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TOWARDS A FINITE QUANTUM GRAVITY

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1. As is well known [1] the Einstein theory of gravity with the action

$$S = -\frac{1}{16\pi G}\int d^4x \sqrt{g}\, R(g) \qquad (1)$$

is nonrenormalizable. The counterterms which remove the ultraviolet divergences do not repeat the structure of an initial Lagrangian and thus are not reduced to the renormalization of a finite number of parameters leading to an infinite-parameter arbitrariness. In this sense the quantum theory of gravity does not exist.

However, what we actually need is a finite and uniquely calculable S-matrix rather than the Green functions. It happens to be that the pure gravity (i.e. without matter fields) possesses the desired property in the lowest one-loop order [2]. Ignoring the nontrivial topology, i.e. the terms with the total derivatives in the Lagrangian, the one-loop divergences are (we use the dimensional regularization)

$$\frac{1}{\varepsilon}\left[a R^2 + b R_{\mu\nu} R^{\mu\nu} \right]. \qquad (2)$$

On mass shell, i.e. taking into account the Hilbert-Einstein equations $R_{\mu\nu}=0$, the divergences vanish, i.e. the one-loop S-matrix in Einstein gravity is finite.

However, already in the two-loop approximation the singularities are proportional to the third power of the Riemann curvature tensor and the following structure is possible [3]

$$R_{\mu\nu}{}^{\rho\sigma} R_{\rho\sigma}{}^{\tau\lambda} R_{\tau\lambda}{}^{\mu\nu}, \qquad (3)$$

which does not vanish whence $R_{\mu\nu}=0$. Explicit calculation [4] shows that this structure really appears and destroys the finiteness of the S-matrix.

The way out of this difficulty have been searched in supergravity, where the structure like eq. (3) is forbidden due to supersymmetry. However, in higher loops the expressions which do not vanish on shell still appear, and the problem remains unsolved [1].

In the present paper we propose the way towards solution of the problem of divergences leading to a finite S-matrix in all orders of perturbation theory.

2. Consider the general structure of ultraviolet divergences in quantum gravity. They can be expressed in the following way

$$\sum_{n=1}^{\infty}\left(\frac{\mu^2}{p^2}\right)^{n\varepsilon} G^{n-1}\, T^{(n)}(g,\varepsilon) = \sum_{n=1}^{\infty}\left(\frac{\mu^2}{p^2}\right)^{n\varepsilon} G^{n-1}\sum_{k=1}^{n}\frac{T^{(n,k)}(g)}{\varepsilon^k}, \qquad (4)$$

where G is the Newton constant. Due to the general covariance the functions $T^{(n,k)}$ are constructed out of Riemann tensors and covariant derivatives being homogeneous functions of order n+1

$$T^{(n,k)}(g) \sim R^{n+1}(g), \qquad \nabla^2 \sim R.$$

Expression (4) includes all the possible singularities in Green functions on mass shell as well as off shell and can be obtained most easily by the background field method [5].

The functions $T^{(n,k)}$ depend on the background metric $g_{\mu\nu}^{\text{Background}}$. Transition to the mass shell usually means that the background metric obeys the classical equations of motion, in our case $R_{\mu\nu}(g)=0$. The metric $g^B_{\mu\nu}$, generally speaking, depends on the space-time dimension. In $4-2\varepsilon$ dimensions $g^B_{\mu\nu} = g^B_{\mu\nu}(\varepsilon)$, where the ε -dependence is nonsingular.

The idea of the proposed method to construct a finite quantum gravity is based on this property.

<u>Statement</u> In quantum theory of gravity with the Lagrangian (I) one can achieve the cancellation of all the ultraviolet divergences in the S-matrix, if the background metric is chosen to be

$$g^B_{\mu\nu} = \sum_{k\geq 0} g^{(k)}_{\mu\nu}\cdot \varepsilon^k , \qquad (5)$$

where the classical metric $g^{(0)}_{\mu\nu}$ obeys the Einstein equation $R_{\mu\nu}(g^{(0)})=0$.

<u>Proof</u>

Consider eq. (4) order by order in the loop expansion and use the method of induction.

$\underline{n=1}$ $\quad \frac{1}{\varepsilon}:\; T^{(1,1)}(g^{(0)}) = 0 \quad$ if $\; R_{\mu\nu}(g^{(0)})=0 \; . \; (6)$

$\underline{n=2}$ $\quad \frac{1}{\varepsilon^2}:\; T^{(2,2)}(g^{(0)}) = 0 \quad$ due to eq. (6) because it is totally defined by $T^{(1,1)}(g)$ for any g. Otherwise, we would have a nonlocal divergence which cannot be removed in the usual fashion due to the absence of one-loop counterterms on shell.

$\frac{1}{\varepsilon}$: Due to eq. (5) we have

$$T^{(2,1)}(g^{(0)}) + \frac{d}{dy} T^{(2,2)}(g^{(0)} + y g^{(1)})\Big|_{y=0} = 0 . \tag{7}$$

Hence, even if $T^{(2,1)}(g^{(0)}) \neq 0$ one can achieve finiteness properly choosing $g^{(1)}_{\mu\nu}$ provided that $T^{(2,2)}$ possesses a first order zero at $g^B_{\mu\nu} = g^{(0)}_{\mu\nu}$ and the operator acting on $g^{(1)}_{\mu\nu}$ is convertable. We return to this question below.

