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A.S.Galperin*, E.A.Ivanov, V.I.Ogievetsky,
E.Sokatchev**

GAUGE FIELD GEOMETRY FROM COMPLEX AND HARMONIC ANALYTICITIES. KÄHLER AND SELF-DUAL YANG-MILLS CASES

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* Institute of Nuclear Research, Tashkent, USSR
** Institute for Nuclear Research and Nuclear Energy, Sofia, Bulgaria

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I. Introduction

The concept of Cauchy-Riemann analyticity has profound implications in gauge theory. The preservation of analytic representations in a gauge field background is the leading principle in such classical problems as the theory of Yang-Mills (YM) instantons, Kähler and hyper-Kähler geometry, etc. [1-6]. On the other hand, it is remarkable that the analogous principle of preservation of Grassmann analyticity [7] governs the geometric structure of supersymmetric gauge theories. The underlying superspace constraints of these theories can be viewed as the integrability conditions for the existence of certain analytic subspaces. Passing to the basis where analyticity is manifest automatically solves the constraints in terms of several unconstrained superfield objects, the prepotentials. The latter are natural carriers of the intrinsic geometry of a given theory. It is essential that the whole theory can be defined entirely within the representation with a manifest analyticity. Recently, a deeper understanding of the role of Grassmann analyticity has led to important developments in the theory of extended supersymmetry. The introduction of the concept of harmonic superspace and harmonic analyticity made it possible to construct unconstrained geometric formulations of all the N=2 supersymmetric theories [8-11] and of N=3 YM theory [12].

The idea of harmonic superspace is intimately related to the famous twistor theory [1- 4][1)]. The latter theory is widely employed also in the problems mentioned in the beginning. This correspondence has led us to realizing that these purely bosonic problems may have a transparent presentation along the lines one normally follows in supersymmetry. In the present and subsequent [15] papers we reformulate a number of these theories (self-dual Yang-Mills theory, Kähler and hyper-Kähler gravities) in a way which readers with experience in supersymmetry will have no difficulties to understand. The general principle we confess is the preservation of certain "flat" analyticities in the full interaction case. That allows us to reveal a <u>fundamental</u> role of corresponding analytic subspaces in the geometry

1) Twistor-like interpretations of the extended supersymmetry constraints have been considered by Witten [13] and Rosly [14].

of problems in question. Surprisingly enough, almost all the notions known from the geometric superspace considerations have immediate analogs in ordinary spaces, thus demonstrating a deep affinity of both classes of theories.

We use a down-to-earth language of conventional differential geometry, no advanced mathematical background is required including the knowledge of subtleties of twistor formalism. Nevertheless, one of our main incentives is to explain the relationship with twistors [1-4] and some other concepts of similar nature [5]. In the process, we discuss a hard problem in twistor theory. There one has to solve a non-linear differential equation for some basic object, out of which one constructs the self-dual YM or gravitational field. We propose an alternative way which consists in keeping closer to the manifest analyticity underlying those theories. This way leads to a linear differential equation. It was first suggested by B.M. Zupnik [16] in the context of N=2 supersymmetry. Another main result of our research is a theorem [15] establishing the explicit one-to-one correspondence between the most general off-shell superfield Lagrangian for N=2 supersymmetric sigma models in four dimensions and the unconstrained prepotentials of hyper-Kähler geometry. This relationship is understood as fully as the one between N=1 sigma and Kähler geometry [17]. The geometric meaning of the Kähler and hyper-Kähler prepotentials is revealed in an extended space with extra central charge coordinates.

As we have already explained, the basic strategy we keep to consists in searching for an appropriate space (or superspace) and its analytic quotient space (and/or a frame in the tangent space) where the underlying analyticity becomes manifest and from which one may induce in full the relevant intrinsic geometry. Actually one could find such spaces by some reasonings and guess. A propos, just in this way the intrinsic geometry of N=1 supergravity (SG) and YM theories were exposed [18,19] and for N=2 theories the harmonic superspace and the harmonic analyticity were invented [8,9]. A more systematic method is to start with the set of properly postulated constraints in some space (or superspace) and then to interpret them as the integrability conditions for the covariant existence of an analytic subspace. This is precisely what we do throughout the paper and what allows us to establish a direct contact with the conventional definition of the problems we are involved in.

For a further use, it seems to the point here to recall the definition of Kähler and hyper-Kähler geometries. Most appropriate

for our purpose is the definition via the restrictions on the holonomy group of the manifold, that is the group generated by the components of Riemann curvature tensor (see, e.g. [20]) . The Kähler and hyper-Kähler geometries are particular branches of general Riemann geometry in real spaces R^{2n} and R^{4n}, respectively, with the holonomy groups in U(n) and Sp(n). Since the Riemann tensor in general appears in the r.h.s. of commutators of covariant derivatives with respect to the manifold coordinates, the above definition amounts to imposing certain constraints on these commutators. As we shall see, those look very similar to the superspace constraints defining the supersymmetric gauge theories and thus can be treated in a similar way.

This work is divided into two parts. The present article has an introductory character. Our purpose here is to introduce the main concepts of analyticity and its preservation starting with the very simple examples of Yang theory [6] and Kähler geometry. These have direct analogs in N=1 YM and supergravity theories which are based upon N=1 Grassmann analyticity (chirality). A further development of the idea of analyticity is harmonic analyticity. It is defined with the help of new harmonic variables and allows us to interpret and solve the self-dual YM equations as integrability conditions for this new analyticity. The complete supersymmetric analog of this theory is N=2 YM theory. We also discuss the relationship with the twistor-type constructions of Ward [4] and Newman [5] .

In the second article [15] the idea of harmonic analyticity is applied to the case of hyper-Kähler geometry. The corresponding constraints are rewritten once again as integrability conditions and then solved in terms of analytic prepotentials. Then it is shown that these prepotentials and their pregauge group have direct images in the theory of N=2 supersymmetric sigma models. There they determine the most general Lagrangian for the off-shell hypermultiplet Grassmann analytic superfields and its (hyper-Kähler) invariance.

II. Complex analyticity and gauge theories

In this section we shall discuss several examples of gauge theories with intrinsic complex structure. We shall show that their common feature is the preservation of certain analytic representations in a gauge field background.

The general framework for Euclidean gauge theories is a real n dimensional space $R^n = \{X^m\}, m=1,\dots,n$. The fields defined in it form representations of some internal symmetry group and of the (Euclidean)

Poincaré group consisting of translations P_m and $O(n)$ rotations. The gauge theories are obtained by making a symmetry group (an internal one or the Poincaré group itself) local, i.e. letting its parameters τ depend on X, $\tau = \tau(X)$. The principal geometric objects are the covariant derivatives $\mathcal{D}_m$. Their commutators define the tensors of the theory:

$$[\mathcal{D}_m, \mathcal{D}_n] = F_{mn} \quad \text{or} \quad R_{mn}$$

(F is the field strength for YM theories, R is the Riemann tensor for gravity, with values in internal symmetry algebra and tangent group algebra, respectively). Note that the gauge fields – the YM connection A_m and the metric g_{mn} (or vielbein e_{am}) – are the unconstrained potentials of those theories.

