\mu \delta^k_l + p^{\rho k} \psi_{\mu l}(x) \right) u^{+l} .$$

$$\partial^+_{A\nu} H^{--\mu+} = A^\rho_\nu(x) \left(\delta^\mu_\rho - p^{\mu+} I^-_\rho(x,u) \right)$$

$$\psi_{\nu i}(x) = \int dw \cdot w^-_i \, p^+_\nu \, \dot{g} , \quad \dot{g} = \frac{\partial g^{++}}{\partial (p^+ x^+)}$$

$$I^-_\nu(x,u) = \int dw \, \frac{1}{u^+ w^+} \, \dot{g} \, p^+_\nu$$

$$A^\rho_\nu(x) = \frac{1}{1 + p\psi + \frac{1}{4} p^2 \psi^2} \left(\delta^\rho_\nu + \psi^{\rho i}(x) p_{\nu i} \right), \quad (ab \equiv a^{\rho i} b_{\rho i}) .$$

Next one finds the vielbein $E^{+\mu k}_\alpha = (\partial H)^{-1\,\nu}_\alpha \partial^+_{A\nu} x^{\mu k}$ (see Appendix) using

$$(\partial H)^{-1\,\nu}_\alpha = \left(\delta^\rho_\alpha + \frac{1}{1 + p\psi} p^{\rho+} I^-_\alpha \right) A^{-1\,\nu}_\rho .$$

Finally, the vielbein $E^{-\mu k}_\alpha$ equals $(\partial H)^{-1\,\nu}_\alpha \cdot \partial^{--} \partial^+_{A\nu} x^{\mu k}$ and one is able to compute the metric

$$g^{\mu k, \nu l} = \varepsilon^{\rho\alpha} \left(E^{+\mu k}_\alpha E^{-\nu l}_\rho - E^{-\mu k}_\alpha E^{+\nu l}_\rho \right) =$$

$$= (1 + p\psi)^{-1} \left[\varepsilon^{\mu\nu} \varepsilon^{kl} + \psi^{\mu k} p^{\nu l} + \psi^{\nu l} p^{\mu k} + (\psi\psi) p^{\mu k} p^{\nu l} \right] .$$

So, the above Ansatz works for this class of hyper-Kähler metrics. However, we are not aware of any suitable Ansatz of this kind which would make it possible to find the most general hyper-Kähler metrics.

IV. Hyper-Kähler geometry and N=2 supersymmetric sigma models

There is a remarkable one-to-one correspondence between the most general N=1 supersymmetric sigma model in 4 dimensions and

Kähler geometry [10]. The superfield Lagrangian for such a sigma model is just the Kähler prepotential $K(\varphi,\bar{\varphi})$ of the manifold with chiral (i.e. N=1 analytic) superfields as its coordinates. The relationship between N=2 sigma models and hyper-Kähler geometry has been investigated in [11]. It was shown that the metric of an N=2 sigma model is necessarily hyper-Kähler and vice versa, given a hyper-Kähler metric one can always construct an on-shell N=2 sigma model. However, unlike the simple case of N=1, it was a nontrivial problem to give a recipe how to explicitly construct N=2 sigma models. The solution to this problem involves finding an adequate way to formulate N=2 off-shell matter multiplets in superspace. This was done in [12] using harmonic superspace. Here we briefly review this approach and establish the precise one-to-one correspondence between the N=2 sigma model superfield Lagrangians and the hyper-Kähler prepotentials introduced in sect. I.

N=2 supersymmetry is usually realized in the superspace $\{x^{\alpha\dot{\alpha}}, \theta^{\alpha i}, \bar{\theta}^{\dot{\alpha} i}\}$ ($x^{\alpha\dot{\alpha}}$ is the coordinate of 4 dimensional Minkowski space and should not be confused with $x^{\mu k}$ of the previous sections). Adding harmonic coordinates $u^{\pm i}$ one becomes able to define analytic superfields satisfying the constraint

$$D^+_{\alpha}\phi = \bar{D}^+_{\dot{\alpha}}\phi = 0, \qquad \text{(IV.1)}$$

where $D^+_{\alpha(\dot{\alpha})} = u^+_i D^i_{\alpha(\dot{\alpha})}$ are the $+$ projections of the spinor covariant derivatives. This constraint is naturally solved in the analytic basis in harmonic superspace:

$$x_A^{\alpha\dot{\alpha}} = x^{\alpha\dot{\alpha}} - 2i\,\theta^{i\alpha}\bar{\theta}^{\dot{\alpha}j}u^+_{(i}u^-_{j)}, \quad \theta^{\pm}_{\alpha(\dot{\alpha})} = \theta^i_{\alpha(\dot{\alpha})}u^{\pm}_i. \qquad \text{(IV.2)}$$

