сообщения
объединенного
института
ядерных
исследований
дубна

E17-87-224

B.Esser

# RANDOM WALK ON ULTRAMETRIC SPACES: A TIME DEPENDENT BOND CONSTRUCTION

1987

Recently, a random walk occuring on an ultrametric space has been studied by several authors [1-3]. In this space states form clusters separated by a system of hierarchically organized barriers. State spaces with this type of structure are of interest in the spin glass problem [4] and may generally be of importance for systems in which conflicting requirements and constraints are met [1]. In [2] (hereafter referred to as OS) a random walk on an ultrametric space with regular multifurcations was investigated by using an eigenmode representation of the relaxational matrix. In particular, for barriers increasing linearly with ultrametric distance, OS obtained a power law relaxation with a temperature dependent exponent. In [3] a special case of irregular multifurcations was considered and the power law behaviour of OS confirmed. As in [2] this consideration was based on an eigenmode representation of the relaxational matrix. For the case of random multifurcations, however, it is desirable to have methods working independently of representations using eigenmodes which generally are not known for disordered systems. In this short note, relaxation for both regularly and randomly multifurcating ultrametric spaces is considered by using a time dependent bond construction. This construction is an approximate method and is based on exponential variations of the transition probabilities between different sets of states [5]. In the case of regular multifurcations, the power law relaxation of OS is rederived. Then, by using the simple geometric interpretation connected with the bond construction the case of random multifurcation is treated.

First, we consider a random walk between states of a regularly multifurcating ultrametric space (see Fig.1), and assuming the initial condition $P_k(t=0) = \delta_{k0}$, calculate the probability $P_0(t)$ of state o to be still occupied at a later time t . The states are connected by the transition probabilities

$$W_k = W_o \, e^{-\frac{\Delta_k}{T}}, \qquad (1)$$

where $W_0$ is the rate prefactor, T is the temperature and $\Delta_k$ is the barrier corresponding to the ultrametric distance k (Fig.1). In the time dependent bond construction, transitions between states are divided in two classes $W_k > W(t)$ and $W_k < W(t)$, where W(t)

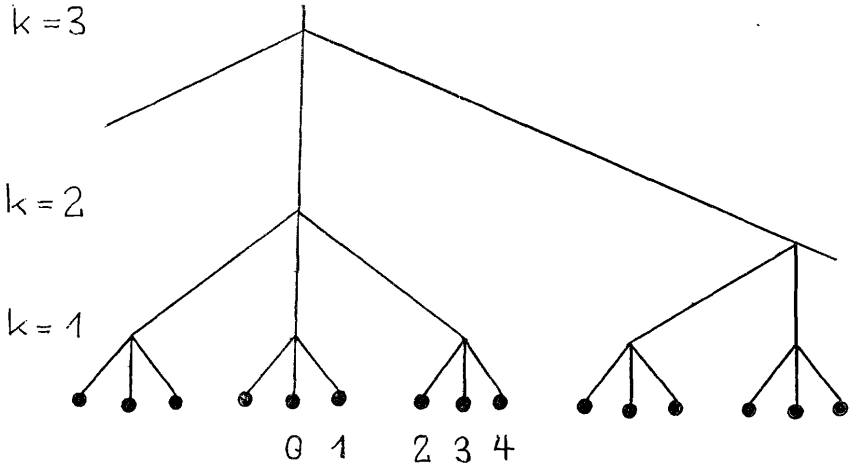


Fig.1. Part of an ultrametric space with regular multifurcations ( $\nu = 3$) . The states are indicated by dots. The tree visualizes the connection of states in a hierarchical organization of barriers $\Delta_k$ depending on the ultrametric distance $k$. The ultrametric distance is defined as the number of branchings one has to rise before lines originating in the corresponding states cross at a branching point, e.g. the ultrametric distance between states 0 and 1 is 1, between 0 and the states 2,3,4 is 2 and so on.

is a prescribed function. For $W_k > W(t)$ a transition is open (bonds are inserted between the corresponding states), and it is assumed that the excitation is almost uniformly spread over the corresponding set of bonded states, whereas for $W_k < W(t)$ transitions between states are blocked. For $W(t)$ one makes the choice $W(t)=A/t$ where A is of the order 1 (see /5/ for details). Here we set A=1 (for $A \neq 1$ the time dependence in eq. (6) is not changed).