In even dimensional space R^{2n} one can introduce a complex structure if one reduces the homogeneous automorphism group of R^{2n} from $O(2n)$ to $U(n)$. Then one can choose the following complex basis:

$$x^\mu = X^\mu + i X^{\mu+n}, \quad x^{\bar\mu} = \overline{(x^\mu)}, \tag{II.1}$$

where $\mu = 1, \dots, n$ is $U(n)$ index. Now it becomes possible to define analytic fields $\varphi(x)$ which satisfy the $U(n)$ covariant Cauchy-Riemann (CR) condition

$$\frac{\partial}{\partial x^{\bar\mu}} \varphi(x, \bar x) = 0 \Rightarrow \varphi = \varphi(x). \tag{II.2}$$

This concept has important implications in gauge theory.

II.I. Analytic YM theory (Yang theory [6]). Suppose that the field $\varphi(x, \bar x)$ transforms under a YM group with real parameters $\tau(x, \bar x)$, $\tau = \bar\tau$, $\varphi' = e^{i\tau}\varphi$. Then the CR condition (II.2) must be covariantized:

$$\mathcal{D}_{\bar\mu}\varphi = 0, \quad \mathcal{D}_{\bar\mu} = \partial_{\bar\mu} + A_{\bar\mu}(x, \bar x). \tag{II.3}$$

Clearly, (II.3) can take place iff the following integrability condition holds:

$$[\mathcal{D}_{\bar\mu}, \mathcal{D}_{\bar\nu}] = 0 \Leftrightarrow F_{\bar\mu\bar\nu} = 0. \tag{II.4}$$

So, the gauge potential $A_{\mu,\bar\mu}$ is now constrained by equation (II.4). This constraint has the general solution

$$A_{\bar\mu} = e^{-iW}\partial_{\bar\mu} e^{iW}, \quad (A_\mu = e^{-i\bar W}\partial_\mu e^{i\bar W}), \tag{II.5}$$

where $W(x, \bar x)$ is a complex Lie-algebra valued scalar field transforming as follows:

$$e^{iW'} = e^{i\lambda} e^{iW} e^{-i\tau}. \tag{II.6}$$

There arise new transformations with analytic parameters λ,

$$\partial_{\bar\mu}\lambda = 0, \quad \lambda = \lambda(x). \tag{II.7}$$

Due to their analyticity they leave the connection $A_{\bar\mu}$ (II.5) invariant and, consequently, can be called pregauge transformations.

The solution (II.5-7) admits the following interpretation. Define the field ϕ

$$\phi = e^{iW}\varphi, \quad \phi' = e^{i\lambda}\phi. \tag{II.8}$$

Then it is easy to check that the covariant CR condition (II.3) implies

$$\partial_{\bar\mu}\phi = 0 \Rightarrow \phi = \phi(x). \tag{II.9}$$

In other words, the new field ϕ is manifestly analytic whereas φ is covariantly analytic. This is consistent with the new gauge transformation law (II.8) with analytic parameters λ.

This simple example illustrates an inportant phenomenon in gauge theories with constraints. There the potentials (e.g. $A_{\bar\mu}$) are expressed in terms of a new, unconstrained object (W) called prepotential. The latter has the geometric meaning of a bridge between two gauge frames: one with real parameters $\tau(x, \bar x)$ and another one with analytic parameters $\lambda(x)$. In the τ frame reality is a manifest property, whereas in the λ frame analyticity becomes manifest. In fact, the τ gauge freedom can be completely fixed by imposing the gauge condition $\mathrm{Re}\, W = 0$. This makes τ a (non-linear) function of $\lambda, \bar\lambda$ and $V = \mathrm{Im}\, W$, and leaves V as the only gauge field in the theory:

$$e^{2V'} = e^{-i\bar\lambda} e^{2V} e^{i\lambda}. \tag{II.10}$$

One can say that V parameterizes the homogeneous space G^c/G, where G^c is the complexification of the YM group G. So the above construction can be interpreted as a kind of nonlinear sigma model defined on G^c/G, with V being a Nambu-Goldstone field which gives a nonlinear realization of G^c (in a complete analogy with the N=1 supersymmetric YM theory [19], see below). Note that the theory can be formulated entirely in the λ frame. There the covariant derivatives are

$$\mathcal{D}^{(\lambda)}_{\mu} = e^{-V}\mathcal{D}_{\mu}e^{V} = \partial_{\mu} + e^{-2V}\partial_{\mu}e^{2V}, \quad \mathcal{D}^{(\lambda)}_{\bar{\mu}} = e^{-V}\mathcal{D}_{\bar{\mu}}e^{V} = \partial_{\bar{\mu}} \tag{II.11}$$

and the components of the λ -covariant field strength are

$$F_{\mu\bar{\nu}} = \partial_{\bar{\nu}}(e^{-2V}\partial_{\mu}e^{2V}) = -F_{\bar{\nu}\mu}, \quad F_{\mu\nu} = F_{\bar{\mu}\bar{\nu}} = 0. \tag{II.12}$$

Recall that the Yang theory [6] was aimed at solving the YM self-duality equations. What we have described here is a merely kinematic part of the whole Yang construction which involves in addition certain dynamical equations for A_{μ}, $A_{\bar{\mu}}$ (and hence for V). We postpone a complete discussion of YM self-duality to Sect.III where this concept will be entirely translated into a "kinematic" language with introducing the analyticity of a new type (the harmonic one).

II.2. Kähler geometry. Another, less trivial manifestation of the principle of preservation of analyticity is Kähler geometry. Consider a 2n-dimensional real Riemannian space parametrized by $x^{\mu}, x^{\bar{\mu}}$ (II.1). The general coordinate transformation (GCT) group is

$$\delta x^{\mu} = \tau^{\mu}(x,\bar{x}), \quad \delta x^{\bar{\mu}} = \tau^{\bar{\mu}}, \quad \tau^{\bar{\mu}} = \overline{(\tau^{\mu})}. \tag{II.13}$$

In addition, one chooses a $U(n)$ tangent space group with parameters

$$\Lambda_{\alpha\bar{\beta}}(x,\bar{x}) = -\Lambda_{\bar{\beta}\alpha} = -\overline{\Lambda_{\beta\bar{\alpha}}} \in U(n). \tag{II.14}$$

Kähler geometry is specified by the following $U(n)$ covariant constraints expressing the requirement that the holonomy group is contained in $U(n)$ (see Introduction):

$$[\mathcal{D}_{\bar{\alpha}}, \mathcal{D}_{\bar{\beta}}] = 0 \quad \text{and c.c.} \tag{II.15}$$

These constraints look identical with the YM ones (II.4) (but now transformations (II.13) involve coordinates themselves). Actually, they admit the same interpretation of integrability conditions for the existence of analytic fields defined by the CR condition

$$\mathcal{D}_{\bar{\alpha}}\varphi = 0. \tag{II.16}$$

This similarity suggests to use the same strategy for solving (II.15). We begin by defining an analytic basis:

$$x^{\mu}_{A} = x^{\mu} + v^{\mu}(x,\bar{x}), \quad x_{A}^{\bar{\mu}} = \overline{(x^{\mu}_{A})} \tag{II.17}$$

in which the GCT group has analytic parameters $\lambda^{\mu}(x_A)$

$$\delta x^{\mu}_{A} = \lambda^{\mu}(x_A). \tag{II.18}$$

The function v^{μ} giving the coordinate change (II.17) is a bridge between the τ and λ bases:

$$\delta v^{\mu}(x,\bar{x}) = \lambda^{\mu}(x_A) - \tau^{\mu}(x,\bar{x}). \tag{II.19}$$

In this case v^{μ} can be gauged away by τ transformations, and the τ group can be identified with the λ one [2].