There $D^+_{\alpha(\dot{\alpha})} = \partial/\partial\theta^{\alpha(\dot{\alpha})-}$ and (IV.1) simply means

$$\Phi = \Phi(x_A, \theta^+, \bar{\theta}^+, u^{\pm}). \qquad \text{(IV.3)}$$

It turns out that the N=2 matter multiplets (hypermultiplets) are adequately described by analytic superfields $q^{\mu+}(x_A, \theta^+, \bar{\theta}^+, u)$ with U(I) charge +1. Here μ is an Sp(n) spinor index, and $q^{\mu+}$ satisfies the pseudo-reality condition

$$\widetilde{q^{\mu+}} = q^+_{\mu} \equiv \Omega_{\mu\nu}\, q^{\nu+}, \quad \mu = 1, \dots 2n. \qquad \text{(IV.4)}$$

The superfield $q^{\mu+}$, as it is, contains an infinite number of fields in its harmonic expansion. However, its free equation of motion

$$D^{++}q^{\mu+} = 0, \quad D^{++} = \partial^{++} - 2i\,\theta^{\alpha+}\bar{\theta}^{\dot{\alpha}+}\partial^A_{\alpha\dot{\alpha}} \qquad \text{(IV.5)}$$

cuts off the infinite tail of auxiliary fields. The remaining 4n real scalar fields ($q^{\mu+} = q^{\mu i}(x)u^+_i + \dots$) and the same number of fermions satisfy the Klein-Gordon and Dirac equations. The equation (IV.5) can be derived from the following action

$$S_{free} = -\int d^4x_A\, d^2\theta^+ d^2\bar{\theta}^+ du\; q^+_{\mu} D^{++} q^{\mu+}. \qquad \text{(IV.6)}$$

The above picture can be generalized to include self-interactions of the hypermultiplets. The most general action propagating 4n scalar fields has the following form [4)]

$$S = \int d^4x_A\, d^2\theta^+ d^2\bar{\theta}^+ du\; H^{++(++)} \qquad \text{(IV.7)}$$

$$H^{++(++)} = -\mathcal{L}^+_{\mu}(q^+, u)D^{++}q^{\mu+} + \mathcal{L}^{(+4)}(q^+, u). \qquad \text{(IV.8)}$$

This action is invariant under reparametrizations of $q^{\mu+}$, $\delta q^{\mu+} = \lambda^{\mu+}(q^+, u)$ provided $\mathcal{L}^+_{\mu}$ and $\mathcal{L}^{(+4)}$ transform as follows:

$$\delta\mathcal{L}^+_{\mu} = -\partial^-_{\mu}\lambda^{\nu+}\mathcal{L}^+_{\nu}, \quad \delta\mathcal{L}^{(+4)} = \mathcal{L}^+_{\mu}\partial^{++}\lambda^{\mu+}. \qquad \text{(IV.9)}$$

Here $\partial^-_{\mu} = \partial/\partial q^{\mu+}$ and ∂^{++} is the partial derivative acting on the argument u of $\lambda^{\mu+}(q,u)$ (but ignoring the u dependence of the argument q^+). In addition, the action does not change under the following redefinitions ($\lambda^{++} = \lambda^{++}(q^+, u)$)

$$\delta\mathcal{L}^+_{\mu} = -\partial^-_{\mu}\lambda^{++}, \quad \delta\mathcal{L}^{(+4)} = \partial^{++}\lambda^{++}. \qquad \text{(IV.10)}$$

Indeed,

$$\delta H^{++(++)} = \partial^{++}\lambda^{++} + D^{++}q^{\mu+}\partial^-_{\mu}\lambda^{++} = D^{++}\lambda^{++} \qquad \text{(IV.11)}$$

and the integral of such a total derivative in (IV.7) vanishes. Here

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4) In [7] we have proposed a form of the action which could contain more than one derivative D^{++}. The corresponding equations of motion are of higher order in D^{++} (although not in $\partial/\partial x$) which is equivalent to propagating more than 4n scalars. So, one can restrict oneself to the action (IV.7,8) without loss of generality. We are grateful to O. Ogievetsky for pointing this out to us.

one may recall a similar phenomenon in N=1 sigma models, where the action is invariant up to a total derivative under Kähler transformations of the Lagrangian.

