'Buddha's light' of cumulative particles

Analytical explanation of the nuclear glory effect in the cumulative particles production, which is similar to the known optical (atmospheric) glory phenomenon, is presented. Employing the small phase space method we arrived at a characteristic angular dependence of the production cross section $d\sigma \sim 1/\sqrt{(\pi - \theta)}$ near the strictly backward direction. This effect takes place for any number $n \geq 3$ of interactions of rescattered particle, either elastic or inelastic (with resonance excitations in intermediate states), when the final particle is produced near corresponding kinematical boundary. In the small final angles interval including the value $\theta = \pi$ the angular dependence of the cumulative production cross section can have the crater-like (or funnel-like) form. Further studies including, probably, certain numerical calculations, are necessary to clear up this point. Such a behaviour of the cross section near the backward direction is in qualitative agreement with available data. Some mathematical aspects of the nuclear glory phenomenon in the cumulative particles production, which are of methodical interest, are discussed as well.

● Vladimir Kopeliovich, leading research scientist (INR of RAS, Moscow), professor of Moscow Institute for Physics and Technology - MIPT (Introduction to Elementary Particle Physics and Quantum Field Theory). Discovered toroidal bound states of skyrmions (with Boris Stern, end of 1986). ● Galina Matushko, research scientist (INR of RAS, Moscow).

978-3-639-76496-3

Scholars' Press

Vladimir Kopeliovich
Galina Matushko

'Buddha's light' of cumulative particles

Nuclear glory phenomenon

Vladimir Kopeliovich
Galina Matushko

'Buddha's light' of cumulative particles

Vladimir Kopeliovich
Galina Matushko

'Buddha's light' of cumulative particles

Nuclear glory phenomenon

Scholar's Press

Impressum / Imprint
Bibliografische Information der Deutschen Nationalbibliothek: Die Deutsche Nationalbibliothek verzeichnet diese Publikation in der Deutschen Nationalbibliografie; detaillierte bibliografische Daten sind im Internet über http://dnb.d-nb.de abrufbar.

Bibliographic information published by the Deutsche Nationalbibliothek: The Deutsche Nationalbibliothek lists this publication in the Deutsche Nationalbibliografie; detailed bibliographic data are available in the Internet at http://dnb.d-nb.de.

Coverbild / Cover image: www.ingimage.com

Verlag / Publisher:
Scholar's Press
ist ein Imprint der / is a trademark of
OmniScriptum GmbH & Co. KG
Heinrich-Böcking-Str. 6-8, 66121 Saarbrücken, Deutschland / Germany
Email: info@scholars-press.com

Herstellung: siehe letzte Seite /
Printed at: see last page
ISBN: 978-3-639-76496-3

"Buddha's light" of cumulative particles (nuclear glory phenomenon)

Vladimir B. Kopeliovich[a,b*] and Galina K. Matushko[a†]

a) Institute for Nuclear Research of RAS, Moscow 117312, Russia

b) Moscow Institute of Physics and Technology (MIPT), Dolgoprudny, Moscow district, Russia

May 29, 2015

Presented at the 18th International Seminar on High Energy Physics (Quarks 2014) 2-8 June 2014. Suzdal', Russia; 11th Quark Confinement Conference, 7-12 September 2014, St Petersburg, Russia; Session of the Nuclear Physics Department of RAS, 17 - 21 November 2014, MEPHI, Moscow; 7th International Conference Quarks and Nuclear Physics, 2 - 6 March 2015, Valparaiso, Chile. Based on the papers J.Phys. G41(2014)12, 125107 (arXiv:1403.1992); Phys. Reports 139 (1986) 51 and Sov. J. Nucl. Phys. 26 (1977) 87.

* **e-mail**: kopelio@inr.ru

† **e-mail**: matushko@inr.ru

Abstract

Analytical explanation of the nuclear glory effect, which is similar to the known optical (atmospheric) glory phenomenon, is presented. Employing the small phase space method we arrived at a characteristic angular dependence of the production cross section $d\sigma \sim 1/\sqrt{\pi - \theta}$ near the strictly backward direction. This effect takes place for any number $n \geq 3$ of interactions of rescattered particle, either elastic or inelastic (with resonance excitations in intermediate states), when the final particle is produced near corresponding kinematical boundary. Rigorous proof of the effect is given for the case of the optimal kinematics, as well as for arbitrary polar scattering angles in the case of the light particle rescattering, but the arguments in favor of the backward azimuthal (axial) focusing are quite general and hold for any kind of the multiple interaction processes. In the small final angles interval including the value $\theta = \pi$ the angular dependence of the cumulative production cross section can have the crater-like (or funnel-like) form. Further studies including, probably, certain numerical calculations, are necessary to clear up this point. Such a behaviour of the cross section near the backward direction is in qualitative agreement with some of available data. Some mathematical aspects of the nuclear glory phenomenon in the cumulative particles production, which are of methodical interest, are discussed as well. Explanation of this effect and the angular dependence of the cross section near $\theta \sim \pi$ are presented for the first time.

Contents

To the memory of Lyonya Kondratyuk, outstanding scientist and person

1 Introduction

Intensive studies of the particles production processes in high energy interactions of different projectiles with nuclei, in regions forbidden by kinematics for the interaction with a single free nucleon, began back in the 70th mostly at JINR (Dubna), headed by A.M.Baldin and V.S.Stavinsky, and at ITEP (Moscow), where during many years the leader and great enthusiast of these studies was professor G.A.Leksin. Relatively simple experiments could provide information about such objects as fluctuations of the nucleus density [1] or, discussed much later, few nucleon (or multiquark) clusters probably existing in nuclei. At JINR such processes have been called "cumulative production" [2, 3] [1], at ITEP the variety of properties of such reactions has been called "nuclear scaling" [4]- [6] because certain universality of these properties has been noted, confirmed somewhat later at much higher energy, $400\,GeV$ incident protons [7, 8] and $40\,GeV/c$ incident pions, kaons and antiprotons [9, 10]. A new wave of interest to this exciting topic appeared lately. New experiment has been performed in ITEP [11] aimed to define the weight of multiquark configurations in the carbon nucleus [2].

The interpretation of these phenomena as being manifestation of internal structure of nuclei assumes that the secondary interactions, or, more generally, multiple interactions processes (MIP) do not play a crucial role in such production [12] - [18]. Generally, the role of secondary interactions in the particles production off nuclei is at least two-fold: they decrease the amount of produced particles in the regions, where it was large (it is, in particular, the screening phenomenon), and increase the production probability in regions where it was small; so, they smash out the whole production picture.

The development of the Glauber approach [19, 20] to the description of particles scattering off nuclei has been considered many years ago as remarkable progress in understanding the particles-nuclei interactions. Within the Glauber model the amplitude of the particle-nucleus scattering is pre-

[1]According to the Oxford dictionary (Oxford Adv. Learn. Dictionary of Current English, A.S.Hornby, Oxford Univ. Press, 1982) "cumulative — increasing in amount by one addition after another".

[2]We do not pretend here to give a comprehensive review of numerous experiments on cumulative particles production.

sented in terms of elementary particle-nucleons amplitudes and the nucleus wave function describing the nucleons distribution inside the nucleus. The Glauber screening correction for the total cross section of particle scattering off deuteron allows widely accepted, remarkably simple and transparent interpretation.

Gribov [21] investigated nontrivial peculiarities of the space-time picture of such scattering processes and concluded that the inelastic shadowing corrections play an important role at high enough energy and should be included into consideration. [3]

In the case of the large angle particle production the background processes which mask the possible manifestations of nontrivial details of nuclear structure, are subsequent multiple interactions with nucleons inside the nucleus leading to the particles emission in the "kinematically forbidden" region. Leonid Kondratyuk was the first who the has noted that rescattering of intermediate particles could lead to the final particles emission in "kinematically forbidden" regions (KFR). The rigorous investigation of the double interaction process in the case of pion production off deuteron (see Fig. 1.1) has been made first by L.Kondratyuk and V.Kopeliovich in [24]. Later the multiple interaction processes leading to nucleons production in KFR were investigated in [25] and in more details in [26] where the magnitude of the cumulative protons production cross sections was estimated as well.

M.A.Braun and V.V.Vechernin with coauthors made many interesting and important observations and investigated processes leading to the particles emission in KFR [27]-[35], including the processes with resonances in intermediate state [27]-[28]. They found also that processes with pions in intermediate state lead to the nucleons emission in KFR due to subsequent processes, like $\pi N \to N \pi$ [31, 32]. Basic theoretical aspects of MIP leading to the cumulative particles emission and some review of the situation in this field up to 1985 have been presented in [36].

Several authors attempted the cascade calculations of the cumulative particles production cross sections relying upon the available computing codes created previously [39] - [44]. The particles production cross section was found to be in reasonable agreement with data. Different kinds of subprocesses

[3]The pion double charge exchange scattering is an interesting example of the reaction where the inelastic intermadiate states give the dominant contribution at high enough energy [22, 23].

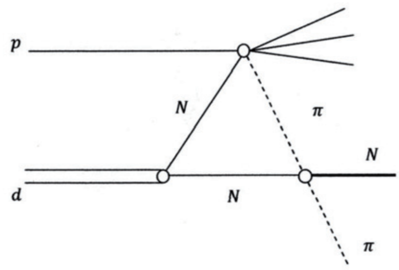

Fig. 1.1. The simplest pion rescattering diagram which leads to the partial fill-up of the "kinematically forbidden" region for the case of the cumulative pion production by protons on the deuteron [24].

play a role in these calculations, and certain work should be performed for detailed comparison. In calculations by NOMAD Collaboration the particles formation time has been considered as a parameter, and results near to the experimental observations have been obtained for this time equal to $\sim 2\,Fm$ [43, 44], see discussion below.

While many authors have admitted the important role of the final state interactions (FSI), most of them did not discuss the active role of such interactions, i.e. their contribution to particles production in KFR, see e.g. [45]. It has been stated in a number of papers that multiple interactions cannot describe the spectra of backwards emitted particles. Such statement in fact has no firm grounds because there were so far no reliable calculations of the MIP contributions to the cross sections and other observables in the cumulative particles production reactions. Moreover, such calculations are hardly possible because, as we argue in the present paper, necessary information about elementary interactions amplitudes is still lacking.

Several specific features of the MIP mechanism have been noted previously experimentally and discussed theoretically [26, 37, 36], among them the presence of the recoil nucleons, which amount grows with increasing energy of the cumulative particle, possible large value of the cumulative baryons polarization, and some other, see [36]. The enhancement of the production cross section near the strictly backward direction has been detected in a number of experiments, first at JINR (Dubna) [46, 47] and somewhat later at ITEP

(Moscow) [48, 49] [4]. This glory-like effect which can be called also the "Buddha's light" of cumulative particles, has been shortly discussed previously in [26, 36] [5]. More experimental evidence of this effect appeared since that time [51, 52]. Here we show analytically that presence of the backward focusing effect is an intrinsic property of the multiple interaction mechanism leading to the cumulative particles production. The detailed treatment of this effect is presented, including the angular dependence of the particles production cross section near the strictly backward direction. To our knowledge, the proof of the existence of the nuclear glory phenomenon in the cumulative production reactions was absent so far in the literature.

In the next section the peculiarities of kinematics of the processes in KFR will be recalled, in section 3 the small phase space method of the MIP contributions calculation to the particles production cross section in KFR is described. In section 4 the focusing effect, similar to the known in optics glory phenomenon, is described in details. Section 5 contains discussion of some characteristic properties of cumulative particles production which find a natural explanation within the multiple interaction mechanism. Final section contains discussion of problems and conclusions. Some mathematical aspects of the nuclear glory phenomenon are presented in section 6; solving the physics problem has certain mathematical consequences of interest.

[4] It should be noted here that the glory-like enhancement of the scattering cross section near the backward direction has been observed in [50] in the deuterons scattering off the ^{40}Ca nuclei at the energy $52\,MeV$.

[5] For the case of the scattering semiclassical explanation of the glory-like effect was given long ago by Ford and Wheeler [53], see also discussion in section 4.

2 Details of kinematics of the processes in "Kinematically Forbidden Regions"

When the particle with 4-momentum $p_0 = (E_0, \vec{p}_0)$ interacts with the nucleus with the mass $m_t \simeq Am_N$, and the final particle of interest has the 4-momentum $k_f = (\omega_f, \vec{k}_f)$ the basic kinematical relation is

$$(p_0 + p_t - k_f)^2 \geq M_f^2, \tag{2.1}$$

where M_f is the sum of the final particles masses, except the detected particle of interest. At large enough incident energy, $E_0 \gg M_f$, we obtain easily

$$\omega_f - zk_f \leq m_t, \tag{2.2}$$

which is the basic restriction for such processes. $z = cos\,\theta < 0$ for particle produced in backward hemisphere. The quantity $(\omega_f - zk_f)/m_N$ is called the cumulative number (more precize, the integer part of this ratio plus one).