The purpose of the introduction of the λ basis was to make analyticity manifest. Indeed, the condition $\partial_{\bar{\mu}}\varphi = 0$ (for a scalar field) is now covariant. This suggests that the covariant derivative $\mathcal{D}_{\bar{\alpha}}$ does not contain $\partial/\partial x^{\mu}$:

$$\mathcal{D}_{\bar{\alpha}} = e_{\bar{\alpha}}^{\bar{\mu}}\frac{\partial}{\partial x^{\bar{\mu}}_{A}} + \omega_{\bar{\alpha}}. \tag{II.20}$$

The vielbein $e_{\bar{\alpha}}^{\bar{\mu}}(x,\bar{x})$ and the connection ω_{α} ($U(n)$ algebra valued) transform as follows:

$$\delta e_{\bar{\alpha}}{}^{\bar{\mu}} = \Lambda_{\bar{\alpha}\beta}e_{\bar{\beta}}{}^{\bar{\mu}} + e_{\bar{\alpha}}{}^{\bar{\nu}}\partial_{\bar{\nu}}\lambda^{\bar{\mu}} \tag{II.21}$$

$$\delta\omega_{\bar{\alpha}\beta\bar{\gamma}} = -e_{\bar{\alpha}}{}^{\bar{\mu}}\partial_{\bar{\mu}}\Lambda_{\beta\bar{\gamma}} + \Lambda_{\bar{\alpha}\rho}\omega_{\bar{\rho}\beta\bar{\gamma}} - \Lambda_{\bar{\rho}\beta}\omega_{\bar{\alpha}\rho\bar{\gamma}} + \Lambda_{\bar{\gamma}\rho}\omega_{\bar{\alpha}\beta\bar{\rho}}.$$

Note the absence of vielbeins $e_{\bar{\alpha}}{}^{\mu}$ in (II.20) which is a consequence of the analyticity of the λ group. The connections ω_{α} are determined by the usual Riemannian torsion constraint

$$T_{\bar{\alpha}\beta\bar{\gamma}} = 0 \rightarrow \omega_{\bar{\alpha}\beta\bar{\gamma}} = -e_{\bar{\alpha}}{}^{\bar{\mu}}\partial_{\bar{\mu}}e^{\nu}_{\beta}\cdot e_{\nu\bar{\gamma}}. \tag{II.22}$$

Finally, one has to plug the conjecture (II.20) into the defining constraint (II.15). The only independent part of it is

$$T_{\bar{\alpha}\bar{\beta}\gamma} = 0 \rightarrow \partial_{[\bar{\mu}}\, g_{\bar{\nu}]\lambda} = 0, \tag{II.23}$$

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[2] Note that this is due to the accidental fact that in the present case the λ group is a subgroup of the τ group. In all the other examples that we consider the bridge carries non-trivial degrees of freedom.

where the metric $g_{\bar{\nu}\lambda}$ is constructed from the vielbeins,

$$g_{\bar{\nu}\lambda} = e_{\bar{\nu}\alpha} e_{\lambda\bar{\alpha}} . \qquad (II.24)$$

Equation (II.23) is easily recognized as the well-known Kählerian condition on the metric. It has the following general solution:

$$g_{\bar{\nu}\lambda} = \partial_{\bar{\nu}} \partial_\lambda K , \qquad (II.25)$$

where arbitrary function $K = K(x, \bar{x})$ is nothing else than the unconstrained prepotential of Kähler geometry. Obviously, (II.25) is invariant under the pregauge transformations

$$\delta K = -\frac{i}{2}\left(\lambda(x_A) - \bar{\lambda}(\bar{x}_A)\right) \qquad (II.26)$$

with an arbitrary analytic parameter $\lambda(x_A)$.

Thus we have succeeded in solving the constraints (II.15). The concept of the analyticity preservation proved crucial once again. The analogy with the Yang theory [6] reveals itself also in the existence of manifestly analytic and anti-analytic world frames in the present case. Converting, e.g., any index $\bar{\alpha}$ with e^{μ}_{α} and β with $e_{\mu\bar{\beta}}$, one passes to the frame where the covariant derivative with respect to $x^{\bar{\mu}}_A$ has no connection

$$(\mathcal{D}_{\bar{\alpha}})_{\beta\bar{\gamma}} \rightarrow e^{\nu}_{\alpha} e_{\rho\bar{\beta}} (\mathcal{D}_{\bar{\alpha}})_{\beta\bar{\gamma}} e^{\mu}_{\gamma} = \delta^{\mu}_{\rho} g^{\nu\bar{\nu}} \frac{\partial}{\partial x^{\bar{\nu}}_A} . \qquad (II.27)$$

In this frame functions with arbitrary indices can be made manifestly analytic. Instead of the tangent group $U(n)$ with real parameters one has now transformations induced by the world ones with parameters $\partial_\mu \lambda^\nu(x_A)$.

Comparing Kähler geometry with the Yang theory of Sect.II.I one can say that the analogs of the bridges e^{iW}, $e^{i\bar{W}}$ are the vielbeins e^{μ}_{α} , $e^{\bar{\mu}}_{\bar{\alpha}}$. At the same time the Kähler potential $K(x,\bar{x})$ and its pregauge group are new concepts without analogs in the Yang theory. They have peculiar dimensions ($[g]=0 \rightarrow [K]=[\lambda]$ = = length^2), and thus do not naturally fit into the customary field geometric framework.

II.3. Central charge as the origin of the Kähler prepotential. In order to incorporate K as a geometric object, we shall extend R^{2n} by adding a new real coordinate $z = \bar{z}$ with dimension length^2 :

$$R^{2n,1} = \{(x^\mu, x^{\bar{\mu}}, z)\} .$$

In this space one can realize the following "central charge" extention of the Poincaré algebra:

$$[P_\mu, P_{\bar{\nu}}] = 2\,\delta_{\mu\bar{\nu}} Z \qquad (II.28)$$

$$[P_\mu, Z] = [P_{\bar{\mu}}, Z] = [P_\mu, P_\nu] = [P_{\bar{\mu}}, P_{\bar{\nu}}] = 0 .$$

This algebra still has $U(n)$ as its automorphism group, with central charge Z being a singlet. The transformations realizing (II.28) in $R^{2n,1}$ are

$$\delta x^\mu = a^\mu ,\ \delta x^{\bar{\mu}} = a^{\bar{\mu}} ,\ \delta z = a + i(a^{\bar{\mu}} x^\mu - a^\mu x^{\bar{\mu}}) . \qquad (II.28')$$

The covariant derivatives in $R^{2n,1}$(they commute with P , $\bar{P}$ and Z) have the following form:

$$D_\mu = \partial_\mu + i\, x^{\bar{\mu}} \partial_z \ ,\ D_{\bar{\mu}} = \partial_{\bar{\mu}} - i\, x^\mu \partial_z \ ,\ D_z = \partial_z$$

and satisfy the algebra

$$[D_\mu, D_{\bar{\nu}}] = -2i\, \delta_{\mu\bar{\nu}} D_z \qquad (II.29)$$

$$[D_\mu, D_z] = [D_{\bar{\mu}}, D_z] = [D_\mu, D_\nu] = [D_{\bar{\mu}}, D_{\bar{\nu}}] = 0 .$$

The crucial observation is that the algebra (II.29) is still consistent with the analyticity (II.2). This becomes obvious in a special analytic basis in $R^{2n,1}$:

$$x^\mu_A = x^\mu ,\ x^{\bar{\mu}}_A = x^{\bar{\mu}} ,\ z_A = z + i\, x^\mu x^{\bar{\mu}} \qquad (II.30)$$

$$\delta x^\mu_A = a^\mu ,\ \delta x^{\bar{\mu}}_A = a^{\bar{\mu}} ,\ \delta z_A = a + 2i\, a^{\bar{\mu}} x^\mu_A .$$

In other words, the subspace (x^μ_A, z_A) is invariant, and one can define analytic functions $\varphi(x_A, z_A)$. Of course, the ordinary z -independent analytic functions $\varphi(x_A)$ are still allowed.