The variation of the action (IV.7) with respect to $q^{\mu+}$ yields the equation of motion

$$(\partial^-_\mu \mathcal{L}^+_\nu - \partial^-_\nu \mathcal{L}^+_\mu) D^{++} q^{\nu+} = \partial^-_\mu \mathcal{L}^{(+4)} + \partial^{++} \mathcal{L}^+_\mu \,. \tag{IV.12}$$

By now the reader should have recognized the remarkable similarity of the theory of q^+ hypermultiplets with the description of hyper-Kähler geometry given in sect.I. The analytic basis coordinates $x^{\mu+}$ are replaced by Grassmann analytic superfields $q^{\mu+}$, the hyper-Kähler prepotentials $\mathcal{L}^+_\mu$, $\mathcal{L}^{(+4)}$ determine the self-interaction. The on-shell derivative $D^{++}q^{\mu+}$ expressed from the equation of motion (IV.12) is exactly the same as the vielbein $H^{++\mu+}$ (I.38). Finally, the Lagrangian in (IV.7,8) coincides on -shell with the vielbein $H^{++(++)}$ (I.37) in front of $\partial/\partial z^{++}_A$ in the covariant derivative $\mathcal{D}^{++}$ (II.7). Thus, the invariance (IV.11) of the sigma model action is associated with the transformations of the central charge coordinate z^{++}_A.

We stress that it is the on-shell hypermultiplet action which corresponds to a hyper-Kähler sigma model. The reason is that off--shell the hypermultiplet superfield $q^{\mu+}$ contains an infinite number of auxiliary fields of the same dimension as the physical ones. The role of the equation of motion (IV.12) (which corresponds to the hyper-Kähler constraint (I.38)) is to eliminate those auxiliary fields in favour of the physical ones. For the latter the equations with the familiar sigma model type of self-interaction emerge. We recall that in [7] we used just this procedure to obtain the Taub-NUT metric. In [7] we also gave another interpretation of the equation of motion for $q^{\mu+}$ which was inspired by the constrained superfield formulation of hyper-Kähler sigma models of [13]. This equation can be viewed as the definition of a bridge to a certain τ basis. Indeed, in the gauge $\mathcal{L}^+_\mu = q^+_\mu$ the equation of motion (IV.12) reads

$$D^{++} q^{\mu+} = -\frac{1}{2}\partial^{\mu-}\mathcal{L}^{(+4)} = H^{++\mu+}(q^+, u) \,. \tag{IV.13}$$

Making the change of variables

$$q^{\mu+} = Q^{\mu i} u^+_i + v^{\mu+}(Q,u) \,, \tag{IV.14}$$

where $Q^{\mu i} = Q^{\mu i}(x^{\alpha\dot\alpha}, \theta^{\alpha}_{j}, \bar\theta^{\dot\alpha}_{j})$, (IV.13) becomes

$$D^{++} v^{\mu+}(Q,u) = H^{++\mu+}[Q^+ + v^+(Q,u), u] \,. \tag{IV.15}$$

This is nothing but the bridge defining equation (I.14). So, we conclude that passing to the new variables $Q^{\mu i}$ (which involves solving (IV.15)) we have eliminated the infinite set of auxiliary fields. However, in the new basis the condition of Grassmann analyticity (IV.1) for $q^{\mu+}$ becomes covariant:

$$D^+_{\alpha(\dot\alpha)} Q^{\mu i} u^+_i + \frac{\partial v^{\mu+}}{\partial Q^{\nu j}} D^+_{\alpha(\dot\alpha)} Q^{\nu j} = 0 \,. \tag{IV.16}$$

This equation generalizes the flat-space equation for the hypermultiplet

$$D^{(i}_{\alpha(\dot\alpha)} Q^{\mu j)} = 0$$

proposed in [14]. Comparing (IV.16) with the definition of inverse vielbein (A.6) we see that this equation can be written as

$$E^{+\mu}_{\nu j} D^+_{\alpha(\dot\alpha)} Q^{\nu j} = 0$$

that differs from the equation given in [13] merely by a rotation of the tangent space index μ with the $Sp(n)$ bridge $M^\rho_\mu(x,u)$ ($E^{+\mu}_{\nu j}$ is covariantly u-independent).