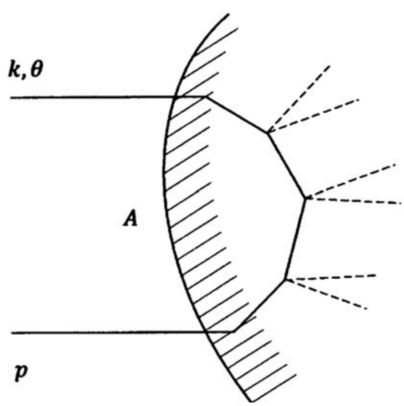

Fig. 2.1. Schematical picture of the multiple interaction process within the nucleus A leading to the emission of the final particle with the momentum k at the angle θ relative to the projectile proton momentum. The binary reactions are assumed to take place in secondary interactions.

Let us recall some peculiarities of the multistep processes kinematics established first in [25, 26] and described in details in [36]. It is very selective kinematics, essentially different from the kinematics of the forward scattering off nuclei when random walking of the particle is allowed in the plane perpendicular to the projectile momentum. Schematically the multistep process is shown on Fig. 2.1.

2.1 Rescatterings of light particles

For light particles (photon, also π-meson) iteration of the Compton formula

$$\frac{1}{\omega_n} - \frac{1}{\omega_{n-1}} \simeq \frac{1}{m}\left[1 - cos(\theta_n)\right] \tag{2.3}$$

allows to get the final energy in the form

$$\frac{1}{\omega_N} - \frac{1}{\omega_0} = \frac{1}{m}\sum_{n=1}^{N}\left[1 - cos(\theta_n)\right] \tag{2.4}$$

The maximal energy of final particle is reached for the coplanar process when all scattering processes take place in the same plane and each angle equals to $\theta_k = \theta/N$. As a result we obtain

$$\frac{1}{\omega_N^{max}} - \frac{1}{\omega_0} = \frac{1}{m}N\left[1 - cos(\theta/N)\right] \tag{2.5}$$

Corresponding kinematical boundaries for N = 1, 2, 3, and 4 are shown in Fig. 2.2. Already at $N > 2$ and for $\theta \leq \pi$ the $1/N$ expansion can be made (it is in fact the $1/N^2$ expansion):

$$1 - cos(\theta/N) \simeq \theta^2/2N^2\left(1 - \theta^2/12N^2\right) \tag{2.6}$$

and for large enough incident energy ω_0 we obtain

$$\omega_N^{max} \simeq \frac{2N\,m}{\theta^2}\left[1 + \frac{\theta^2}{12\,N^2} + \frac{\theta^4}{240\,N^4}\right]. \tag{2.7}$$

This expression works quite well beginning with $N = 2$. This means that the region, kinematically forbidden for interaction with single nucleon, is partly filled up due to elastic rescatterings, see Fig. 2.2. Remarkably, that this rather simple property of rescattering processes has not been even mentioned in the pioneer papers [2] - [6] [6].

[6]This property was well known, however, to V.M.Lobashev, who observed experimentally that the energy of the photon after 2-fold interaction can be substantially greater than the energy of the photon emitted at the same angle in 1-fold interaction.

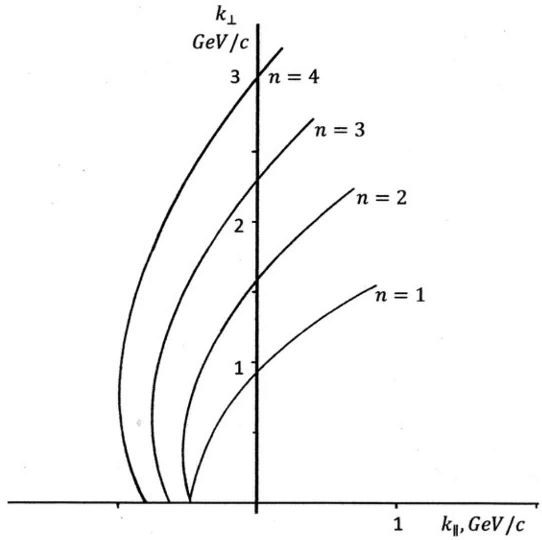

Fig. 2.2. Kinematical boundaries for the light particle rescatterings on the nucleons which are supposed to be at rest in the nucleus. The number of interactions (n in the figure) are $N = 1$, 2, 3 and 4.

2.2 Rescatterings of nucleons on the nucleons inside of nucleus

In the case of the nucleon-nucleon scattering (scattering of particles with equal nonzero masses in general case) it is convenient to introduce the factor

$$\zeta = \frac{p}{E+m}, \qquad 1-\zeta^2 = \frac{2m}{E+m}, \tag{2.8}$$

where p and E are spatial momentum and total energy of the particle with the mass m. When scattering takes place on the particle which is at rest in the laboratory frame, the ζ factor of scattered particle is multiplied by $cos\,\theta$, where θ is the scattering angle in the laboratory frame. So, after N rescatterings we obtain the ζ factor

$$\zeta_N = \zeta_0 cos\,\theta_1 cos\,\theta_2 ... cos\,\theta_N. \tag{2.9}$$

As in the case of the small mass of rescattered particle, the maximal value of final ζ_N is obtained when all scattering angles are equal

$$\theta_1 = \theta_2 = ... = \theta_N = \theta/N, \tag{2.10}$$

and the coplanar process takes plase. So, we have

$$\zeta_N^{max} = \zeta_0 \left[cos(\theta/N)\right]^N . \tag{2.11}$$

Corresponding kinematical boundaries are presented in Fig. 2.3 for $N = 1, 2, 3$, and 4.

The final momentum is from (2.11)

$$k^{max} = 2m \frac{\zeta^{max}}{1 - (\zeta^{max})^2} \tag{2.12}$$

Again, at large enough N and large incident energy ($\zeta_0 \to 1$) the $1/N$ expansion can be made at $k \gg m$, and we obtain the first terms of this expansion

$$k_N^{max} \simeq N \frac{2m}{\theta^2} \left[1 - \frac{\theta^2}{6\,N^2} \left(1 + \frac{\theta^2}{4}\right)\right], \tag{2.13}$$

which coincides at large N with previous result for the rescattering of light particles, but preasymptotic corrections are negative in this case, and convergence of this expansion in the case of nucleons rescatterings is worse than in previous case.

The normal Fermi motion of nucleons inside the nucleus makes these boundaries wider [36]:

$$k_N^{max} \simeq N \frac{2m}{\theta^2} \left[1 + \frac{p_F^{max}}{2m} \left(\theta + \frac{1}{\theta}\right)\right], \tag{2.14}$$

where it is supposed that the final angle θ is large, $\theta \sim \pi$. For numerical estimates we took the step function for the distribution in the Fermi momenta of nucleons inside of nuclei, with $p_F^{max}/m \simeq 0.27$, see [36] and references there. At large enough N normal Fermi motion makes the kinematical boundaries for MIP wider by about 40 %.

There is characteristic decrease (down-fall) of the cumulative particle production cross section due to simple rescatterings near the strictly backward direction. However, inelastic processes with excitations of intermediate particles, i.e. with intermediate resonances, are able to fill up the region at $\theta \sim \pi$.

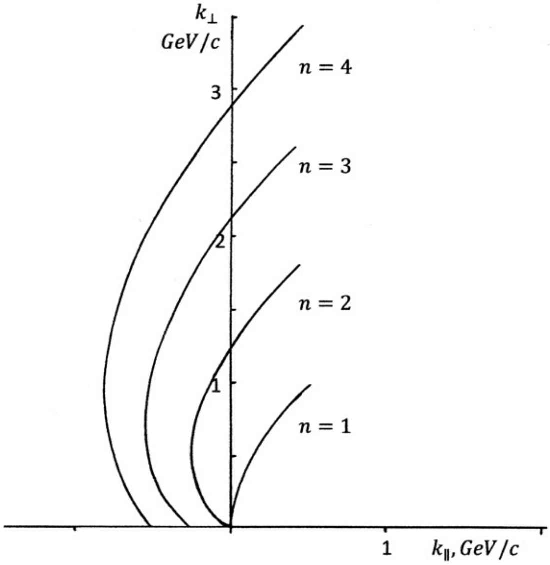

Fig. 2.3. Kinematical boundaries for the nucleon rescatterings on the nucleons which are supposed to be at rest in the nucleus. The number of interactions (n in the figure) are $N = 1$, 2, 3 and 4.

2.3 Resonances excitations in intermediate states

The elastic rescatterings themselves are only the "top of the iceberg". Excitations of the rescattered particles, i.e. production of resonances in intermediate states which go over again into detected particles in subsequent interactions, provide the dominant contribution to the production cross section. Simplest examples of such processes may be $NN \to NN^* \to NN$, $\pi N \to \rho N \to \pi N$, etc. The important role of resonances excitations in intermediate states for cumulative particles production has been noted first by M.Braun and V.Vechernin [27] and somewhat later in [26], see Figs. (2.4) and (2.5). At incident energy about few GeV the dominant contribution into cumulative protons emission provide the processes with $\Delta(1232)$ excitation and reabsorption, see [36] and [41]. Experimentally the role of dynamical excitations in cumulative nucleons production at intermediate energies has been established in [54] and, at higher energy, in [55].

When the particles in intermediate states are slightly excited above their ground states, approximate estimates can be made. Such resonances could be $\Delta(1232)$ isobar, or $N^*(1470)$, $N^*(1520)$ etc. for nucleons, two-pion state or $\rho(770)$, etc for incident pions, $K^*(880)$ for kaons. This case has been investigated previously with the result for the relative change (increase) of the final momentum k_f (Eq. (8) of [26])

$$\frac{\Delta k_f}{k_f} \simeq \frac{1}{N} \sum_{l=1}^{N-1} \frac{\Delta M_l^2}{k_l^2}, \tag{2.15}$$

or

$$\Delta k_f^2 \simeq \frac{2}{N^3} \sum_{l=1}^{N-1} l^2 \Delta M_l^2, \tag{2.16}$$

with $\Delta M_l^2 = M_l^2 - \mu^2$, k_l is the value of 3-momentum in the l-th intermediate state. This effect can be explained easily: the additional energy stored in the mass of intermediate particle is transfered to the kinetic energy of the final (cumulative) particle.

The number of different processes for the N-fold MIP is $(N_R + 1)^{N-1}$, where N_R is the number of resonances making important contribution to the process of interest. The greatest kinematical advantage has the process with resonance production at the $(N-1)$-th step of the whole process with subsequent its deexcitation at the last step [7] . To calculate contributions of all these processe one needs not only to know cross sections and the spin structure of the amplitudes $NN_1^* \to NN_2^*$ at the energies up to several Gev, but also consider correctly possible interference betwee amplitudes of different processes. Such information is absent and hardly will be available in nearest future.

To produce the final particle at the absolute boundary available for the nucleus as a whole one needs to have the masses of intermediate resonances (or some particles system) of the order of incident energy, $s \sim E_0 m_A$.

In this extreme case

$$M_l^2(max) \simeq s_A \frac{l}{A}\left(1 - \frac{l}{A}\right) \tag{2.17}$$

[7]In some of cascade calculations the important contribution to the cumulative nucleons production gives the process with production of pions of not high energy with its subsequent absorption by two-nucleon pair. This process can be, at least partly, to the processes with resonance formation and reabsorption, because pions of moderate energies are produced mostly via resonance formation and decay to nucleon and pion.

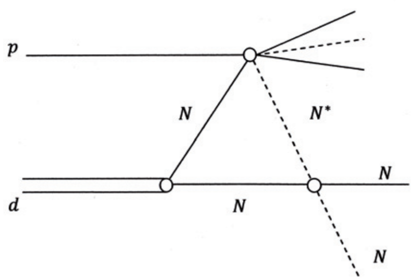

Fig. 2.4. The diagram of the two-fold interaction process on the deuteron with the nucleon resonances (or Δ isobars) excitations in intermediate states.

where $s_A \simeq 2AE_0m$ (see Appendix of [26]). Interaction with all A nucleons should take place, and the intermediate mass is maximal at $l \sim A/2$. For the deuteron the intermediate mass at the absolute boundary should be

$$M_1^D(max) \sim s_D/4 \simeq E_0m/2. \tag{2.18}$$

This case is of academic interest, only. Our aim is to show that the whole region of final particles momenta allowed for interaction with the nucleus as a compact object can be covered due to MIP, but the price for this are the extremely large masses of intermediate states.