In the case of a hierarchy of barriers bonds open stepwise at the discrete times $t_k$ given by the equation $W_k \cdot t_k = 1$, i.e. at

$$t_k = W_o^{-1} e^{\frac{\Delta_k}{T}}. \qquad (2)$$



Assuming the occupation as uniformly spread over the bonded states, one finds that after $t_1, t_2, t_3 \ldots$ the excitation spreads over $\nu^1, \nu^2, \nu^3 \ldots$ sites, i.e.

$$P_o(t_k) \sim \frac{1}{\nu^k}, \qquad (3)$$

where $\nu$ is the number of branchings ( $\nu = 3$ in the Fig.1). Assuming a linear variation of $\Delta_k$ with the ultrametric distance $k$

$$\Delta_k = k \cdot \Delta \qquad (4)$$

and inserting (4) into (2), one obtains

$$k = \frac{T}{\Delta} \ln(W_o t_k). \qquad (5)$$

Now expressing the step $k$ in eq. (3) by $t_k$ and using (5), one finds

$$P_o(t_k) \sim \frac{1}{(W_o t_k)^{(T/\Delta)\ln\nu}}. \qquad (6)$$

Eq. (6) is identical to the asymptotic expression of OS obtained for $W_o t \gg 1$ from the exact eigenvector representation for $P_o(t)$ ( in /2/ $W_o = 1$, $\nu = 2$ ), if the discrete index k in (6) is dropped. Dropping of $k$ is indeed justified because the inserting of bonds should not be taken too literally: in fact it means that the spreading of the excitation by including the next bond in the hierarchy occurs at times of an order of $t \sim t_k$. The inclusion of an arbitrary A into the bonding criterion is easily seen to result in a change of $W_o$ in (6): $W_o \to W_o/A$, leaving the time dependence unchanged.

The main conclusion of the simple calculation given above is that eq. (6) is equivalent to (3), i.e. it is related to the number of sites $N_k = \nu^k$ over which the excitation spreads in a stepwise process of inserting bonds. Now, this observation is used to consider $P_o(t)$ for the case of an ultrametric space with random multifurcations (see Fig.2). The transition probabilities are again assumed to follow (1) but now the number of branchings is a random function. After the $k^{th}$ step one obtains

$$P_o(t_k) \sim \left\langle \frac{1}{N_k} \right\rangle. \qquad (7)$$

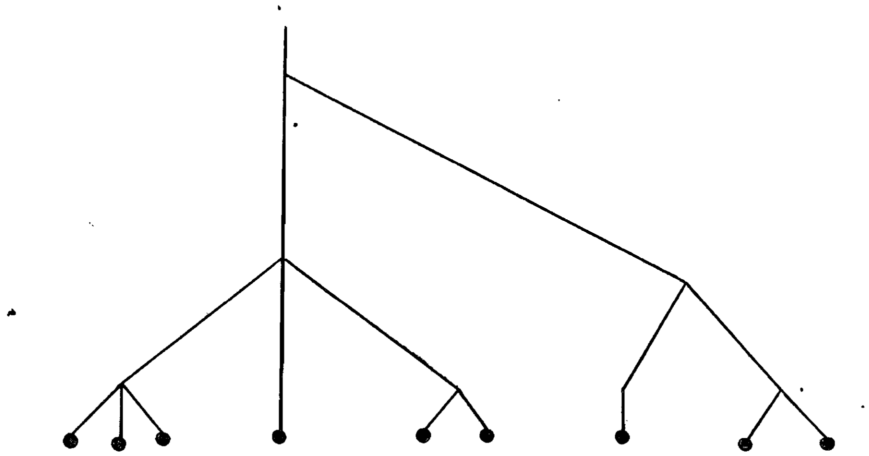

Fig. 2. Part of an ultrametric space with random multifurcations.