$n = N$ All the coefficient functions of higher poles should vanish when $g^B_{\mu\nu} = \sum_{k=0}^{N-2} g^{(k)}_{\mu\nu} \cdot \varepsilon^k$ for the same reason as above. This is a consequence of the absence of divergences in lower orders of perturbation theory and hence the absence of lower order counterterms for a given choice of $g^B_{\mu\nu}$. As is well known, the counterterms of higher poles are not independent but are defined by the coefficients of the lowest pole in previous orders of perturbation theory. This is true both in renormalizable and nonrenormalizable theories and is a consequence of the renormalization group /6/. The ε-dependence of $g^B_{\mu\nu}$ does not change the situation since the above is true for in arbitrary value of $g^B_{\mu\nu}$.

Thus, the only singularity left is that of a simple pole

$$\frac{1}{\varepsilon}: \quad T^{(N,1)}(g^{(0)}) + \dots + \frac{d}{dy} T^{(N,N)}(g^{(0)} + y g^{(N-1)})\Big|_{y=0} = 0 . \tag{8}$$

Again, if $T^{(N,N)}$ has a simple zero at $g^B_{\mu\nu} = g^{(0)}_{\mu\nu}$ and the operator is convertable, one can achieve finiteness of the S-matrix.

To show this, we note that

$$T^{(N,N)}(g) = (-)^N K^{(N,N)}(g), \tag{9}$$

where $K^{(N,N)}(g)$ is the N-th order counterterm. It obeys the equation /6/

$$N K^{(N,N)}(R(g)) = \frac{d}{dy} K^{(N-1,N-1)}(R(g) + y K^{(1,1)}(R(g)))\Big|_{y=0} , \tag{10}$$

where $K^{(1,1)}(R(g)) = - T^{(1,1)}(g)$. It follows that if $K^{(1,1)}(g^{(0)}) = 0$ then $K^{(N,N)}(g^{(0)}) = 0$. Moreover, the structures with a first order zero is always present /6/. They are

$$R \underbrace{R_{\text{"}} R_{\text{"}} \cdots R_{\text{"}}}_{N} , \qquad R_{\text{"}} \underbrace{R_{\text{"}} R_{\text{"}} \cdots R_{\text{"}}}_{N} .$$

Here to find $g^{(N)}_{\mu\nu}$, we have the second order linear differential equation with the operator defined by the variation of the Ricci tensor

$$\delta R_{\mu\nu}(g^{(N)}) = \frac{1}{2}\left[\nabla^2 g^{(N)}_{\mu\nu} + \nabla_\mu \nabla_\nu g^{(N)\alpha}{}_{\alpha} - \nabla_\mu \nabla^\alpha g^{(N)}_{\alpha\nu} - \nabla_\nu \nabla^\alpha g^{(N)}_{\alpha\mu} \right],$$

where the covariant derivative is defined with respect to the metric $g^{(0)}_{\mu\nu}$.

This completes our proof.

3. Some comments concerning the proposed procedure to construct a finite S-matrix in quantum gravity are in order:

1) Equations defining $g^{(N)}_{\mu\nu}$ are obtained uniquely.
2) The physical meaning is attached to the metric $g^{(0)}_{\mu\nu}$, the classical equations being unchanged.
3) As far as all the poles in matrix elements are cancelled in every order, the powers of $(\mu^2/p^2)^\varepsilon$ vanish when $\varepsilon \to 0$ and does not give $\ln p^2/\mu^2$. Thus the usual arbitrariness in the choice of μ is absent. The dimensional transmutation does not occur, and the theory remains conformally invariant.
4) Addition of matter fields to eq. (1) with spin 0, 1/2 and 1 destroys the one-loop finiteness on shell /1/. It can be saved only in the presence of spin 3/2 field. Thus the incorporation of matter fields inevitably leads to supergravity, where the proposed algorithm enables us to spread the finiteness to all order of perturbation theory.

The described procedure has a more general validity. The statement is that any theory (renormalizable or nonrenormalizable) finite in the one-loop order (as a whole or on mass shell) possesses this property in all orders of perturbation theory. Some examples have been already considered /7,8/.

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К конечной квантовой теории гравитации

Показано, что в квантовой теории эйнштейновской гравитации можно добиться сокращения всех ультрафиолетовых расходимостей на массовой поверхности надлежащим выбором фоновой метрики. В рамках размерной регуляризации $g^B_{\mu\nu} = \sum_{k\geq o} g^{(k)}_{\mu\nu}\, \epsilon^k$, причем классическая метрика $g^{(0)}_{\mu\nu}$ удовлетворяет уравнению Эйнштейна $R_{\mu\nu}(g^{(0)}) = 0$.

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Towards a Finite Quantum Gravity

It is shown that in Einstein quantum gravity one can achieve the cancellation of all the ultraviolet divergences on mass shell by an appropriate choice of the background metric. In the framework of dimensional regularization $g^B_{\mu\nu} = \sum_{k\geq o} g^{(k)}_{\mu\nu} \cdot \epsilon^k$, where the classical metric $g^{(0)}_{\mu\nu}$ obeys the Einstein equation $R_{\mu\nu}(g^{(0)}) = 0$.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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