2.4 Features of the space-time picture of the MIP leading to cumulative particles production

What is the most important: at arbitrary high incident energy the kinematics of all subsequent processes is defined by the momentum and the angle of the outgoing particle. In other words, for the nucleus fragmentation with particles emitted backwards with probably large but limited by few GeV energies, the fragmentation of nucleon takes place in the first interaction act of the MIP, according to kinematics analysed above. The slight dependence of the whole MIP on the incident projectile, hadron or lepton, follows from this observation, as it was noted long ago by Leksin et al [4]-[6].

The theory of elementary particles based on the S-matrix approach operates with so called $|in>$ and $<out|$ states as initial and final states of the process under consideration. It is assumed that there is time enough

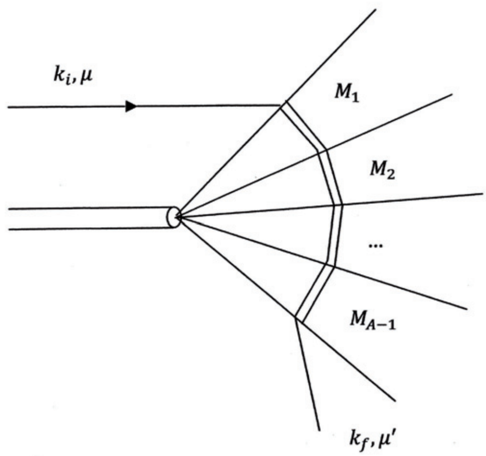

Fig. 2.5. Schematical picture of the multiple interaction process with the resonances excitations in intermediate states. The resonances may be either different or the same in different intermediate states.

for the formation of the outgoing particles and the fields surrounding it. Usually it is in complete correspondence with experimental conditions, when the elementary interaction amplitude is studied by means of cross sections, polarization observables, etc. measurements.

Situation may be, however, quite different when the interaction of the projectile with nucleons inside the nucleus takes place. The role of the formation time in the interaction of the particle within some medium has been discussed long ago, one of the pioneer paper is the paper by Landau and Pomeranchuk [56] where the electromagnetic processes of the photon emission and pair production by electrons has been considered. Similar to the case of electromagnetic interactions, the hadron formation time is of the order of

$$\tau^{form} \sim 1/(\omega - k_z) \tag{2.20}$$

if the incident energy is large enough, where ω and k_z are the energy and the longitudinal momentum of the produced particle, the axis z is defined by the momentum of the incident particle. When the particle is produced in the forward direction with large enough energy (momentum), the formation time becomes

$$\tau^{form} \sim \frac{2\omega}{\mu^2}, \tag{2.21}$$

where μ is the mass of the produced particle. So, formation time, or coherence length in forward direction, become very large for the energetic particle produced in the direction of the projectile momentum (see, e.g. [57] for review of the history of this problem and references. The nuclei fragmentation region has not been discussed in [57]).

As noted above, for the production of a particle on a target with the mass m_t at high enough incident energy the inequality takes place:

$$\omega - k_z \leq m_t, \tag{2.22}$$

at the kinematical boundary the equality takes place. As we have shown in this section, to produce a final particle beyond the kinematical boundary due to multiple interaction process, in the first interaction act the particle should be produced near the kinematical boundary, i.e.

$$\omega_1 - cos\theta_1 k_1 \sim m_N, \tag{2.23}$$

therefore, the formation time of the first produced particle

$$\tau_1^{form} \sim 1/(\omega_1 - cos\theta_1 k_1) \sim 1/m \tag{2.24}$$

is necessarily small, and the whole production picture is of quasiclassical character. The interesting phenomena observed in the high energy particles - nuclei interaction reactions and widely discussed in the literature [57], connected with the large formation time of the particles produced in forward direction, do not take place in the cumulative production processes.

3 The small phase space method for the MIP probability calculations

This method, most adequate for analytical and semi-analytical calculations of the MIP probabilities, has been proposed in [26] and developed later in [36]. It is based on the fact that, according to established in [25, 26] and presented in previous section kinematical relations, there is a preferable plane of the whole MIP leading to the production of energetic particle at large angle θ, but not strictly backwards. Also, the angles of subsequent rescatterings are close to θ/N. Such kinematics has been called optimal, or basic kinematics. The deviations of real angles from the optimal values are small, they are defined mostly by the difference $k_N^{max} - k$, where $k_N^{max}(\theta)$ is the maximal possible momentum reachable for definite MIP, and k is the final momentum of the detected particle. $k_N^{max}(\theta)$ should be calculated taking into account normal Fermi motion of nucleons inside the nucleus, and also resonances excitation — deexcitation in the intermediate state. Some high power of the difference $(k_N^{max} - k)/k_N^{max}$ enters the resulting probability.

3.1 The probability product approximation

Within the quasiclassical treatment adequate for our case, the probability product approximation is valid, and the starting expression for the inclusive cross section of the particle production at large angles contains the product of the elementary subprocesses matrix elements squared, see, e.g., Eq. (4.11) of [36].

After some evaluation, introducing differential cross sections of binary reactions $d\sigma_l/dt_l(s_l, t_l)$ instead of the matrix elements of binary reactions $M_l^2(s_l, t_l)$, we came to the formula for the production cross section due to the N-fold MIP [26, 36]

$$f_N(\vec{p}_0, \vec{k}) = \pi R_A^2 G_N(R_A, \theta) \int \frac{f_1(\vec{p}_0, \vec{k}_1)(k_1^0)^3 x_1^2 dx_1 d\Omega_1}{\sigma_1^{leav} \omega_1} \times$$
$$\times \prod_{l=2}^{N} \left(\frac{d\sigma_l(s_l, t_l)}{dt_l} \right) \frac{(s_l - m^2 - \mu_l^2)^2 - 4m^2\mu_l^2}{4\pi m \sigma_l^{leav} k_{l-1}}$$
$$\times \prod_{l=2}^{N-1} \frac{k_l^2 d\Omega_l}{k_l(m + \omega_{l-1} - z_l \omega_l k_{l-1})} \frac{1}{\omega_N'} \delta(m + \omega_{N-1} - \omega_N - \omega_N'). \tag{3.1}$$

Here $z_l = cos\,\theta_l$, σ_l^{leav} is the cross section defining the removal (or leaving) of the rescattered object at the corresponding section of the trajectory, it is smaller than corresponding total cross section. $G_N(R_A, \theta)$ is the geometrical factor which enters the probability of the N-fold multiple interaction with definite trajectory of the interacting particles (resonances) inside the nucleus. This trajectory is defined mostly by the final values of $\vec{k}$ (k, θ), according to the kinematical relations of previous section. Inclusive cross section of the rescattered particle production in the first interaction is $\omega_1 d^3\sigma_1/d^3k_1 = f_1(\vec{p}_0, \vec{k}_1)$ and $d^3k_1 = (k_1^0)^3 x_1^2 dx_1$, $\omega_N = \omega$ — the energy of the observed particle.

To estimate the value of the cross section (3.1) one can extract the product of the cross sections out of the integral (3.1) near the optimal kinematics and multiply by the small phase space avilable for the whole MIP under consideration [25, 36]. Further details depend on the particular process. For the case of the light particle rescattering, π-meson for example, $\mu_l^2/m^2 \ll 1$, we have

$$\frac{1}{\omega'_N}\delta(m + \omega_{N-1} - \omega_N - \omega'_N) = \frac{1}{kk_{N-1}}\delta\left[\frac{m}{k} - \sum_{l=2}^{N}(1 - z_l) - \frac{1}{x_1}\left(\frac{m}{p_0} + 1 - z_1\right)\right] \tag{3.2}$$

To get this relation one should use the equality $\omega'_N = \sqrt{m^2 + k^2 + k_{N-1}^2 - 2kk_{N-1}z_N}$ for the recoil nucleon energy and the well known rules for manipulations with the δ-function. When the final angle θ is considerably different from π, there is a preferable plane near which the whole multiple interaction process takes place, and only processes near this plane contribute to the final output. At the angle $\theta = \pi$, strictly backwards, there is azimuthal symmtry, and the processes from the whole interval of azimuthal angle $0 < \phi < 2\pi$ provide contribution to the final output (azimuthal focusing, see next section).

3.2 Expansion in angular variables near the optimal kinematics

A necessary step is to introduce azimuthal deviations from this optimal kinematics, φ_k, $k = 1, ..., N-1$; $\varphi_N = 0$ by definition of the plane of the process, $(\vec{p}_0, \vec{k})$. Polar deviations from the basic values, θ/N, are denoted as ϑ_k, obviously, $\sum_{k=1}^{N}\vartheta_k = 0$. The direction of the momentum $\vec{k}_l$ after l-th interaction, $\vec{n}_l$, is defined by the azimuthal angle φ_l and the polar angle

$\theta_l = (l\theta/N) + \vartheta_1 + ... + \vartheta_l$, $\theta_N = \theta$.

Then we obtain making the expansion in φ_l, ϑ_l up to quadratic terms in these variables:

$$z_k = (\vec{n}_k \vec{n}_{k-1}) \simeq cos(\theta/N)(1 - \vartheta_k^2/2) - sin(\theta/N)\vartheta_k +$$
$$+ sin(k\theta/N) sin[(k-1)\theta/N](\varphi_k - \varphi_{k-1})^2/2. \qquad (3.3)$$

In the case of the rescattering of light particles the sum enters the phase space of the process

$$\sum_{k=1}^{N}(1 - cos\vartheta_k) = N[1 - cos(\theta/N)] + cos(\theta/N)\sum_{k=1}^{N}\Big[- \varphi_k^2\, sin^2(k\theta/N) +$$
$$+ \frac{\varphi_k\varphi_{k-1}}{cos(\theta/N)} sin(k\theta/N) sin((k-1)\theta/N)\Big] - \frac{cos(\theta/N)}{2}\sum_{k=1}^{N}\vartheta_k^2 \qquad (3.4)$$

To derive this equality we used that $\varphi_N = \varphi_0 = 0$ — by definition of the plane of the MIP, and the mentioned relation $\sum_{k=1}^{N}\vartheta_k = 0$. We used also the identity, valid for $\varphi_N = \varphi_0 = 0$:

$$\frac{1}{2}\sum_{k=1}^{N}\left(\varphi_k^2 + \varphi_{k-1}^2\right) sin(k\theta/N) sin[(k-1)\theta/N] = cos(\theta/N)\sum_{k=1}^{N}\varphi_k^2 sin^2(k\theta/N). \qquad (3.5)$$

It is possible to present the quadratic form in angular variables which enters (3.4) in the canonical form and to perform integration easily, see Appendix B and Eq. (4.23) of [36], and also section 6 of present paper. As a result, we have the integral over angular variables of the following form:

$$I_N(\Delta_N^{ext}) = \int \delta\Big[\Delta_N^{ext} - z_N^\theta\Big(\sum_{k=1}^{N}\varphi_k^2 - \varphi_k\varphi_{k-1}/z_N^\theta + \vartheta_k^2/2\Big)\Big]\prod_{l=1}^{N-1} d\varphi_l d\vartheta_l =$$
$$= \frac{\left(\Delta_N^{ext}\right)^{N-2}(\sqrt{2}\pi)^{N-1}}{J_N(z_N^\theta)\sqrt{N}(N-2)!\left(z_N^\theta\right)^{N-1}}, \qquad (3.6)$$

$z_N^\theta = cos(\theta/N)$. Since the element of a solid angle $d\Omega_l = sin(\theta\, l/N) d\vartheta_l d\varphi_l$, we made here substitution $sin(\theta\, l/N)\, d\varphi_l \to d\varphi_l$ and $d\Omega_l \to d\vartheta_l d\varphi_l$, $z_N^\theta = cos(\theta/N)$. The whole phase space is defined by the quantity

$$\Delta_N^{ext} \simeq \frac{m}{k} - \frac{m}{p_0} - N(1 - z_N^\theta) - (1 - x_1)\frac{m}{p_0} \qquad (3.7)$$

which depends on the effective distance of the final momentum (energy) from the kinematical boundary for the N-fold process. The Jacobian of the azimuthal variables transformation squared is

$$J_N^2(z) = Det\,||a_N||, \tag{3.8}$$

where the matrix $||a_N||$ defines the quadratic form $Q_N(z, \varphi_k)$ which enters the argument of the δ-function in Eq. (3.6):

$$Q_N(z, \varphi_k) = a_{kl}\varphi_k\varphi_l = \sum_{k=1}^{N-1} \varphi_k^2 - \frac{\varphi_k\varphi_{k-1}}{z}. \tag{3.9}$$

For example,

$$Q_3(z, \varphi_k) = \varphi_1^2 + \varphi_2^2 - \varphi_1\varphi_2/z; \quad Q_4(z, \varphi_k) = \varphi_1^2 + \varphi_2^2 + \varphi_3^2 - (\varphi_1\varphi_2 + \varphi_2\varphi_3)/z,$$

$$Q_5(z, \varphi_k) = \varphi_1^2 + \varphi_2^2 + \varphi_3^2 + \varphi_4^2 - (\varphi_1\varphi_2 + \varphi_2\varphi_3 + \varphi_3\varphi_4)/z, \tag{3.9a}$$

etc, see next section and section 6.