where $N_k$ is a random function corresponding to a particular realization of multifurcations after k steps and the average $\langle\ \rangle$ corresponds to an ensemble of random walkers experiencing different multifurcations. Introducing an auxiliary integration (7) is represented in the form

$$P_0(t_k) \sim \int_0^\infty dx \langle e^{-N_k x} \rangle \equiv \int_0^\infty dx\, G_k(e^{-x}) , \qquad (8)$$

where the generating function $G_k(z)$ of the random variable $N_k$

$$G_k(z) = \langle z^{N_k} \rangle \equiv \sum_N P(N_k = N) \cdot z^N \qquad (9)$$

was introduced. In (9) $P(N_k=N)$ is the probability of $N_k=N$. According to a basic theorem of the theory of branching processes (see, e.g. /6/ ) $G_k(z)$ can be expressed as

$$G_k(z) = G^{(k)}(z) \equiv G(G..G(z))..), \qquad (10)$$



i.e. $G_k(z)$ is the kth iterate $G^{(k)}(z)$ of the generating function $G(z)$ of the random variable $N_1$, the number of branchings after the first step. By definition

$$G(z) = \sum_\nu p_\nu z^\nu . \qquad (11)$$

where $p_\nu$ is the probability of having exactly $\nu$ branchings in one step.

The calculation of $P_0(t_k)$ now proceeds in two steps:

1. Assuming a distribution $p_\nu$ one finds $G(z)$ and the kth iterate $G^{(k)}(z)$ with which the integral in the r.h.s. of (8) is calculated.
2. Assuming an explicit $\Delta_k$ dependence in (2) one expresses k by $t_k$ .

Here, we consider the simple case of $p_\nu$ obeying the geometric progression $p_\nu = (1/2)^\nu$, $\nu = 1,2, \ldots$ , where the kth iterate $G^{(k)}(z)$ is given by the explicit expression

$$G^{(k)}(z) = \frac{z}{2^k - (2^k - 1) z} . \qquad (12)$$

The mean value of branchings for this distribution is $\bar{\nu} = 2$ . With (12) in (8) the integral is easily done. One obtains

$$P_0(t_k) \sim \frac{\ln 2^k}{2^k - 1} . \qquad (13)$$

Assuming for $\Delta_k$ again (4) and expressing k by $t_k$, one finds

$$P_0(t_k) \sim \frac{\ln (W_0 t_k)^{(T/\Delta)\ln\bar{\nu}}}{(W_0 t_k)^{(T/\Delta)\ln\bar{\nu}}} , \qquad (14)$$

where 1 in the denominator was neglected. For comparison with (6) the mean value of $\nu$ was inserted. From (14) it is seen that a random branching distributed according to the geometric progression introduces an additional logarithmic time dependence compared to (6).

References

1. R.G.Palmer, D.L.Stein, E.Abrahams and P.W.Anderson, Phys.Rev.Lett. 53, 958 (1984).
2. A.T.Ogielski and D.L.Stein, Phys.Rev.Lett. 55, 1634 (1985).
3. D.Kumar and S.R.Shenoy, Phys.Rev. B34, 3547 (1986).
4. M.Mezard, G.Parisi, N.Sourlas, G.Toulouse and M.Virasoro, Phys.Rev.Lett. 52, 1156 (1984).
5. B.Esser and I.P.Zvyagin, phys.stat.sol. (b) 104, 523 (1981).
6. T.E.Harris, The Theory of Branching Processes, Springer Verlag, 1963.


Received by Publishing Department
on April 7, 1987.