The next step is to generalize the above rigid framework to the curved case. Since the new coordinate z is auxiliary and the geometric objects of R^{2n} do not depend on it, we choose the GCT group in $R^{2n,1}$ to be z independent too. In particular, z undergoes τ transformations $\delta z = \tau(x, \bar{x})$, $\tau = \bar{\tau}$.

To make analyticity manifest we need a new basis in which an analytic λ group acts. In addition to (II.17-19) we change z :

$$z_A = z + v(x, \bar{x}) \qquad (II.31)$$

$$\delta z_A = \lambda(x_A) ,\ \ \delta v = \lambda(x_A) - \tau(x, \bar{x}) .$$

Unlike $v^{\mu}(x,\bar{x})$ (II.19), the bridge v cannot be gauged away completely (v is complex, while τ is real). A possible gauge is (cf. the Yang theory of subsection II.1):

$$\mathrm{Re}\, v = 0 \rightarrow \tau(x,\bar{x}) = \frac{1}{2}\left(\lambda(x_A) + \bar{\lambda}(\bar{x}_A)\right). \qquad \text{(II.32)}$$

The remaining part of v is

$$K = \mathrm{Im}\, v\,, \quad \delta K = -\frac{i}{2}\left(\lambda(x_A) - \bar{\lambda}(\bar{x}_A)\right). \qquad \text{(II.33)}$$

Note that this is precisely the transformation law of the Kähler prepotential (II.26).

To reveal the relation between K and the metric we proceed to the construction of covariant derivatives. We shall show that the scheme developed above provides the solution of the following generalization of the R^{2n} constraints (II.15):

$$\begin{aligned} &[\mathcal{D}_{\bar{\alpha}}, \mathcal{D}_{\bar{\beta}}] = 0 && \text{(a)} \\ &[\mathcal{D}_{\alpha}, \mathcal{D}_{\bar{\beta}}] = -2i\,\delta_{\alpha\bar{\beta}}\mathcal{D}_z + R_{\alpha\bar{\beta}} && \text{(b)} \\ &[\mathcal{D}_{\alpha}, \mathcal{D}_z] = [\mathcal{D}_{\bar{\alpha}}, \mathcal{D}_z] = 0\,. && \text{(c)} \end{aligned} \qquad \text{(II.34)}$$

The new torsion term in (II.34b) is prompted by the rigid space algebra (II.29). In order to keep as close as possible to the original Riemann geometry in R^{2n} one is led to choose

$$\mathcal{D}_z = \frac{\partial}{\partial z}\,. \qquad \text{(II.35)}$$

This option is allowed by the gauge group though is not obligatory. The point is that any other choice would give rise to new tensors which were not present originally. The analytic basis form of $\mathcal{D}_{\bar{\alpha}}$ (II.20) remains unchanged since the gauge group does not require a $\frac{\partial}{\partial z_A}$ term. Changing back to the τ basis we obtain (in the gauge (II.32)):

$$\mathcal{D}_{\bar{\alpha}} = e_{\bar{\alpha}}^{\ \bar{\mu}}\left(\partial_{\bar{\mu}} - i\,\partial_{\bar{\mu}}K\frac{\partial}{\partial z}\right) + \omega_{\bar{\alpha}}\,. \qquad \text{(II.36)}$$

In this basis $\mathcal{D}_{\alpha}$ is the conjugate of $\mathcal{D}_{\bar{\alpha}}$. Putting all this into (II.34b) we derive the old expression (II.22) for the connection $\omega_{\bar{\alpha}}$. Further, the relation

$$T^{z}_{\alpha\bar{\beta}} = -2i\,\delta_{\alpha\bar{\beta}} \qquad \text{(II.37)}$$

is not a constraint any more, it becomes the definition of the metric $g_{\mu\bar{\nu}}$ in terms of K exactly as in (II.25). The constraint (II.34a) is now a consequence of the Bianchi identities.

In conclusion we can say that the extension of R^{2n} to $R^{2n,1}$ allowed us to find a formulation of Kähler geometry in which the prepotential arises as an object with a clear geometric meaning. It comes out as the imaginary part of the bridge from the original τ basis to the analytic λ basis (involving z_A). The Kähler invariance is identified with the analytic GCT of the extra central charge coordinate. This is to be compared with the standard formulation of Kähler geometry in R^{2n} (see sect. II.2), where the prepotential and its invariance emerge as solutions of the constraint on the metric. In accompanying paper [15] we shall extend this approach to include the case of hyper-Kähler geometry. The structure of the latter becomes much clearer after introducing an SU(2) triplet of central charges.

We would like to note that the central charge coordinate was regarded here as purely auxiliary, and none of the fields and parameters depended on it. However, the constraints do not rule out that some analytic matter fields with non-vanishing central charge exist. It is interesting to find out the relevance of such representations.

II.4. <u>Analogies with N=1 supersymmetry.</u> The two examples of gauge theories based on analyticity in x-space considered above bear profound analogies with N=1 supersymmetric theories in four dimensions. Those theories are formulated in superspace with even coordinates x^m and Grassmann odd coordinates θ^{α}, $\bar{\theta}^{\dot{\alpha}} = \overline{(\theta^{\alpha})}$. The algebra of the rigid covariant derivatives D_{α} , $\bar{D}_{\dot{\alpha}}$ and D_m resembles (II.34)

$$\{D_{\alpha}, D_{\beta}\} = \{\bar{D}_{\dot{\alpha}}, \bar{D}_{\dot{\beta}}\} = [D_{\alpha}, D_m] = [\bar{D}_{\dot{\alpha}}, D_m] = 0 \qquad \text{(II.38)}$$

$$\{D_{\alpha}, \bar{D}_{\dot{\beta}}\} = 2i\,\sigma^{m}_{\alpha\dot{\beta}}D_m\,.$$

Clearly, it allows one to define Grassmann analytic superfields satisfying the constraint

$$\bar{D}_{\dot{\alpha}}\,\phi(x,\theta,\bar{\theta}) = 0\,. \qquad \text{(II.39)}$$

It is solved in a special left-handed chiral basis in superspace:

$$\{(x^m_L = x^m + i\theta\sigma^m\bar{\theta}, \theta^{\alpha});\ \bar{\theta}^{\dot{\alpha}}\}\,, \qquad \text{(II.40)}$$

where $\bar{D}_{\dot{\alpha}} = \partial/\partial\bar{\theta}^{\dot{\alpha}}$ and (II.39) means that $\phi = \phi(x_L, \theta)$. Note the close similarity of this picture to the one of subsection II.3 with ($x^{\mu}, x^{\bar{\mu}}, z$) replaced by ($\theta^{\alpha}, \bar{\theta}^{\dot{\alpha}}, x^m$).