3.3 The phase space of the MIP near the optimal kinematics

The phase space of the process in (3.1) which depends strongly on Δ_N^{ext}, after integration over angular variables can be presented in the form

$$\Phi_N^{pions} = \int \frac{1}{\omega'_N}\delta(m + \omega_{N-1} - \omega_N - \omega'_N) \prod_{l=1}^{N} d\Omega_l = \frac{I_N(\Delta_N^{ext})}{kk_{N-1}} =$$

$$= \frac{(\sqrt{2}\pi)^{N-1}(\Delta_N^{ext})^{N-2}}{kk_{N-1}(N-2)!\sqrt{N} J_N(z_N^\theta)\left(z_N^\theta\right)^{N-1}} \tag{3.10}$$

The normal Fermi motion of target nucleons inside of the nucleus increases the phase space considerably [26, 36]:

$$\Delta_N^{ext} = \Delta_N^{ext}|_{p_F=0} + \vec{p}_l^{\,F}\vec{r}_l/2m, \tag{3.11}$$

where $\vec{r}_l = 2m(\vec{k}_l - \vec{k}_{l-1})/k_l k_{l-1}$. A reasonable approximation is to take vectors $\vec{r}_l$ according to the optimal kinematics for the whole process, and the Fermi momenta distribution of nucleons inside of the nucleus in the form of the step function. Integration over the Fermi motion leads to increase of the power of Δ_N^{ext} and change of numerical coefficients in the expression for the phase space. Details can be found in [26, 36], but they are not importanr for our mostly qualitative treatment here.

For the case of the nucleons rescattering there are some important differences from the light particle case, but the quadratic form which enters the angular phase space of the process is essentially the same, with additional coefficient:

$$\Phi_N^{nucleons} = \frac{1}{k(m+\omega_{N-1})} \int \delta \left[\Delta_{N,nucl}^{ext} - \left(z_N^\theta\right)^N Q_N(\varphi_k) - \frac{\left(z_N^\theta\right)^{N-2}}{2} \sum_{l=1}^N \vartheta_l^2 \right] \prod_{l=1}^N d\Omega_l =$$

$$= \left(\frac{\sqrt{2}\pi}{\zeta_0 z^{N-1}} \right)^{N-1} \frac{(\Delta_{N,nucl}^{ext})^{N-2}}{(N-2)!\sqrt{N} J_N(z_N^\theta)} \frac{(1-\zeta_N^2)(1-\zeta_{N-1}^2)}{4m^2\zeta_N} \tag{3.12}$$

where

$$\Delta_{N,nucl}^{ext} = \zeta_N - (1-x_1)\zeta_N \frac{1-\zeta_1^2}{1+\zeta_1^2} - \frac{k}{m+\omega}, \tag{3.13}$$

with $\zeta_N = \zeta_0 \left(z_N^\theta\right)^N$, $\zeta_1 = \zeta_0 z_N^\theta$. As in the case of the light particle rescattering, the normal Fermi motion of nucleons inside the nucleus can be taken into account.

4 The backward focusing effect (Buddha's light of cumulative particles)

This is the sharp enhancement of the production cross section near the strictly backward direction, $\theta = \pi$. This effect has been noted first experimentally in Dubna (incident protons, final particles pions, protons and deuterons) [46, 47] and somewhat later by Leksin's group at ITEP (incident protons of 7.5 GeV/c, emitted protons of 0.5 GeV/c) [48]. This striking effect was not well studied previously, both experimentally and theoretically. In the papers [26, 36] where the small phase space method has been developed, it was noted that this effect can appear due to multiple interaction processes (see p.122 of [36]). However, the consideration of this effect was not detailed enough, the explicit angular dependence of the cross section near backward direction, $\theta = \pi$, has not been established, estimates and comparison with data have not been made.

One of the authors (VBK) discussed the cumulative (backward) particles production off nuclei with Yakov Abramovich Smorodinsky who noted its analogy with known optical phenomenon - glory, or "Buddha's light". The glory effect has been mentioned by Leksin and collaborators in paper [51], however, it was not clear to authors of [51], can it be related to cumulative production, or not. In the case of the optical (atmospheric) glory phenomenon the light scatterings take place within droplets of water, or another liquid. A variant of the atmospheric glory theory can be found in [58]. However, the optical glory is still not fully understood, the existing explanation is still incomplete, see, e.g. http://www.atoptics.co.uk/droplets/glofeat.htm. In nuclear physics the glory-like phenomenon has been observed for elastic deuterons scattering off ^{40}Ca nuclei at the energy $52\,MeV$ in [50] and has been studied in [59] for the case of low energy antiprotons (energy up to few KeV) interacting with heavy nuclei (due to Coulomb interaction).

The backward focusing effect in the cumulative particles production has been observed and confirmed later in a number of papers for different projectiles and incident energies [49, 51, 52]. It seems to be difficult to explain the backward focusing effect as coming from interaction with dense few nucleon clusters existing inside the nucleus. Similar effect has been explained semiclassically in quantum mechanics in [53], but the angular dependence of the elastic differential cross section obtained in [53] is different from that obtained in our papers, see below.

Mathematically the focusing effect comes from the consideration of the small phase space of the whole multiple interaction process by the method described in previous section and in [26, 36]. It takes place for any MIP, regardless the particular kind of particles or resonances in the intermediate states. As it was explained in section 2, when the angle of cumulative particle emission is large, but different from $\theta = \pi$, there is a prefered plane for the whole process. When the final angle $\theta = \pi$, then integration over one of azimuthal angles takes place for the whole interval $[0, 2\pi]$, which leads to the rapid increase of the resulting cross section when the final angle θ approaches the value $\theta = \pi$.

4.1 The proof of the backward focusing for arbitrary polar scattering angles in the case of the light particles rescattering

We show first that the azimuthal focusing takes place for any values of the polar scattering angles θ_k, not only for the optimal values deduced above for some particular processes. For arbitrary angles θ_k the cosine of the angle between directions $\vec{n}_k$ and $\vec{n}_{k-1}$ is

$$z_k = (\vec{n}_k \vec{n}_{k-1}) \simeq cos(\theta_k - \theta_{k-1})(1 - \vartheta_k^2/2) -$$

$$-sin(\theta_k - \theta_{k-1})\vartheta_k + sin(\theta_k) sin\theta_{k-1}(\varphi_k - \varphi_{k-1})^2/2. \tag{4.1}$$

After substitution $sin\theta_k \varphi_k \to \varphi_k$ we obtain

$$z_k = (\vec{n}_k \vec{n}_{k-1}) \simeq cos(\theta_k - \theta_{k-1})(1 - \vartheta_k^2/2) - sin\,(\theta_k - \theta_{k-1})\,\vartheta_k +$$

$$+ \frac{s_{k-1}}{2s_k}\varphi_k^2 + \frac{s_k}{2s_{k-1}}\varphi_{k-1}^2 - \varphi_{k-1}\varphi_k, \tag{4.2}$$

where we introduced shorter notations $s_k = sin\theta_k$.

It follows from Eq. (4.2) that tin general case of arbitrary polar angles θ_k the quadratic form depending on the small azimuthal deviations φ_k which enters the sum $\sum_k (1 - z_k)$ for the N-fold process is

$$Q_N^{gen}(\varphi_k, \varphi_l) = \frac{s_2}{s_1}\varphi_1^2 + \frac{s_1 + s_3}{s_2}\varphi_2^2 + \frac{s_2 + s_4}{s_3}\varphi_3^2 + ... + \frac{s_{N-2} + s_N}{s_{N-1}}\varphi_{N-1}^2 -$$

$$-2\varphi_1\varphi_2 - 2\varphi_2\varphi_3 - ... - 2\varphi_{N-2}\varphi_{N-1} = ||a||^{gen}(\theta_1, ..., \theta_{N-1})_{kl}\varphi_k\varphi_l, \tag{4.3}$$

with $s_N = sin\theta$. E.g., for $N = 5$ we have the matrix

$$||a||^{gen}_{N=5}(\theta_1, \theta_2, \theta_3, \theta_4) = \begin{bmatrix} s_2/s_1 & -1 & 0 & 0 \\ -1 & (s_1+s_3)/s_2 & -1 & 0 \\ 0 & -1 & (s_2+s_4)/s_3 & -1 \\ 0 & 0 & -1 & (s_3+s_\theta)/s_4 \end{bmatrix}, \tag{4.4}$$

$s_\theta = s_5$, and generalization to arbitrary N is straightforward.

Determinant of this matrix can be easily calculated. It can be shown by induction that at arbitrary N

$$Det\,(||a||^{gen}_N) = \frac{s_\theta}{s_1}, \quad s_\theta = s_N. \tag{4.5}$$

It follows from the generalized expression (4.4) for the matrix $||a||$ that

$$Det||a||^{gen}_{N+1}(\theta) = \frac{s_{N-1}+s_\theta}{s_N} Det\,(||a||^{gen}_N)\,(\theta_N) - Det\,(||a||^{gen}_{N-1})\,(\theta_{N-1}), \tag{4.6}$$

where $\theta_{N+1} = \theta$. Since $Det\,(||a||^{gen}_N)\,(\theta_N) = s_N/s_1$ and $Det\,(||a||^{gen}_N)\,(\theta_{N-1}) = s_{N-1}/s_1$, we obtain easily

$$Det||a||^{gen}_{N+1}(\theta) = \left(\frac{s_{N-1}+s_\theta}{s_N}\right)\frac{s_N}{s_1} - \frac{s_{N-1}}{s_1} = \frac{s_\theta}{s_1}. \tag{4.7}$$

After integration the delta-function containing the quadratic form over the small azimuthal deviations we obtain

$$\int \delta\,(\Delta - ||a||^{gen}_N(\theta_1, ..., \theta_{N-1})_{kl}\varphi_k\varphi_l)\,d\varphi_1...d\varphi_{N-1} =$$
$$= \frac{\Delta^{(N-3)/2}}{Det||a||^{gen}_N(N-3)!!}(2\pi)^{(N-3)/2}c_{N-3} = \sqrt{\frac{s_1}{s_\theta}}\frac{\Delta^{(N-3)/2}}{(N-3)!!}(2\pi)^{(N-3)/2}c_{N-3}, \tag{4.8}$$

$c_n = \pi$ for odd n, and $c_n = \sqrt{2\pi}$ for even n, and $N - 3 \geq 0$, see section 6.

We obtain from above expressions the characteristic angular dependence of the cumulative particles production cross section near $\theta = \pi$:

$$d\sigma \sim \sqrt{\frac{s_1}{s_\theta}} \simeq \sqrt{\frac{s_1}{\pi - \theta}}, \tag{4.9}$$

since $sin\theta \simeq \pi - \theta$ for $\pi - \theta \ll 1$.

This formula does not work at $\theta = \pi$, because integration over the azimuthal angle which defines the plane of the whole MIP takes place in the interval $(0, 2\pi)$. The result for the cross section is final, of course, as we show in details for the case of the optimal kinematics.