To avoid a possible misunderstanding, we point out that there is no direct correlation between choices of bases in the target harmonic space $\{q, u\}$ and in the harmonic superspace $\{x, \theta, u\}$, where q (and Q) are defined as superfields. For instance, we might stay in the λ-world with manifest analyticity as regards q^+ and, at the same time, choose the central basis in superspace where D^{++} equals ∂^{++} and Grassmann harmonic analyticity is non-manifest. It is crucial that both manifolds, $\{q, u\}$ and $\{x, \theta, u\}$, intersect by the pure harmonic part $\{u\}$, having the common set of harmonic variables.

Our next remark concerns an alternative way of describing off-shell hypermultiplets [12]. It makes use of chargeless analytic real superfield $\omega(x_A, \theta^+, \bar\theta^+, u)$ satisfying the second-order equation of motion

$$(D^{++})^2 \omega = F(\omega, u) \,. \tag{IV.17}$$

In fact, the ω description can be obtained by a duality transformation from the q^+ one [15]. To this end one decomposes q^+ into (we consider the case of R^4)

$$q^{\mu+} = u^{\mu+}\omega + u^{\mu-} f^{++}$$

and inserts it into (IV.7). Eliminating the auxiliary superfield f^{++} one obtains an action for ω alone, from which the equation of motion (IV.17) follows. It is remarkable that (IV.17) looks almost identical with the "good cut" equation of the so-called "H-space" approach to self-dual Einstein equations [4].

In conclusion we point out that besides hyper-Kähler manifolds in $\mathbb{R}^{4n}$ one may consider quanternionic manifolds as well. According to a theorem in [16], such manifolds necessarily emerge when coupling N=2 supersymmetric sigma models to N=2 supergravity. Recently the general N=2 supergravity-matter couplings were explicitly constructed employing harmonic superspace [17]. In a separate publication we shall describe how this coupling gives rise to quaternionic sigma models and shall find the prepotentials for that kind of quaternionic geometry.

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APPENDIX. Vielbeins and metric in the τ world

The τ basis components of the vielbein can be read off from the expressions for covariant derivatives $\mathcal{D}^+_\alpha$, $\mathcal{D}^-_\alpha$ (I.10), (I.23), (I.39) by passing in the latter to central basis coordinate $x^{\rho k}$ (we impose the gauge (I.45)):

$$E^{+\rho k}_\alpha = (\partial H)^{-1\ \mu}_\alpha \partial^+_\mu x^{\rho k}, \quad \mathcal{D}^{++} E^{+\rho k}_\alpha = 0, \tag{A.1}$$

$$E^{-\rho k}_\alpha = \mathcal{D}^{--} E^{+\rho k}_\alpha = -\partial^-_\alpha x^{\rho k} - (\partial H)^{-1\ \varphi}_\alpha \partial^+_\varphi x^{\beta i} \partial_{\beta i} \partial^{--} x^{\nu +} \partial^-_\nu x^{\rho k} =$$
$$= (\partial H)^{-1\ \varphi}_\alpha \partial^{--} \partial^+_\varphi x^{\rho k}. \tag{A.2}$$

Here $\mathcal{D}^{\pm\pm} = \partial^{\pm\pm} + \omega^{\pm\pm}$ where $\omega^{\pm\pm}$ are given by eqs. (I.27), (I.22) and $(\partial H)^{\ \rho}_\mu = \partial^+_\mu H^{--\rho +} = \partial^+_\mu x^{\beta i} \partial_{\beta i} \partial^{--} x^{+\rho}$.
Note the useful relations

$$\partial^{++} \partial^+_\nu x^{\rho k} = 0 \Rightarrow (\partial^{--})^2 \partial^+_\nu x^{\rho k} = 0 \tag{A.3}$$

which follow from the analyticity of $H^{++\nu +}$ and from the gauge choice (I.15). The metric is defined as

$$g^{\rho k, \lambda s}(x) = \left(E^{+\rho k}_\alpha E^{-\lambda s}_\beta - E^{-\rho k}_\alpha E^{+\lambda s}_\beta\right) \Omega^{\beta\alpha}. \tag{A.4}$$