The Grassmann analytic representations of supersymmetry (II.39) are fundamental objects. N=1 matter is described by a set of scalar chiral superfields $\Phi(x_L, \theta)$ (and their conjugates $\overline{\Phi}(x_R, \overline{\theta})$, $x_R = \overline{x_L}$). Their general self-coupling is given by the action

$$S = \int d^4x\, d^2\theta\, d^2\overline{\theta}\; K(\Phi, \overline{\Phi}) + \int d^4x_L\, d^2\theta\, P(\Phi) + \int d^4x_R\, d^2\overline{\theta}\; \overline{P}(\overline{\Phi}). \qquad \text{(II.41)}$$

where K is an arbitrary real function of Φ and $\overline{\Phi}$, and P is an analytic function of Φ. Note that the first term in (II.41) is invariant under the replacement $K \to K - \frac{i}{2}(\Lambda(\Phi) - \overline{\Lambda}(\overline{\Phi}))$ since $\int d^4x\, d^2\theta d^2\overline{\theta}\, \Lambda(\Phi(x_L, \theta)) = 0$. This resemblance of Kähler geometry is not accidental. Indeed, as shown by Zumino [17] the action (II.41) contains the sigma model action

$$S = \int d^4x\; g_{\mu\bar\nu}(\varphi, \overline{\varphi}) \partial_m \varphi^\mu \partial^m \varphi^{\bar\nu},$$

where

$$\varphi^\mu(x) = \Phi^\mu(x_L, \theta)\big|_{\theta=0}, \quad \varphi^{\bar\mu} = \overline{(\varphi^\mu)}$$

and the metric is

$$g_{\mu\bar\nu}(\varphi, \overline{\varphi}) = \frac{\partial^2 K}{\partial \Phi^\mu \partial \Phi^{\bar\nu}} \bigg|_{\theta=\overline{\theta}=0} \qquad \text{(II.42)}$$

(cf. (II.25)). Thus one observes the one-to-one correspondence between N=1 supersymmetric sigma models and Kähler geometry. The crucial point is that the fundamental characteristic of Kähler geometry, the unconstrained prepotential K comes out as the leading part of the general unconstrained superfield Lagrangian of N=1 matter. In [15] we shall extend this correspondence to the case of N=2 sigma models and hyper-Kähler geometry.

N=1 supersymmetric YM theory is also based upon Grassmann analyticity. The covariantization of (II.39) leads to the integrability condition

$$\{\overline{\mathcal{D}}_{\dot\alpha}, \overline{\mathcal{D}}_{\dot\beta}\} = 0. \qquad \text{(II.43)}$$

It looks almost identical with the constraint (II.4) of Yang theory, and can be solved in exactly the same way. Once again, the prepotential is the bridge between the real YM group and its chiral complexification [19].

The preservation of chirality is the leading principle in N=1 SG as well [18]. There one repeats the main steps of the formulation

of Kähler geometry in subsection II.3, the space-time coordinate x^m being now an analog of Kähler central charge coordinate z. The non-trivial bridge connects x^m_L and x^m $(x^m_L = x^m + iH^m(x, \theta, \overline{\theta})$;

$$\delta x^m_L = \lambda^m(x_L, \theta), \quad \delta\theta^\alpha = \lambda^\alpha(x_L, \theta);$$

$$\delta H^m = -\frac{i}{2}(\lambda^m - \overline{\lambda}^m)\ ).$$

Once again, the vielbeins and connections are expressed in terms of H^m and thus the constraints of the theory are solved.

The conclusion is that all the fundamental objects of N=2 supersymmetry have direct analogs in the gauge theories with intrinsic complex analyticity in R^{2n} (or $R^{2n,1}$). We shall see (Sect. III and the accompanying paper [15]) that this close relationship persists in the case of N=2 supersymmetry on the one hand, and self-dual YM and gravity (and their higher dimensional generalizations in R^{4n}) on the other hand. (There, the underlying analyticity is the harmonic analyticity).

III. Harmonic analyticity and Yang-Mills theory

In this section we shall consider an example of a gauge theory based on a new principle of analyticity, which is intimately related to the harmonic variables $u^{\pm i}$ parametrizing the sphere S^2. This is the YM theory in the Euclidean space R^{4n} with a constraint generalizing the self-duality condition in R^4 (and coinciding with the latter for n=1).

We begin by choosing the group Sp(n) x SU(2)$\subset$O(4n) as the "Lorentz" group of R^{4n}. Then the coordinates of R^{4n} can be naturally denoted by $x^{\mu i}$ where μ is an Sp(n) spinor index ($\mu = 1, \ldots, 2n$) and i is an SU(2) spinor index (i=1,2). The reality condition on $x^{\mu i}$ is $\overline{x^{\mu i}} = \Omega_{\mu\nu} \varepsilon_{ij} x^{\nu j}$ where $\Omega_{\mu\nu}$ and ε_{ij} are the antisymmetric invariant tensors of Sp(n) and SU(2) respectively ($\Omega^{\mu\nu}\Omega_{\nu\rho} = \delta^\mu_\rho$, $\varepsilon^{ij}\varepsilon_{jk} = \delta^i_k$). Next we consider matter fields $\varphi(x)$ transforming under a YM group with parameters $\tau(x)$,

$$\varphi' = e^{i\tau}\varphi. \qquad \text{(III.1)}$$

The covariant YM derivative is

$$\mathcal{D}_{\mu i} = \partial_{\mu i} + iA_{\mu i}(x), \quad A'_{\mu i} = \frac{1}{i} e^{i\tau}(\partial_{\mu i} + iA_{\mu i})e^{-i\tau}. \qquad \text{(III.2)}$$

The purpose of the reduction from O(4n) to Sp(n) x SU(2) is to be able to impose the following covariant constraint on the YM potentials $A_{\mu i}$:

$$[\mathcal{D}_{\mu i}, \mathcal{D}_{\nu j}] = i\,\varepsilon_{ij}\, F_{(\mu\nu)} . \qquad (III.3)$$

To explain the meaning of eq. (III.3) we note that the general covariant strength $[\mathcal{D}_{\mu i}, \mathcal{D}_{\nu j}] = i\, F_{\mu i \nu j}$ can always be decomposed as

$$[\mathcal{D}_{\mu i}, \mathcal{D}_{\nu j}] = i\,\varepsilon_{ij}\, F_{(\mu\nu)} + i\, F_{(ij)[\mu,\nu]} .$$

Then (III.3) is equivalent to requiring $F_{(ij)[\mu,\nu]}$ to vanish:

$$[\mathcal{D}_{\mu(i}, \mathcal{D}_{\nu j)}] = 0 . \qquad (III.3')$$

In the case n=1, $O(4) \sim$ Sp(1) x SU(2) and $F_{(ij)[\mu,\nu]} \equiv \varepsilon_{\mu\nu} F_{(ij)}$. So (III.3) becomes the familiar self-duality condition

$$n=1: \qquad F_{(ij)} = 0 .$$

Note that (III.3) and the Bianchi identity, like in the conventional case n=1, imply the equation of motion $\mathcal{D}^{\mu i} F_{(\mu\nu)} = 0$, $\mathcal{D}^{\mu i} \equiv \Omega^{\mu\nu}\varepsilon^{ij}\mathcal{D}_{\nu j}$.

Now the problem is to solve the constraint (III.3). Unlike the Yang theory of subsection II.1, this constraint is not immediately recognized to be an integrability condition. However, as we shall see, it can be equivalently rewritten in an extended space, where it does become an integrability condition.