4.2 The case of the optimal kinematics (equal scattering angles)

For the optimal kinematics with equal polar scattering angles $\theta_k = k\theta/N$ (see section 2), and the general quadratic form goes over into quadratic form obtained in [36] with some coefficiens:

$$Q^{gen} \to 2z_N^\theta Q(z_N^\theta, \varphi_k, \varphi_l), \quad z_N^\theta = cos(\theta/N), \tag{4.10}$$

and

$$Det(||a||_N^{gen}) = \left(2z_N^\theta\right)^{N-1} Det(||a||_N). \tag{4.10a}$$

It is convenient to present the quadratic form which enters the δ - function in (3.6) as

$$Q_N(z_N^\theta, \varphi_k, \varphi_l) = J_2^2\left(\varphi_1 - \frac{\varphi_2}{2zJ_2^2}\right)^2 + \frac{J_3^2}{J_2^2}\left(\varphi_2 - \frac{J_2^2\varphi_3}{2zJ_3^2}\right)^2 + ...$$

$$... + \frac{J_{N-1}^2}{J_{N-2}^2}\left(\varphi_{N-2} - \frac{J_{N-2}^2\varphi_{N-1}}{2zJ_{N-1}^2}\right)^2 + \frac{J_N^2}{J_{N-1}^2}\varphi_{N-1}^2. \tag{4.11}$$

For the sake of brevity we omitted here the dependence of all J_k^2 on their common argument z_N^θ. The recurrent relation

$$J_N^2(z) = J_{N-1}^2(z) - \frac{1}{4z^2}J_{N-2}^2(z) \tag{4.12}$$

can be obtained from (4.11), since, as it follows from(3.6) and (3.9)

$$Q_{N+1}(z, \varphi_k, \varphi_l) = Q_N(z, \varphi_k, \varphi_l) + \varphi_N^2 - \varphi_N\,\varphi_{N-l}/z \tag{4.13}$$

(recall that for the $N+1$-fold process $\varphi_{N+1} = 0$ by definition of the whole plane of the process), The proof of relation (4.12) is given in final section.

The following formula for $J_N^2(z_N^\theta)$ has been obtained in [36]:

$$Det||a_{kl}|| = J_N^2(z_N^\theta) = 1 + \sum_{m=1}^{m<N/2}\left(-\frac{1}{4\left(z_N^\theta\right)^2}\right)^m \frac{\Pi_{k=1}^m(N-m-k)}{m!} =$$

$$= 1 + \sum_{m=1}^{m<N/2}\left(-\frac{1}{4\left(z_N^\theta\right)^2}\right)^m C_{N-m-1}^m. \tag{4.14}$$

Recurrent relations for Jacobians with subsequent values of N and with same argument z:

$$J_{N+1}^2(z) = J_N^2(z) - \frac{1}{4z^2}J_{N-1}^2(z) = J_{N-1}^2(z)\left(1 - \frac{1}{4z^2}\right) - \frac{1}{4z^2}J_{N-2}^2(z) \tag{4.15}$$

can be continued easily to lower values of N and also used for calculations of J_N^2 at any N starting from two known values, $J_2^2(z) = 1$ and $J_3^2(z) = 1 - 1/(4z^2)$ (see section 6). The Eq. (4.14) can be verified in this way.

The condition $J_N(\pi/N) = 0$ leads to the equation for z_N^π which solution (one of all possible roots) provides the value of $cos(\pi/N)$ in terms of radicals. In other words, we found a way to get polynomials in $1/z^2$ with rational coefficients one of roots of which is just $cos(\pi/N)$. The following expressions for these polynomials (Jacobians) take place [26, 36]

$$J_2^2(z) = 1; \qquad J_3^2(z) = 1 - \frac{1}{4z^2}; \qquad J_4^2(z) = 1 - \frac{1}{2z^2}, \tag{4.16}$$

$J_3(\pi/3) = J_3(z = 1/2) = 0$, $J_4(\pi/4) = J_4(z = 1/\sqrt{2}) = 0$. Let us give here less trivial examples. For $N = 5$

$$J_5^2 = 1 - \frac{3}{4z^2} + \frac{1}{16z^4}, \qquad (J_5^2)'_z = \frac{3}{2z^3} - \frac{1}{4z^5} \tag{4.17}$$

and one obtains $cos^2(\pi/5) = (3 + \sqrt{5})/8$, $J_5(\pi/5) = 0$.

At $N = 6$

$$J_6^2 = 1 - \frac{1}{z^2} + \frac{3}{16z^4} = J_3^2\left(1 - \frac{3}{4z^2}\right), \qquad (J_6^2)'_z = \frac{2}{z^3} - \frac{3}{4z^5}. \tag{4.18}$$

see also Eq. (6.9). For $N = 7$

$$J_7^2 = 1 - \frac{5}{4z^2} + \frac{3}{8z^4} - \frac{1}{64z^6}, \qquad (J_7^2)'_z = \frac{5}{2z^3} - \frac{3}{2z^5} + \frac{3}{32z^7}. \tag{4.19}$$

$J_7(\pi/7) = 0$.

$$J_8^2 = 1 - \frac{3}{2z^2} + \frac{5}{8z^4} - \frac{1}{16z^6} = J_4^2\left(1 - \frac{1}{z^2} + \frac{1}{8z^4}\right), \qquad (J_8^2)'_z = \frac{3}{z^3} - \frac{5}{2z^5} + \frac{3}{8z^7}, \tag{4.20}$$

see Eq. (6.9); $J_8(\pi/8) = 0$.

$$J_9^2 = 1 - \frac{7}{4z^2} + \frac{15}{16z^4} - \frac{5}{32z^6} + \frac{1}{256z^8} = J_3^2\left(1 - \frac{3}{2z^2} + \frac{9}{16z^4} - \frac{1}{64z^6}\right), \tag{4.21}$$

$J_9(\pi/9) = 0$.

$$J_{10}^2 = J_5^2\left(J_6^2 - \frac{1}{4z^2}J_4^2\right), \tag{4.22}$$

see section 6 where the proof of Eq. (4.12) is given. It can be obtained easily from (4.22) that $cos^2(\pi/10) = (5+\sqrt{5})/8$. For arbitrary N, J_N^2 is a polinomial in $1/4z^2$ of the power $|(N-1)/2|$ (integer part of $(N-1)/2$), see Eq. (4.14). These equations can be obtained using the elementary mathematics methods as well, see section 6, Eqs (6.14) – (6.16). The case $N = 2$ is a special one, because $J_2(z) = 1$ - is a constant. In this case the 2-fold process at $\theta = \pi$ (strictly backwards) has no advantage in comparison with the direct one, see Eq. (2.5), if we consider the elastic rescatterings.

For particles emitted strictly backwards the phase space has different form, instead of $J_N(\theta/N)$ enters $J_{N-1}(\theta/N)$ which is different from zero at $\theta = \pi$, and we have instead of Eq. (3.6)

$$I_N(\varphi,\vartheta) = \int \delta\Big[\Delta_N^{ext} - z_N^\pi\Big(\sum_{k=1}^{N}\varphi_k^2 - \varphi_k\varphi_{k-1}/z_N^\pi + \vartheta_k^2/2\Big)\Big]\left[\prod_{l=1}^{N-2} d\varphi_l d\vartheta_l\right] 2\pi d\vartheta_{N-1} =$$

$$= \frac{\left(\Delta_N^{ext}\right)^{N-5/2}(2\sqrt{2}\pi)^{N-1}}{J_{N-1}(z_N^\pi)\sqrt{N}(2N-5)!!\,(z_N^\pi)^{N-3/2}}, \tag{4.23}$$

This follows from Eq. (4.11) where at $\theta = \pi$ the last term disappears, since $J_N(\pi/N) = 0$ and integration over $d\varphi_{N-1}$ takes place over the whole 2π interval.

To illustrate the azimuthal focusing which takes place near $\theta = \pi$ the ratio is useful of the phase spaces near the backward direction and strictly at $\theta = \pi$. The ratio of the observed cross sections in the interval of several degrees slightly depends on the elementary cross sections and is defined mainly by this ratio of phase spaces. It is

$$R_N(\theta) = \frac{\Phi(z)}{\Phi(\theta=\pi)} = \sqrt{\frac{\Delta_N^{ext}}{z_N^\pi}\frac{(2n-5)!!}{2^{N-1}(N-2)!}\frac{J_{N-1}(z_N^\pi)}{sin(\pi/N)J_N(z_N^\theta)}} \tag{4.24}$$

Near $\theta = \pi$ we use that

$$J_N(z_N^\theta) \simeq \sqrt{\frac{\pi-\theta}{N}[J_N^2]'(z_N^\pi)sin\frac{\pi}{N}} \tag{4.25}$$

and thus we get

$$R_N(\theta) = C_N\sqrt{\frac{\Delta_N^{ext}}{\pi-\theta}} \tag{4.26}$$

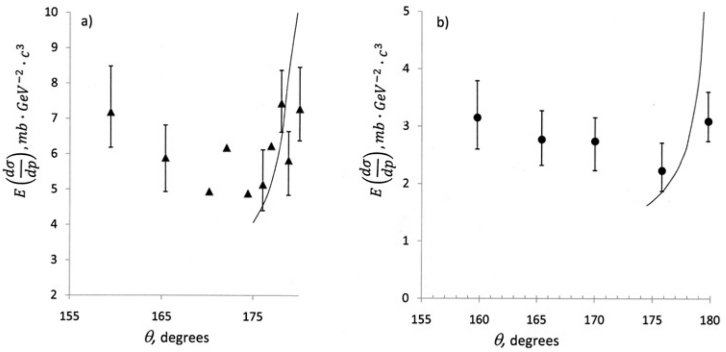

Fig. 4.1. The angular dependence of inclusive cross section of the production of positive pions by projectile protons with momentum $8.9\,GeV/c$. a) pions with momentum $0.5\,GeV/c$ emitted from Pb nucleus. The error bars at some points have not been clearly indicated in the original paper; b) pions with momentum $0.3\,Gev/c$ emitted from He nucleus. The data are taken from Fig. 18 of the paper [47].

with

$$C_N = \frac{J_{N-1}(z_N^\pi)\sqrt{N}}{[(J_N^2)'(z_N^\pi)]^{1/2}[sin(\pi/N)]^{3/2}} \frac{(2N-5)!!}{\sqrt{z_N^\pi}(N-2)!2^{N-1}} \tag{4.27}$$

We need also values of $J_{N-1}[\pi/N]$ to estimate the behaviour of the cross sction near $\theta = \pi$, they are given in Table 1 [8]. Integration over variable x_1 leads to multiplication C_N by factor $(2N-3)/(2N-2)$, i.e. it makes it smaller, increasing the effect under consideration. The connection between the case of the optimal kinematics when the result is presented in Eq. (4.26), and the general case considered at the beginning of this section, see Eq. (4.9), can be established with the help of Eq-s (6.17) – (6.20).

4.3 Comparison with some of available data

According to Eq. (4.15), the differential cross section of the cumulative particle production increases with increasing angle θ. At the critical value

$$\theta^{crit} \simeq \pi - C_N^2 \Delta_N^{ext}, \quad \epsilon^{crit} = \pi - \theta^{crit} \simeq C_N^2 \Delta^{ext} \tag{4.28}$$

[8] The angular dependence of the cross section obtained in [53] in semiclassical quantum mechanical treatment of the elastic scattering is $d\sigma \sim 1/(\pi - \theta)$, after averaging over several oscillations (eq.(34) of [53]), i.e. more singular than we obtained. However, it was not indicated clearly in [53] which types of potential can lead to the glory-like behaviour of the cross section.

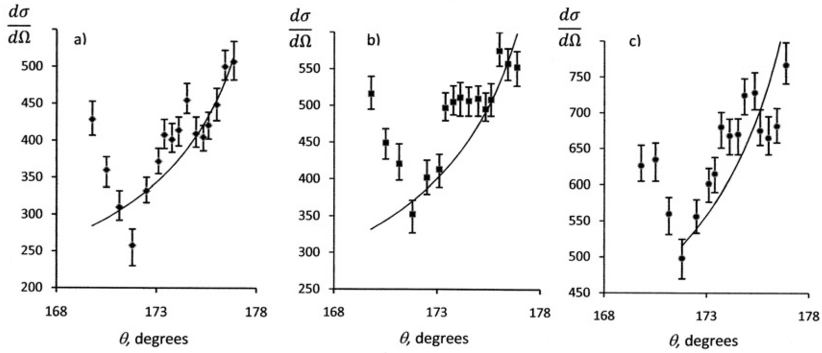

Fig. 4.2. Angular distributions of secondary protons with kinetic energy between 0.06 and $0.24\,GeV$ emitted from the Pb nucleus, in arbitrary units. The momentum of the projectile protons is $4.5\,GeV/c$. a) The energy of emitted protons in the interval $0.11 - 0.24\,GeV$; b) the energy interval $0.08 - 0.11\,GeV$; c) the energy interval $0.06 - 0.08\,GeV$. Data obtained by G.A.Leksin group at ITEP, taken from Fig. 3 of paper [51].

it becomes equal to the cross section at $\theta = \pi$ which is proportional to Eq. (4.12), and may slightly increase further with increasing θ. But near $\theta = \pi$ it should derease, to become again $d\sigma/d\Omega|_{\theta=\pi}$ at $\theta = \pi$. So, the differential cross section has a crater-like (or funnel-like) form near the backward direction. We do not provide here the detailed description of the cross section in the transition region between θ^{crit} and $\theta = \pi$: this is technically rather complicated problem, and not so important for us now.