It involves no dependence on $u^\pm$

$$\partial^{++} g^{\rho k, \lambda s} = 0 \tag{A.5}$$

that can be checked by using the equation (A.1) together with

$$\mathcal{D}^{++} E^{-\rho k} = E^{+\rho k}.$$

Recall that the $Sp(n)$-connections of the τ basis derivatives $\mathcal{D}^{++}, \mathcal{D}^{--}$ can be entirely removed by rotating tangent space indices with the help of the $Sp(n)$-bridge $M^\alpha_\gamma(x, u)$ (the latter drops out from the metric (A.4)).

It is useful also to quote here the vielbeins which are inverse to (A.1), (A.2) and produce the conventional metric (with covariant world indices):

$$E^{+\alpha}_{\rho k} = -\partial_{\rho k} x^{+\alpha}$$
$$E^{-\alpha}_{\rho k} = \partial_{\rho k} x^{-\gamma} (\partial H)^{\ \alpha}_\gamma - \partial_{\rho k} x^{+\sigma} (\partial H)^{-1\ \varphi}_\sigma \partial^+_\varphi x^{\beta i} \partial_{\beta i} \partial^{--} x^{-\gamma} (\partial H)^{\ \alpha}_\gamma \tag{A.6}$$

$$E^{+\beta}_{\rho k} E^{-\lambda i}_\beta + E^{-\beta}_{\rho k} E^{+\lambda i}_\beta = \delta^\lambda_\rho \delta^i_k,$$
$$E^{-\beta}_{\rho k} E^{-\rho k}_\alpha = 0, \quad E^{+\beta}_{\rho k} E^{+\rho k}_\alpha = 0, \tag{A.7}$$
$$E^{-\rho k}_\alpha E^{+\beta}_{\rho k} = \delta^\beta_\alpha, \quad E^{+\rho k}_\alpha E^{-\beta}_{\rho k} = \delta^\beta_\alpha,$$
$$\mathcal{D}^{++} E^{+\alpha}_{\rho k} = 0, \quad \mathcal{D}^{--} E^{+\alpha}_{\rho k} = -E^{-\alpha}_{\rho k}.$$

It is easy to check that the metric

$$g_{\rho k, \lambda s} = E^{-\beta}_{\rho k} E^{+}_{\lambda s\, \beta} - E^{+\beta}_{\rho k} E^{-}_{\lambda s\, \beta} \tag{A.8}$$

is u-independent, $\partial^{++} g_{\rho k, \lambda s} = 0$, and is indeed inverse to $g^{\rho k, \lambda s}$ (A.4).

As a final remark, we point out that the skew-symmetric twofold charged object

$$\Omega^{++}_{[\rho k,\lambda s]} = \Omega^{(ij)}_{[\rho k,\lambda s]} u^+_i u^+_j = E^{+\beta}_{\rho k} E^{+}_{\lambda s\,\beta} . \qquad (A.9)$$

contains just the triplet of covariantly constant 1-forms the existence of which is characteristic of hyper-Kähler geometry [11,13].

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Гальперин А.С. и др. E2-87-264
Геометрия калибровочных полей из комплексной и гармонической аналитичностей. Гиперкэлеров случай.

Принцип сохранения гармонической аналитичности применен для нахождения препотенциалов гиперкэлеровой геометрии. Их геометрический смысл выявлен в рамках расширенного пространства с координатами центрального заряда. Установлено взаимно-однозначное соответствие между гиперкэлеровыми препотенциалами и наиболее общим лагранжианом d==4, N=2 суперсимметричной σ модели вне массовой оболочки.

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Galperin A.S.et al. E2-87-264
Gauge Field Geometry from Complex and Harmonic Analyticities. Hyper-Kähler Case

The concept of preservation of harmonic analyticity is applied to find unconstrained prepotentials of hyper-Kähler geometry. The geometric meaning of prepotentials is revealed with introducing extra central charge coordinates. Finally, we establish the one-to-one correspondence between hyper-Kähler geometry and off-shell d=4, N=2 supersymmetric σ-models. Their most general Lagrangian is shown to be uniquely composed of hyper-Kähler prepotentials, with the analytic space coordinates replaced by analytic hypermultiplet superfields defined on the same set of harmonic variables.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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