III.1. Harmonic space and harmonic analyticity. Along the lines of ref. [8-12,21] , the space R^{4n} , where the problem was formulated, can be regarded as the quotient:

$$\{x^{\mu i}\} \sim \mathcal{P}/Sp(n)\times SU(2) . \qquad (III.4)$$

where $\mathcal{P}$ is the Poincaré group with Sp(n) x SU(2) as its Lorentz subgroup. Now we consider the quotient

$$\{x^{\mu i}, u^{\pm i}\} \sim \mathcal{P}/Sp(n) . \qquad (III.5)$$

Here the coordinates $u^{\pm i}$ parameterize SU(2) [8]

$$u^{+i}u^-_i = 1 , \quad \overline{(u^{+i})} = u^-_i . \qquad (III.6)$$

The functions of $u^{\pm i}$ are defined as harmonic expansions in terms of the irreducible products of $u^{\pm i}$ so we call $u^{\pm i}$ harmonic variables [8-12,21].

The introduction of the new variables $u^{\pm i}$ will help us to reformulate the old problem (III.3) in a new, more transparent way . To this end we multiple (III.3) by u^{+i}, u^{+j}

$$[\mathcal{D}^+_\mu, \mathcal{D}^+_\nu] = 0 , \quad \mathcal{D}^+_\mu \equiv u^{+i}(\partial_{\mu i} + iA_{\mu i}(x)) . \qquad (III.7)$$

Doing this we do not lose any solutions of (III.3). Indeed, u^{+i}, u^{+j} in (III.7) are arbitrary variables, so (III.7) simply means that the part of the full field strength symmetric in its SU(2) indices i,j vanishes, and we get just eq. (III.3') which is the same as (III.3) . Remarkably, in the form (III.7) our constraint is nothing but the integrability condition for the existence of analytic fields satisfying the covariant CR condition

$$\mathcal{D}^+_\mu \varphi(x,u) = 0 . \qquad (III.8)$$

In the rigid case the analyticity condition

$$\partial^+_\mu \varphi = 0 \qquad (III.9)$$

simply states that $\varphi = \varphi(x^+, u)$, where $x^{\mu +} \equiv u^+_i x^{\mu i}$. This property is natural to call harmonic analyticity, because its definition essentially involves the harmonic variables (like the complex analyticity which is associated with the imaginary unit i).

Note that the analytic harmonic space

$$\{x^{\mu +}, u^{\pm i}\} \qquad (III.10)$$

is closed under the full Poincaré group. This space is not real in the sense of ordinary complex conjugation ($\overline{x^+} = x^-$, see (III.6)). However, one can define another conjugation $\sim$ (in [8-12,21] conjugation of such a kind was denoted by $*$)

$$\widetilde{x^{\mu i}} = \overline{x^{\mu i}} , \quad \widetilde{u^\pm_i} = u^{\pm i} , \quad \widetilde{u^{\pm i}} = -u^\pm_i . \qquad (III.11)$$

which leaves (III.10) invariant. This will allow us to define real analytic functions in what follows.

The covariantization (III.8) of the new kind of analyticity (III.9) gives the clear meaning to the constraint (III.7) and suggests the way to solve it, much like the analyticity (II.3) which underlies the Yang theory. Moreover, the same concept will turn out to be in the basis of hyper-Kähler geometry too (see accompanying paper [15]).

III.2. Harmonic derivatives and analytic prepotential. The general solution of the constraint (III.7) is

$$A^+_\mu(x,u) = -i\, e^{-iv(x,u)}\, \partial^+_\mu\, e^{iv(x,u)} , \qquad (III.12)$$

where $v(x,u)$ is an arbitrary (for the time being) real scalar field. The reality of v , $\widetilde{v} = v$, follows from the property $\overline{A_{\mu i}}(x) = -A^{\mu i}(x)$ and (III.11). However, the original connection A^+_μ in (III.7) is a linear function of u^+, $A^+_\mu = u^{+i}A_{\mu i}(x)$.

This is not automatically so in (III.12), therefore we have to impose certain restrictions on $v(x,u)$. To this end we consider the derivatives with respect to the harmonic variables. There are three operators consistent with the SU(2) defining conditions (III.6):

$$D^{++}=u^{+i}\frac{\partial}{\partial u^{-i}}\,,\quad D^{--}=u^{-i}\frac{\partial}{\partial u^{+i}}\,,\quad D^{0}=u^{+i}\frac{\partial}{\partial u^{+i}}-u^{-i}\frac{\partial}{\partial u^{-i}}\,. \qquad \text{(III.13)}$$

They satisfy the SU(2) algebra

$$[D^{++},D^{--}]=D^{0},\quad [D^{0},D^{++}]=2D^{++},\quad [D^{0},D^{--}]=-2D^{--}. \qquad \text{(III.14)}$$

Since the gauge parameters $\tau(x)$ (III.1) do not depend on $u^{\pm}$, one needs not to covariantize the harmonic derivatives (III.13). Further, the original derivative (III.7) obviously satisfies the relations

$$[D^{0},\mathcal{D}^{+}_{\mu}]=\mathcal{D}^{+}_{\mu} \qquad \text{(a)}$$
$$[D^{++},\mathcal{D}^{+}_{\mu}]=0\,. \qquad \text{(b)} \qquad \text{(III.15)}$$

Moreover, if (III.15) holds, one can show that the connection A^{+}_{μ} is linear in u^{+i}. First of all, (III.15a) yields

$$D^{0}A^{+}_{\mu}(x,u)=A^{+}_{\mu}(x,u) \qquad \text{(III.16)}$$

which means that A^{+}_{μ} is an eigenfunction of D^{0}. Then, taking account of the explicit form of D^{0} the harmonic expansion of A^{+}_{μ} is as follows

$$A^{+}_{\mu}(x,u)=A_{\mu i}(x)u^{+i}+A_{\mu(ijk)}(x)u^{+i}u^{+j}u^{-k}+\dots \qquad \text{(III.17)}$$

It is important, that the number of u^{+} exceeds that of u^{-} by 1 in each term. In other words, A^{+}_{μ} is a homogeneous function of $u^{\pm}$ of degree +1 :

$$A^{+}_{\mu}(x,e^{i\alpha}u^{+},e^{-i\alpha}u^{-})=e^{i\alpha}A^{+}_{\mu}(x,u)\,.$$

This means that A^{+}_{μ} is actually defined on the quotient $SU(2)/U(1)\sim S^{2}$. In what follows we shall deal only with such functions on S^{2} (which can be viewed as functions on $S^{3}\sim SU(2)$ but having definite U(I) charge).

Given the expansion (III.17), one immediately sees that the condition (III.15b) ($D^{++}A^{+}_{\mu}=0$) leaves only the first term

$A_{\mu i}(x)u^{+i}$. So, we have recovered the connection $A_{\mu i}(x)$. The conclusion is that the full set of constraints (III.7),(III.15) is equivalent to the original constraint (III.3).