N	$(J_N^2(z_N^\pi))'$	$sin(\pi/N)$	$\left[(J_N^2(z_N^\pi))'\,sin^3(\pi/N)\right]^{1/2}$	$J_{N-1}[z_N^\pi]$	C_N
3	4	0.866	1.612	1	0.38
4	2.83	0.707	0.999	0.707	0.32
5	2.11	0.588	0.655	0.486	0.29
6	1.540	0.5	0.438	0.333	0.27
7	1.087	0.434	0.298	0.229	0.26

Table 1. Numerical values of the quantities which enter the particles production cross section near backward direction, $\theta = \pi$. Here $z_N^\pi = cos(\pi/N)$.

Characteristic values of Δ^{ext} are defined by kinematical boundaries described in section 2, Eq. (2.7), (2.13), and we obtain easily

$$\Delta^{ext}_{typical} \sim \theta^2/2N(N+1) < \pi^2/2[N(N+1)], \tag{4.29}$$

so it is not greater than ~ 0.5 for $N = 3$ and decreases rapidly with increasing N. Therefore, the values of ϵ^{crit} may be quite small, about several degrees.

The inclusion into consideration of the Fermi motion of nucleons inside the nucleus leads to the change of the Δ^{ext} in (3.7), (3.13) and subsequent formulas, and the constant C_N in (4.26), but the angular dependence of the cross section given by (4.26) does not change. Inclusion of resonance excitation in one (or several) intermediate states leads to the increase of the quantity Δ_N^{ext} according to formulas of section 2, and to the increase of the phase space of the whole MIP, but the effect of azimuthal focusing persists. Quite similar results can be obtained for the case of nucleons, only some technical detaols are different, see section 3. The behaviour given by Eq. (4.15) is in good agreement with available data, the value of the constants C_N is not important for our semiquantiatative treatment. The comparison of the observed behaviour with predicted one according to the simple law $d\sigma \sim A + B/\sqrt{\pi - \theta}$ is presented in Fig.4.1, Fig. 4.2 and Fig. 4.3.

We selected several examples where qualitative agreement of data with predicted behaviour takes place. In Fig. 4.1 the inclusive cross section of the production of positive pions by projectile protons with momentum $8.9\,GeV/c$ is presented for pions with momentum $0.5\,GeV/c$ (Pb as a target) and for pions with momentum $0.3\,Gev/c$ (He as a target)[47]. In Fig 4.2 angular distributions of secondary protons with kinetic energy between 0.06 and $0.24\,GeV$ emitted from the Pb nucleus are presented, in arbitrary units. The momentum of the projectile protons is $4.5\,GeV/c$ [51]. In Fig 4.3 angular distributions of secondary pions with kinetic energy greater $0.14\,GeV$ emitted from the Pb nucleus, are presented, also in arbitrary units. The momentum of the projectile protons is 4.5 GeV/c. Data are taken from Fig. 5 of paper [51].

There are other data where the glory-like effect is clearly seen. In many other cases the flat behaviour of the differential cross section near $\theta \sim \pi$ takes place, but it was probably not sufficient resolution to detect the enhancement of the cross section near $\theta = \pi$, see e.g. [60]. In some experiments the deviation of the final angle from 180 deg. is large, therefore, further measurements near $\theta = \pi$ are desirable, also for kaons, hyperons as

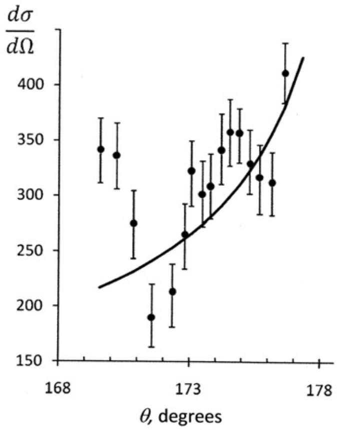

Fig. 4.3. Angular distributions of secondary pions with kinetic energy greater $0.14\,GeV$ emitted from the Pb nucleus, in arbitrary units. The momentum of the projectile protons is 4.5 GeV/c. Data obtained by G.A.Leksin group at ITEP, taken from Fig. 5 of paper [51].

cumulative particles.

5 A-number dependence of cumulative particles production cross sections; mixing effect and polarization phenomena

For completeness we discuss here several most striking properties of cumulative production reactions which found a natural explanation within the multiple interaction mechanism: the sharp A-number dependence of the cumulative particles production cross sections, the mixing between the produced particles which belong to the same isomultiplet, and some polarization phenomena.

5.1 A-number dependence of the cumulative particles production cross sections

Let us begin with the sharp A-number dependence of the cumulative production cross sections. It is a specific property of such reactions investigated previously within the multiple interactions picture in [38]. According to [38] the sharp A-number dependence of the cumulative particles production cross sections finds a natural explanation within the MIP approach. The A-number dependence of the cross section is defined by geometrical factor $\Phi_N(R,\theta)$

$$\Phi_N(R,\theta) = \pi R_A^2 G_N(R,\theta) = \int dxdyd\left[exp\left\{-\int_{-\infty}^{l_1}\sigma_0\rho(\vec{l})dl\right\}\right]\times$$

$$\times d\left[exp\left\{-\int_{l_1}^{l_2}\sigma_1\rho(\vec{l})dl\right\}\right]...d\left[exp\left\{-\int_{l_{N-1}}^{l_N}\sigma_{N-1}\rho(\vec{l})dl\right\}\right]exp\left\{-\int_{l_N}^{\infty}\sigma_N\rho(\vec{l})dl\right\}, \tag{5.1}$$

where x, y are the coordinates in the plane perpendicular to the momentum of the projectile, and the integration takes place along the trajectory of the object (particle or resonance) propagating within the nucleus. It is important that this trajectory has a protuberant (convex) form. This expression can be essentially simplified if, as a zero approximation, one takes all cross sections equal, performs integration over intermediate parts of the trajectory, and takes the model of the nucleus with a sharp edge:

$$\Phi_N(R_A,\theta) = \int\left[\int_0^{L(R_A,\theta,x,y)/l_0}\frac{l^{N-1}exp(-l)}{(N-1)!}\,dl\right]dx\,dy, \tag{5.2}$$

$l_0 = 1/(\rho.\sigma)$. For the rough estimates we made substitution $L(R_A, \theta, x, y) \rightarrow \bar{L}(R_A)$, and we came to [38, 36]

$$\Phi_N(R_A, \theta) \simeq \pi R_A^2 \int_0^{\bar{L}(R_A,\theta)/l_0} \frac{l^{N-1} exp(-l)}{(N-1)!} dl, \tag{5.3}$$

with $L_{max} \simeq \theta R/sin(\theta/2)$ in the case of the particle propagation along the circle (evidently, for $\theta = 0$ $L_{max} = 2R_A$, for $\theta = \pi$ $L_{max} = \pi R_A$). For the crude estimate one can substitute, at $\theta >\sim 3\pi/4$, $L(R_A, \theta) \rightarrow \bar{L} \simeq (0.3 - 0.4)R_A\theta$.

If the absorption of the rescattered particles would absent at all, then we would obtain the dependence $\Phi_N \sim R_A^2(R_A\theta)^N \sim A^{(N+2)/3}$ which is the sharpest possible dependence of the cross section due to the N-fold interaction. For $N = 1$ we would have $d\sigma \sim A$ — the well known result for the weekly interacting particles.

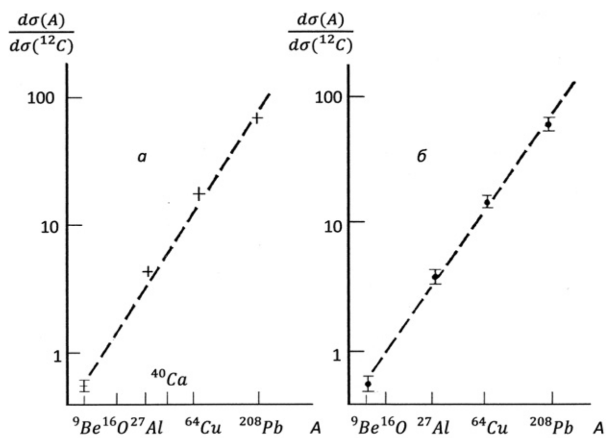

Fig. 5.1. A-number dependence of the proton production cross sections at $\theta = 137^o$, $E_0 = 400 Gev$ [7]. a) $k = 0.95\, Gev/c$, $N = 4$. Calculations performed with the Woods-Saxon distribution of the nuclei density, $\sigma_0 \simeq 0$; $\sigma_1 \simeq 30\, mb$; $\sigma_2 \simeq 25\, mb$; $\sigma_3 \simeq 20\, mb$; b) $k = 0.37\, Gev/c$, $N = 3$, $\sigma_1 \simeq 20\, mb$; $\sigma_2 \simeq 15\, mb$, [38].

The data obtained at the incident energy $400\, GeV$ [7, 8] are in good qualitative agreement with theoretical expectations, as shown in Figs. 5.1

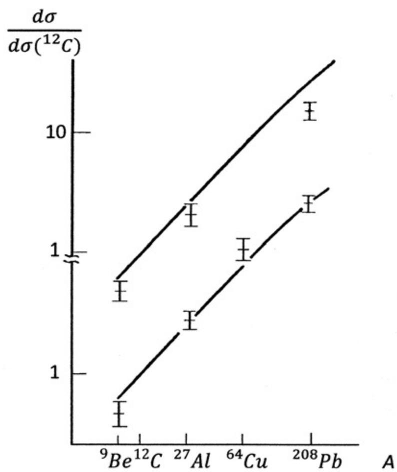

Fig. 5.2. A-number dependence of $\pi^{\pm}$ production cross sections at $\theta = 160^o$, $E_0 = 400 Gev$ [8]. $k = 0.67\, Gev/c$, $N = 3$, $\sigma_1 \simeq 28\, mb$; $\sigma_2 \simeq 34\, mb$; $\sigma_3 \simeq 20\, mb$, or all $\sigma_k \simeq 27\, mb$. The error bars indicated do not include the uncertainty of measurements on the ^{12}C nucleus, same in Fig. 5.1.

and 5.2 [38]. The calculated $A-$number dependence of the production cross sections weekly depends on the particular values of interaction cross sections in intermediate states σ_k, but the protuberant form of the trajectory dictated by the kinematics of the MIP is very important.

5.2 Mixing effect in the yields of the components of isomultiplets

The ratio of neutron to proton yields for different target nuclei, including isotopes of Sn and Ni, was studied in [61, 62]. It was observed that this ratio, averaged over the sign of incident pions [61], is smaller than the N/Z ratio for the nucleus which could be expected in the simplest spectator picture. As it was shown in [37], several kinds of MIP naturally lead to mixing in the yield of neutrons and protons and to decrease of the ratio n/p, relative to the N/Z ratio.

It was obtained in [37] for the isobar mechanism $NN \to N\Delta \to NN$

that the ratio

$$R(n/p)_2 = \frac{R(n/p)_1 + 2NA/(10NZ + A^2)}{1 + R(n/p)_1 2ZA/(10NZ + A^2)}, \tag{5.4}$$

where $R(n/p)_1$ ($R(n/p)_2$) are the neutron to proton ratios before and after the isobar interaction mechanism. The limiting value of the neutron to proton ratio is

$$R(n/p)_{lim} = \sqrt{\frac{N}{Z}}$$

The nucleon-nucleon interaction itself also leads to mixing. Let us define three independent cross sections

$$\sigma_1 = \sigma(pp \to pp) = \sigma(nn \to nn); \quad \sigma_2 = \sigma(pn \to pn) = \sigma(np \to np);$$

$$\sigma_3 = \sigma(pn \to np).$$

After one scattering act we obtain

$$R(n/p)_2 = \frac{R(n/p)_1 + (N/Z)\sigma_3/(\sigma_1 N/Z + \sigma_2)}{(\sigma_1 + \sigma_2 N/Z)/(\sigma_1 N/Z + \sigma_2) + R(n/p)_1\sigma_3/(\sigma_1 N/Z + \sigma_2)} \tag{5.5}$$

The inequality should take place

$$\sigma_3 > |\sigma_1 - \sigma_2|$$

for mixing to occur. The limiting value of $R(n/p)$ is

$$R(n/p)_{lim} = \frac{1}{2\sigma_3}\left[(\sigma_1 - \sigma_2)(N/Z - 1) + \sqrt{(\sigma_1 - \sigma_2)^2(N/Z - 1)^2 + 4(N/Z)\sigma_3^2}\right]. \tag{5.6}$$

For $\sigma_1 - \sigma_2 = \sigma_3$ we obtain $R(n/p)_{lim} = \sqrt{N/Z}$, similar to the isobar mechanism.

For the case of reaction $\pi N \to N\pi$ the mixing proceeds effectively in the channel with isospin $I_{N\pi} = 1/2$.

Experimental value of $R^{exp}_{n/p} = 1.32 \pm 0.07$ for Pb and ^{238}U nuclei [61] in comparison with the $\sqrt{N/Z}$ values 1.236 and 1.260 correspondingly. The mixing in the yields of the particles which belong to other isomultiplets, e.g. (π^-, π^+), also can be explained within the MIP [37].

5.3 Polarization phenomena

Relatively large polarization of cumulative baryons has been observed in a number of experiments, for Λ hyperons and for protons [63] - [67]. For Λ hyperons [63] polarization reached a value of the order 1 in modulo for emission angle between 80^o and 110^o, but with large uncertainty $\sim 35\%$, for protons polarization reached the values $\sim 0.4 - 0.5$ [64]. Polarization of baryons emitted backwards can appear due to subsequent rescatterings within the nucleus. It is well known that the scattering off some target is a regular way to get polarization of fermions - protons, electrons, etc. But, moreover, in the case of cumulative baryons production the final polarization can be enhanced if rescatterings proceed in similar kinematical conditions, as it was stressed in [26]. Let P be the polarization after 1-fold interaction with unpolarized target, D - parameter which defines the dependence of the final polarization on the incident, then the polarization after the n-th interaction is given by [9]

$$P_n = \frac{P + DP_{n-1}}{1 + P\,P_{n-1}}, \tag{5.7}$$

If the parameters P, D are equal in all N interactions, then the final polarization is

$$P_N = \frac{(D + x_1 P)^N - (D - x_2 P)^N}{(D + x_1 P)^{N-1}(x_1 + P) + (D - x_2 P)^{N-1}(x_2 - P)} \tag{5.8}$$

where

$$x_{1,2} = \left[\sqrt{(1-D)^2 + 4P^2} \pm (1 - D)\right],$$

At $D = 1$ we have $x_1 = x_2 = 1$, and the final polarization will be

$$P_N = \frac{(1 + P)^N - (1 - P)^N}{(1 + P)^N + (1 - P)^N}, \tag{5.9}$$

if the parameters P are equal for all interaction acts [26]. In general case, when there are differences between interaction acts, instead of $(D + x_1)^N$ will enter product $\prod_{l=1} (D_l + x_{1,l} P_l)$, etc. According to Eq. (5.8), (5.9) there is a potential possibility of enhancement of the polarization of emitted cumulative baryons due to the MIP mechanism. Further measurements of polarization of different baryons in KFR are of great interest.

[9] One of the authors (VK) is greatly indebted to Lev Iosifovich Lapidus for enlightning discussion of the polarization phenomena in nucleon-nucleon interactions.

There is no striking prediction for the asymmetry of emitted particles from polarized projectile within the multiple interaction mechanism. It has been observed that this asymmetry is different from that predicted if the 1-fold interaction takes place, and it has been found to be small in a number of experiments [68] - [74]. So, it is a clear indication that the nature of the cumulative baryons is more complicated than the simple spectator mechanism with the high momentum components in the nucleons distribution within the nucleus taken into account.

6 Useful mathematical relations and formulas

Here we present for the readers convenience some formulas and relations which have been used in present paper (sections 3 and 4). The nuclear glory phenomenon is an example when solving the physics problem leads to mathematical consequences of interest.

6.1 The integrals of the δ-functions

In sections 3 and 4 the following integrals of the δ-functions have been used:

$$I_n(\Delta) = \int \delta(\Delta - x_1^2 - ... - x_n^2) dx_1 ... dx_n = \pi \frac{(2\pi)^{(n-2)/2}}{(n-2)!!} \Delta^{(n-2)/2} \qquad (6.1)$$

for integer even n.

$$I_n(\Delta)_n = \int \delta(\Delta - x_1^2 - ... - x_n^2) dx_1 ... dx_n = \frac{(2\pi)^{(n-1)/2}}{(n-2)!!} \Delta^{(n-2)/2} \qquad (6.2)$$

for integer odd n. Relations

$$\int_0^\pi sin^{2m}\theta \, d\theta \; = \; \pi \frac{(2m-1)!!}{(2m)!!}; \qquad \int_0^\pi sin^{2m-1}\theta \, d\theta \; = \; 2 \frac{(2m-2)!!}{(2m-1)!!}, \qquad (6.3)$$

m — integer, allow to check (6.1) and (6.2) easily.

The equality takes place

$$\int \delta(\Delta - x_1^2 - ... - x_n^2) \delta(x_1 + x_2 + ... + x_n) dx_1 ... dx_{n-1} dx_n = \frac{1}{\sqrt{n}} I_{n-1}(\Delta) \quad (6.4)$$

More generally, for any quatratic form in variables $x_k,\ k=1,...n$ after diagonalization we obtain

$$\int \delta(\Delta - a_{kl}x_k x_l) dx_1 \dots dx_n = \int \delta(\Delta - x_1'^2 - \dots - x_n'^2) \frac{dx_1' ... dx_n'}{\sqrt{det||a||}} = \frac{1}{\sqrt{det||a||}} I_n(\Delta). \tag{6.5}$$

6.2 Properties of the quadratic forms in angular variables

Let t be the transformation (matrix) which brings our quadratic form to the canonical form:

$$\tilde{t}\, a\, t = \mathcal{I}, \tag{6.6}$$

where $\mathcal{I}$ is the unit matrix $n \times n$, and $\tilde{t}_{kl} = t_{lk}$. Then the equality takes place for the Jacobian of this transformation

$$(det\,||t||)^{-2} = J_a^2(z) = det\,||a||, \qquad (det\,||t||)^{-1} = J_a(z) = \sqrt{det\,||a||}. \tag{6.7}$$

To obtain the relation (4.12) we write first the recurrent relation for the quadratic form

$$Q_{N+1}(z, \varphi_k, \varphi_l) = Q_N(z, \varphi_k, \varphi_l) + \varphi_N^2 - \varphi_N\, \varphi_{N-l}/z, \tag{6.8}$$

then rewrite it similar to Eq. (4.11), and write down the equality for the last several terms

$$\frac{J_N^2}{J_{N-1}^2}\varphi_{N-1}^2 + \varphi_N^2 - \frac{\varphi_N \varphi_{N-1}}{z} = \frac{J_N^2}{J_{N-1}^2}\left(\varphi_{N-1} - \frac{J_{N-1}^2}{J_N^2}\frac{\varphi_N}{2z}\right)^2 + \frac{J_{N+1}^2}{J_N^2}\varphi_N^2. \tag{6.9}$$

From equality of coefficients before φ_N^2 in the left and right sides we obtain

$$1 = \frac{J_{N-1}^2}{4z^2 J_N^2} + \frac{J_{N+1}^2}{J_N^2}, \qquad (6.10),$$

and equation (4.12) follows immediately.

The relation can be obtained from Eq. (4.12)

$$J_N^2(z) = J_{N-k}^2(z) J_{k+1}^2(z) - \frac{1}{4z^2} J_{N-k-1}^2(z) J_k^2(z) \tag{6.11}$$

which, at $N = 2m,\ k = m$ (m is the integer), leads to remarkable relation

$$J_{2m}^2(z) = J_m^2(z)\left(J_{m+1}^2(z) - \frac{1}{4z^2} J_{m-1}^2(z)\right). \tag{6.12}$$

Relation (6.10) can be verified easily for J^2_4, J^2_6 and J^2_8, see section 4. It follows from (6.12) that at $N = 2m$ not only $J_N(\pi/N) = 0$, but also $J_N(2\pi/N) = 0$ which has quite simple explanation ($m \geq 3$; the case of $m = 2$ is an exception).

For the odd values of N another useful factorization property takes place:

$$J^2_{2m+1}(z) = \left(J^2_{m+1}(z)\right)^2 - \frac{1}{4z^2}\left(J^2_m(z)\right)^2 =$$
$$= \left(J^2_{m+1}(z) - \frac{1}{2z}J^2_m(z)\right)\left(J^2_{m+1}(z) + \frac{1}{2z}J^2_m(z)\right), \qquad (6.13)$$

which can be easily verified for J^2_7 and J^2_5 given in section 4.

The polinomials J^2_N and equations for $z^\pi_N = cos(\pi/N)$ can be obtained in more conventional way. There is an obvious equality

$$[exp(i\pi/N)]^N = exp(i\pi) = -1 \qquad (6.14)$$

It can be written in the form

$$[cos(\pi/N) + isin(\pi/N)]^N = -1, \qquad (6.15)$$

or separately for the real and imaginary parts

$$Re\left\{[cos(\pi/N) + isin(\pi/N)]^N\right\} = -1,$$
$$Im\left\{[cos(\pi/N) + isin(\pi/N)]^N\right\} = 0. \qquad (6.16)$$

The polinomials in $z^\pi_N = cos(\pi/N)$ which are obtained in the left side of (6.13) coincide with polinomials obtained in section 4. However, some further efforts are necessary to get recurrent relations (6.9), (6.10).

6.3 Cross-checking of basic relations with the help of trigonometrical identities

The following useful relations can be verified:

$$\left(2z^\theta_N\right)^{N-1} J^2_N\left(z^\theta_N\right) sin\frac{\theta}{N} = sin\theta. \qquad (6.17)$$

Using expressions for $J^2_N(z)$ given in (4.16) and (4.17) it is possible to reproduce the following equalities known from trigonometry:

$$\left(2\,cos\frac{\theta}{3}\right)^2\left(1 - \frac{1}{4cos^2(\theta/3)}\right) sin\frac{\theta}{3} = sin\,\theta, \qquad N = 3; \qquad (6.18)$$

$$\left(2\,cos\frac{\theta}{4}\right)^3\left(1-\frac{1}{2cos^2(\theta/4)}\right)sin\frac{\theta}{4}=sin\,\theta, \qquad N=4; \tag{6.19}$$

$$\left(2\,cos\frac{\theta}{5}\right)^4\left(1-\frac{3}{4cos^2(\theta/5)}+\frac{1}{16cos^4(\theta/5)}\right)sin\frac{\theta}{5}=sin\,\theta, \quad N=5; \tag{6.20}$$

etc. Obviously, the right side of these equalities equals zero at $\theta=\pi$, but $sin(\pi/N)$ is different from zero for any integer $N\geq 2$. Therefore, the polinomial in $cos(\pi/N)$ in the left side of (6.17) $-$ (6.20) should be equal to zero. These relations provide the link between the general case considered at the beginning of section 4 and the particular case of the optimal kinematics with all scattering angles equal to θ/N.

Now, using Eq. (6.17), we can cross-check the basic recurrent relation (4.12): $J^2_{N+1}(z)=J^2_N(z)-J^2_{N-1}(z)/4z^2$. We have from (6.17)

$$J^2_{N+1}=\frac{sin[(N+1)\phi]}{sin\phi\;(2cos\phi)^N};\; J^2_N=\frac{sin(N\phi)}{sin\phi\;(2cos\phi)^{N-1}};\; J^2_{N-1}=\frac{sin[(N-1)\phi]}{sin\phi\;(2cos\phi)^{N-1}}.$$

We are using the notation here $z=cos\,\phi$. After removal of some common factors Eq. (4.12) reduces to

$$sin(N+1)\phi=2cos\phi\;sin(N\phi)-sin(N-1)\phi, \tag{4.12a}$$

which can be easily checked with well known trigonometrical identities.