Now we have to study the consequences of the constraint (III.7). For this purpose we shall define a new gauge frame where the gauge parameters are the analytic ones $\lambda(x^{+},u)$ (compare with the Yang theory, (II.5-9)). The bridge between the τ and λ frames is the quantity $v(x,u)$ (III.12) which transforms as follows

$$e^{iv'}=e^{i\lambda}e^{iv}e^{-i\tau}. \qquad \text{(III.18)}$$

Here λ is a real ($\lambda=\tilde{\lambda}$) analytic parameter related to the freedom of the choice of e^{iv} in (III.12). In the new frame all the matter fields are redefined: $\phi=e^{iv}\varphi$, $\phi'=e^{i\lambda}\phi$. Correspondingly, the covariant derivative $\mathcal{D}^{+}_{\mu}$ becomes simply ∂^{+}_{μ}

$$\mathcal{D}^{+}_{\mu}=\partial^{+}_{\mu} \qquad \text{(III.19)}$$

so the analyticity (III.8) is manifest. On the contrary, the harmonic derivative D^{++} acquires a connection term (the parameters $\lambda(x^{+},u)$ depend on $u^{\pm}$ unlike $\tau(x)$):

$$\mathcal{D}^{++}=D^{++}+iV^{++}$$
$$V^{++\prime}=-ie^{i\lambda}(D^{++}+iV^{++})e^{-i\lambda}, \qquad \text{(III.20)}$$

where

$$V^{++}=-ie^{iv}D^{++}e^{-iv}. \qquad \text{(III.21)}$$

Putting this into (III.15b) one proves analyticity of V^{++}

$$\partial^{+}_{\mu}V^{++}=0\ \rightarrow\ V^{++}=V^{++}(x^{+},u)\,. \qquad \text{(III.22)}$$

In what follows we shall use a natural gauge where in the λ basis the connection V^{0} of covariant derivative $\mathcal{D}^{0}$ vanishes and both the bridge v and the parameters λ have zero U(I) charge

$$V^{0}=-ie^{iv}D^{0}e^{-iv}\rightarrow 0\,,\quad D^{0}v=D^{0}\lambda=0\,. \qquad \text{(III.23)}$$

In this gauge all the quantities in the theory have a definite U(I) charge, e.g. $D^{0}V^{++}=2V^{++}$, etc.

So far we have assumed that the bridge v was given, and everything else was constructed from it. However, unlike the Yang theory, the bridge is a constrained object, since V^{++} (III.21) must be analytic. In fact, the genuine prepotential of the theory is the unconstrained connection V^{++}. The relation (III.21) should be considered as an equation for v in terms of V^{++}, rather than as a definition for V^{++} (as in the N=2 YM theory [8,9])

$$(D^{++}+iV^{++})e^{iv}=0. \qquad (III.21')$$

If one is able to solve this equation, one can construct the full solution of the constraints (III.3) by going back to the original τ frame. In principle, it is always possible to solve (III.21). The way to find a pertrubative solution was discussed in [9]. However, this is in general a very difficult task. Equation (III.21) is highly non-linear.[3] Note also that its solutions are defined up to τ gauge freedom.

Here we are going to follow a different line advocated by B.M.Zupnik [16]. After introducing the prepotential V^{++} as the λ frame connection for $\mathcal{D}^{++}$, we will not go back to the τ frame. Staying in the λ frame we will be able to construct all the other connections and tensors directly in terms of V^{++}, without referring to the existence of the bridge v.

III.3. Analytic frame geometry. To complete the λ frame formalism we need expressions for $\mathcal{D}^{--}$ and $\mathcal{D}^{-}_{\mu}$. The connection V^{--} for $\mathcal{D}^{--}$ can be found from the constraint

$$[\mathcal{D}^{++},\mathcal{D}^{--}]=D^{0}. \qquad (III.24)$$

In the τ frame it is automatically satisfied (see (III.13)), but in the λ frame it reads:

$$D^{++}V^{--}-D^{--}V^{++}+i[V^{++},V^{--}]=0. \qquad (III.25)$$

Once again, this is a differential equation for V^{--} in terms of V^{++}. However, unlike the bridge equation (III.21), this one is linear in V^{--}. Moreover, it has a unique solution (because the equation $D^{++}V^{--}=0$ only has the trivial solution $V^{--}=0$). The perturbative solution of (II.25) was found by B.M.Zupnik [16]:

$$V^{--}(x,u)=\sum_{n=1}^{\infty} i^n\int du_1\ldots du_n \frac{V^{++}(1)V^{++}(2)\ldots V^{++}(n)}{(u^+u_1^+)(u_1^+u_2^+)\cdots(u_n^+u^+)}. \qquad (III.26)$$

---

[3] With respect to v itself. Though eq. (II.21') is linear with respect to $\exp\{iv\}$, its solution is not automatically of the exponential form and one should worry about this by imposing the nonlinear unitarity constraint.

Here $V^{++}(1)=V^{++}(x^{\mu i}u^+_{1i}, u^{\pm}_1)$, etc.; the harmonic distributions $(u^+u_1^+)^{-1}$, etc., are defined in [9] and obey the following relations:

$$D^{++}\frac{1}{(u^+u_1^+)}=\delta^{(1,-1)}(u,u_1)$$
$$D^{++}\frac{1}{(u^+u_1^+)^2}=D^{--}\delta^{(2,-2)}(u,u_1). \qquad (III.27)$$

Of course, it is difficult to judge the global properties of such a perturbative solution (III.25). Instead, one can try to solve the linear differential equation (III.25) non-perturbatively (see the example at the end of this subsection).

Finally, connection A^-_{μ} of the covariant derivative $\mathcal{D}^-_{\mu}$ is defined by the relation

$$\mathcal{D}^-_{\mu}=[\mathcal{D}^{--},\mathcal{D}^+_{\mu}] \rightarrow A^-_{\mu}=-\partial^+_{\mu}V^{--}. \qquad (III.28)$$

Then the field strength $F_{\mu\nu}$ (III.3) is

$$F_{\mu\nu}=-i[\mathcal{D}^+_{\mu},\mathcal{D}^-_{\nu}]=-\partial^+_{\mu}\partial^+_{\nu}V^{--}. \qquad (III.29)$$

The field strength in the τ frame does not depend on the harmonic variables. In the λ frame this independence becomes covariant

$$\mathcal{D}^{++}F_{\mu\nu}=0 \qquad (III.30)$$

which can be checked using (III.29),(III.25),(III.22). However, the Lagrangian density $\mathcal{L}=\mathrm{tr}\,(F_{\mu\nu})^2$ is manifestly $u^{\pm}$-independent. The reason is that the λ and τ frame field strengths are related by $F^{\lambda}=e^{iv}F^{\tau}e^{-iv}$, so $\mathrm{tr}\,(F^{\lambda})^2=\mathrm{tr}\,(F^{\tau})^2$. The conclusion is that one can indeed work entirely in the λ basis thus avoiding the necessity to solve the non-linear bridge equation (III.21). This especially concerns the calculation of the topological invariant $S=\int d^{4n}x\,\mathrm{tr}\,(F_{\mu\nu})^2$, which is frame independent.

We illustrate the above approach on the very simple example of the one-instanton solution in R^4 for an SU(2) YM group [21]. We choose the prepotential $(V^{++})_i{}^j$ in the form

$$(V^{++})_i{}^j=-\frac{i}{\rho^2}x^+_i x^{+j}, \quad \rho=\mathrm{const}\ ([\rho]=cm^1) \qquad (III.31)$$

The solution of (III.25) is easily found

$$(V^{--})_i{}^j=\frac{i\,x^-_i x^{-j}}{x^2+\rho^2}, \quad x^2=x^{+i}x^-_i. \qquad (III.32)$$

Finally, from (III.29) one gets:

$$\mathrm{tr}\, F_{\mu\nu}^2 \propto \frac{\rho^4}{(x^2+\rho^2)^4} \tag{III.33}$$

which coincides with the well-known one-instanton result [22].

It would be most interesting to try to find the prepotentials V^{++} which generate the multi-instanton solutions of [23]. This might help to get an insight into the analogous but much more difficult problem of finding all hyper-Kähler metrics.

In conclusion we point out the similarity of the problem discussed in this section with the N=2 supersymmetric YM theory. There one has the constraints [24]

$$\{\mathcal{D}^i_\alpha, \mathcal{D}^j_\beta\} = \varepsilon^{ij}\varepsilon_{\alpha\beta} W$$
$$\{\mathcal{D}^i_\alpha, \bar{\mathcal{D}}_{\dot\beta j}\} = \delta^i_j \mathcal{D}_{\alpha\dot\beta}$$

which look very much like (III.3). Actually, they can be solved in precisely the same way [8,9]. The crucial concept now is that of harmonic Grassmann analytic superfields $\Phi(x,\theta,\bar\theta,u)$

$$\mathcal{D}^+_\alpha \Phi = \bar{\mathcal{D}}^+_{\dot\alpha}\Phi = 0\,, \quad \mathcal{D}^+_{\alpha,\dot\alpha} = u^+_i \mathcal{D}^i_{\alpha,\dot\alpha}\,. \tag{III.34}$$

The unconstrained prepotential of N=2 SYM is once again the analytic connection V^{++} of the covariant derivative $\mathcal{D}^{++}$ in the analytic λ frame.

III.4. Relationship between harmonic and twistor space approaches. As mentioned in the Introduction, there is a close relationship between our approach to the self-dual YM equations (and their higher dimensional generalization) and the construction of Ward [4] (which is an application of Penrose's twistor program [1] to the YM case). Common for both approaches is the interpretation of the self-dual YM equations as integrability conditions obtained with the help of additional variables related to the sphere S^2. We describe S^2 globally by considering functions defined on $SU(2)=\{u^\pm_i\}$, and requiring them to possess a definite U(I) charge. Thus we avoid using an explicit parametrization of S^2. Another advantage is the manifest covariance with respect to SU(2) acting on the indices $i, j, \ldots$. It allows us to easily control the patterns of SU(2) breaking.

In the twistor approach one considers functions of two complex variables π^i (i=1,2), which are analytic (i.e. do not depend on $\bar\pi^i$) and homogeneous, so they effectively depend on, e.g., $\xi = \pi^1/\pi^2$. This clearly provides a parametrization of only one part of S^2 (leaving out the point $\pi^2=0$). In our language this would correspond to using only the single complex variable $\xi = u^{+1}/u^{+2}$ (but not u^-) and replacing the U(I) charge by a degree of homogeneity in u^{+i}. Consequently, we would lose the possibility to use the derivative $D^{++} = u^{+i}\frac{\partial}{\partial u^{-i}} \sim \partial/\partial\bar\xi$. Therefore the way in which we introduce the prepotential $V^{++}(x^+, u^+, u^-)$ as the connection for $\mathcal{D}^{++}$, and the basic equation (III.21) determining the bridge $v(x^\pm, u^\pm)$ in terms of V^{++} are not directly applicable in the twistor approach. Instead, there one does the following. The self-duality equations imply that the connection $A^+_\mu = A_{\mu i}\pi^i$ can be represented in the "pure gauge" form

$$A^+_\mu = H^{-1}\partial^+_\mu H\,. \tag{III.35}$$

The condition that A^+_μ depends on π^i just linearly (which in our language has the form of a differential constraint, $D^{++}A^+_\mu = 0$) is now formulated as the condition that A^+_μ is regular in π (taking into account the homogeneity in π) on the whole sphere S^2. Such a restriction imposed on H would be too strong (H would become a gauge transformation and A^+_μ would be empty). Therefore it is weakened by requiring that H should be analytic (regular) only on one part Ω_1 of S^2 (the approximate equivalent in our language is that we do not require $D^{++}v(x,u)=0$). However, the presentation (III.35) is not unique, since one can consider another function $\hat H$ giving the same A^+_μ

$$A^+_\mu = \hat H^{-1}\partial^+_\mu \hat H \tag{III.36}$$

but analytic on another region $\Omega_2 \subset S^2$, $\Omega_1 \cup \Omega_2 = S^2$. Comparing (III.35) and (III.36) one concludes that

$$G = \hat H H^{-1}\,, \quad \partial^+_\mu G = 0\,, \tag{III.37}$$

i.e., G is analytic in π on $\Omega_1 \cap \Omega_2$ and depends on $x^{\mu +} = x^{\mu i}\pi_i$ only, $G = G(x^+, \pi)$. The argument can now be reversed: given a prepotential $G(x^+,\pi)$ one may try to perform the "splitting" (III.37) into H and $\hat H$ with overlapping regions of analyticity (a variant of the famous Riemann-Hilbert problem), and subsequently construct the connection A^+_μ (III.31) or (III.32) which will automatically satisfy the self-dual YM equation. We conclude that this procedure is analogous to ours, with H replacing the bridge v, G replacing the $\mathcal{D}^{++}$ connection V^{++}, and the splitting (III.37) replacing the nonlinear differential relation (III.21) between V^{++} and v.

Closest to ours seems to be the approach developed by Newman [5] in parallel with the twistor one. Its central point is the reduction of the above splitting problem to that of solving certain differential equation (referred to in [5] as the Sparling equation). The latter looks almost identical with our eq. (III.21'), involving harmonic differentiation D^{++} and unconstrained YM prepotential V^{++} denoted, respectively, by $\eth$ and A. Accordingly, an essential ingredient of that approach is the use of harmonic expansions on sphere S^2 (with the term "spin - weight" standing for the U(I)-charge and with the vector harmonics instead of the spinor ones in our formalism). Though refs.[5] actually treat the case of Minkowski space, all the things are easy to continue to, the Euclidean case. It would be of interest to establish a step-by-step correspondence between Newman's construction and the one presented here.

We have already mentioned that the problem of solving the nonlinear differential equation (III.21) is difficult. We believe that the problem of finding the splitting (III.37) is of comparable complexity. However, in the harmonic space formulation there is a different approach which only involves the geometry of the λ (analytic in x^+, but not in $u^{\pm}$) frame. There the central problem is to solve the linear differential equation (III.25) for V^{--}. We hope that this path may prove easier (all the said equally concerns the Newman approach which is based on the nonlinear equation of the type (III.21')).

We conclude this paper which largely has an introductory and illustrative nature. The concepts and methods developed here will be applied in the second part of this work [15]. They will help to define the novel notions of the unconstrained prepotentials of hyper-Kähler geometry and to establish their one-to-one correspondence with N=2 off-shell supersymmetric sigma models.

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Геометрии калибровочных полей из комплексной и гармонической аналитичностей.
Кэлеров и самодуальный янг-миллсов случаи

Исходя из принципа сохранения аналитичности, продемонстрировано единство полевых теорий с внутренней комплексной структурой и суперполевых калибровочных теорий. Связи этих теорий интерпретируются как условия интегрируемости для существования аналитических подпространств и решаются переходом в аналитический базис. Продемонстрированы удобства использования этого базиса. В частности, удается заменить проблему сплиттинга, характерную для теории твисторов, решением линейного уравнения.

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Gauge Field Geometry from Complex and Harmonic Analyticities.
Kähler and Self-Dual Yang-Mills Cases

The analyticity preservation principle is employed to demonstrate an impressive affinity between the field theories with intrinsic analytic structure and superfield gauge theories. The defining constraints of the former theories are interpreted as the integrability conditions for the existence of appropriate analytic subspaces and are solved by passing to the basis with manifest analyticity. We prefer to work within the analytic basis. This allows, e.g., to replace the nonlinear splitting problem of twistor approach by solving a linear equation.

The investigation has been perofrmed at the Laboratory of Theoretical Physics, JINR.

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