Another relation of interest which can be cheked in similar way is (6.12). From (6.17)

$$J^2_{2m}=\frac{sin(2m\phi)}{sin\phi\;(2cos\phi)^{2m-1}};\; J^2_{m+1}=\frac{sin(m+1)\phi}{sin\phi\;(2cos\phi)^m};\; J^2_{m-1}=\frac{sin((m-1)\phi}{sin\phi\;(2cos\phi)^{m-2}}$$

and (6.12) takes the form

$$\frac{sin(2m\phi)}{sin\phi\;(2cos\phi)^{2m-1}}=\frac{sin(m\phi)}{sin\phi\;(2cos\phi)^{m-1}}\times$$

$$\times\left[\frac{sin(m+1)\phi}{sin\phi\;(2cos\phi)^m}-\frac{sin(m-1)\phi}{sin\phi\;(2cos\phi)^m}\right], \tag{6.12a}$$

and after cancellation of common factors we come to

$$sin(2m\phi)=\frac{sin(m\phi)}{sin\phi}\left[sin(m+1)\phi-sin(m-1)\phi\right], \tag{6.12b}$$

which can be easily verified using well known trigonometrical relations.

The relation (6.13) can be cross-checked in similar way. From (6.17) we obtain

$$J^2_{2m+1} = \frac{sin(2m+1)\phi}{sin\phi\ (2cos\phi)^{2m}};\ J^2_m = \frac{sin(m\phi)}{sin\phi\ (2cos\phi)^{m-1}},$$

and J^2_{m+1} was presented above as function of sines and cosines. After cancellation of some common factors we come to the following relation

$$sin(2m+1)\phi\ \ sin\,\phi = sin^2(m+1)\phi - sin^2(m\phi) =$$

$$= cos^2(m\phi) - cos^2(m+1)\phi. \qquad (6.13a)$$

The left side should be transformed according to the known identity

$$sin\alpha\ sin\beta = \left[cos(\alpha-\beta) - cos(\alpha+\beta)\right]/2 = cos^2\frac{\alpha-\beta}{2} - cos^2\frac{\alpha+\beta}{2},$$

with $(\alpha-\beta)/2 = m\phi$, $(\alpha+\beta)/2 = (m+1)\phi$, and relation (6.13) is confirmed.

7 Discussion and conclusions

The nature of the cumulative particles is complicated and not well understood so far. There are different possible sources of their origin, including the color forces [75], one of them are the multiple collisions inside the nucleus, i.e. elastic or inelastic rescatterings. We have shown that the enhancement of the particles production cross section off nuclei near the backward direction, the glory-like backward focusing effect, is a natural property of the multiple interaction mechanism for the cumulative particles production. It takes place for any multiplicity of the process, $N \geq 3$, when the momentum of the emitted particle is close to the corresponding kinematical boundary. The universal dependence of the cross section, $d\sigma \sim 1/\sqrt{\pi - \theta}$ near the final angle $\theta \sim \pi$, takes place regardless the multiplicity of the process. This statement by itself is quite rigorous and presented for the first time in [76] and in present paper. The competition of the processes of different multiplicities can make this effect difficult for observation in some cases. Presently we can speak only about qualitative, in some cases semiquantitative agreement with data. It is not clear yet how the transition to strictly backward direction proceeds. The angular distribution of emitted particles near $\theta = \pi$ can have a narrow dip, i.e. it may be of a crater- (funnel)-like form. Further studies are necessary for better understanding of this point.

The glory-like backward focusing effect, observed in a number of experiments at JINR and ITEP, is a clear manifestions of the fact that multiple interactions make important contribution to the cumulative particles production probability, although it does not exclude the contribution of interaction of the projectile with few-nucleon, or multiquark clusters possibly existing in nuclei. Few characteristic features of the cumulative production reactions discussed in section 5 also confirm the importance of the MIP mechanism: the sharp A-number dependence of the cumulativeproduction cross sections, the mixing between produced cumulative neutrons and protons, and the relatively large polarization of cumulative baryons. We have proved the existence of the azimuthal focusing for arbitrary polar angles (rescattering of the light particles) and for the case of the optimal (basic) configuration of the MIP, also for nucleons rescattering. Investigation of other possible variants of the optimal kinematical configurations, besides those considered in present paper may be of interest, but obviously, the azimuthal focusing, discussed e.g. in

[58] for the optical glory phenomenon, takes place for any kind of MIP; only some technical details are different.

It would be important to detect the focusing effect for different types of produced particles, baryons and mesons. This effect can be considered as a "smoking gun" of the MIP mechanism. If this nuclear glory-like phenomenon is observed for all kinds of cumulative particles, its universality would be a strong argument in favor of importance of MIP. Reactions where such effect is not observed would provide more chances for revealing nontrivial peculiarities of nuclear structure. The role of the multiple interaction processes leading to the large angle particles production off nuclei is certainly underestimated, still, by many authors, theoreticians and experimentalists. Further efforts are necessary to settle this extremely difficult and important challenge of disentangling between the nontrivial effects of the nuclear structure and the MIP contributions.

8 Acknowledgements

We are indebted to academician Vladimir M.Lobashev who supported strongly the main idea that the background multiple interaction processes should be investigated and their contribution should be subtracted from measured cross sections to determine the weight of few-nucleon or multiquark clusters in nuclei.

We are thankful to Stepan Shimansky, whose questions, remarks and activity stimulated appearance of present paper, and also to A.B.Kurepin, V.L.Matushko for useful discussions. We thank Boris Kopeliovich, Anna Krutenkova for careful reading the manuscript and many useful remarks and suggestions. It was a pleasure for us to discuss the results of our investigation with academicians S.S.Gershtein and V.A.Matveev. We thank Prof. H.Clement for informing us about results of the experiment [50] and theoretical investigation [53].

The present article is based on the papers J. Phys. G; Particles and Nuclei, 41 (2014) 125107 (supported by Fondecyt, Chile, grant N 1130549), Sov. J. Nucl. Phys. 26 (1977) 87 and Phys. Reports 139 (1986) 51.

9 References

1. D.I. Blokhintsev. On the fluctuations of nuclear matter. JETP 33 (1957) 988 [ZhETF 33 (1957) 1295]
2. A.M. Baldin. Cumulative meson production in interactions of relativistic deuterons with nuclei. Published in eConf C720906V1 (1972) 277-278
3. A.M. Baldin, S.B. Gerasimov, N. Guiordenescu, V.N. Zubarev, L.K. Ivanova, A.D. Kirillov, V.A. Kuznetsov, N.S. Moroz, V.B. Radomanov, V.N. Ramzhin et al. Cumulative meson production. Yad.Fiz. 18 (1973) 79
4. Yu.D. Bayukov, L.S. Vorobev, G.A. Leksin, V.L. Stolin, V.B. Fedorov, V.D. Khovanskii. Effects of scaling invariance in backward proton-nucleus scattering at energies of few GeV. Sov.J.Nucl.Phys. 18 (1974) 639 [Yad.Fiz. 18 (1973) 1246]
5. Yu.D. Bayukov, L.S. Vorobev, G.A. Leksin, L.S. Novikov, V.L. Stolin, V.B. Fedorov, V.D. Khovanskii, N.N. Shishov. Invariant effects in backward emission of p, d, t by nuclei bombarded by 0.7-GeV to 6-GeV elementary particles. Sov.J.Nucl.Phys. 19 (1974) 648 [Yad.Fiz. 19 (1974) 1266]
6. V.B. Gavrilov, G.A. Leksin. On the Phenomenological Description of Nuclear Scaling. 1975. 8 pp. ITEP-36-1975
7. Yu.D. Bayukov, V.I. Efremenko, S. Frankel, W. Frati, M. Gazzaly, G.A. Leksin, N.A. Nikiforov, C.F. Perdrisat, V.I. Chistilin, Yu.M. Zaitsev. Backward Production of Protons in Nuclear Reations with 400-GeV Protons. Phys.Rev. C20 (1979) 764
8. N.A. Nikiforov, Yu.D. Bayukov, V.I. Efremenko, G.A. Leksin, V.I. Chistilin, Yu.M. Zaitsev, S. Frankel, W. Frati, M. Gazzaly, C.F. Perdrisat. Backward Production Of Pions And Kaons In The Interaction Of 400-Gev Protons With Nuclei Phys.Rev. C22 (1980) 700
9. Yu.M. Antipov, V.A. Batarin, V.A. Bezzubov, M.S. Bilenky et al. Inclusive Cross-section For Production Of Cumulative Protons In Pi- (K, Anti-p) A Interactions At 40-gev/c. Sov.J.Nucl.Phys. 53 (1991) 274 [Yad.Fiz. 53 (1991) 439]
10. Yu.M. Antipov, V.A. Batarin, V.A. Bezzubov et al. Cross-sections of backward proton production in 40-GeV/c pi- (K-, anti-p) A interactions. Nucl.Phys. A536 (1992) 637
11. B.M. Abramov, P.N. Alekseev, Yu. A. Borodin, S.A. Bulychjov, I.A. Dukhovskoy, A.I. Khanov, A.P. Krutenkova, V.V. Kulikov, M.A. Martemyanov, M.A. Matsyuk, E.N.Tourdakina. Quark cluster contribution to cumulative proton emission in frag-

mentation of carbon ions. Pis'ma v ZhETF (2013) 97, 509; e-Print: arXiv:1304.6220 [nucl-ex]

12. L.L. Frankfurt, M.I. Strikman High-Energy Phenomena, Short Range Nuclear Structure and QCD. Phys.Rept. 76 (1981) 215
13. L.A. Kondratyuk, M.Zh. Shmatikov. Multiquark Clusters in Nuclei. Distributions Of U And D Quarks. Z.Phys. A321, 301 (1985)
14. L.A. Kondratyuk, M.Zh. Shmatikov. Cumulative Processes And Quark Distributions In Nuclei. Sov.J.Nucl.Phys. 41 (1985) 317 [Yad.Fiz. 41 (1985) 498]
15. A.V. Efremov, A.B. Kaidalov, V.T. Kim, G.I. Lykasov, N.V. Slavin. Cumulative Hadron Production In Quark Models Of Flucton Fragmentation. Sov.J.Nucl.Phys. 47 (1988) 868 [Yad.Fiz. 47 (1988) 1364]
16. L.L. Frankfurt, M.I. Strikman. Hard Nuclear Processes and Microscopic Nuclear Structure. Phys.Rept. 160 (1988) 235
17. W. Cassing, E.L. Bratkovskaya. Hadronic and electromagnetic probes of hot and dense nuclear matter. Phys.Rept. 308 (1999) 65
18. A.S.Galoian, G.L.Melkumov, V.V.Uzhinsky. Analysis of charged particle production in nucleus nucleus interactions up to and out of kinematical limit of free NN collisions in the frame of FRITIOF model. Phys.Atom.Nucl. 65 (2002) 1722 [Yad.Fiz. 65 (2002) 1766]
19. R.J. Glauber. Cross Sections in Deuterium at High Energies. Phys. Rev. 100 (1955) 242
20. R.J. Glauber and V. Franco. High-Energy Deuteron Cross Sections: Charge-Exchange Effects. Phys. Rev. 156 (1967) 1685
21. V.N. Gribov. Glauber Corrections and the Interaction between High-energy Hadrons and Nuclei. JETP, 29 (1969) 483 [ZhETF 56 (1969) 892]
22. A.B. Kaidalov, A.P. Krutenkova. Implications of inelastic rescatterings for pion double charge exchange on nuclei . J.Phys. G27 (2001) 893
23. A.P. Krutenkova, T. Watanabe, D. Abe, Y. Fujii, O. Hashimoto,V/V. Kulikov et al, Inclusive pion double charge exchange on ^{16}O above the Delta resonance, Phys. Rev. C72 (2005) 037602
24. L.A. Kondratyuk, V.B. Kopeliovich. Concerning one mechanism of 'cumulative' meson production. JETP Lett. 21 (1975) 40 [Pisma Zh.Eksp.Teor.Fiz. 21 (1975) 88]
25. V.B. Kopeliovich. Multiple Processes with Proton Production on Nuclei in the Range Forbidden for NN Interactions According to Kinematics. JETP Lett. 23 (1976) 313

[Pisma Zh.Eksp.Teor.Fiz. 23 (1976) 348]

26. V.B. Kopeliovich. Production of Relativistic Particles on Nuclei at Large Angles and Role of Multiple Processes. Sov.J.Nucl.Phys. 26 (1977) 87 [Yad.Fiz. 26 (1977) 168]
27. M.A. Braun, V.V Vechernin. Contribution of Inelastic Rescattering to the Cumulative Meson Production. Yad.Fiz. 25 (1977) 1276
28. M.A. Braun, Yu.A. Evlashev, V.V Vechernin . Inelastic Rescattering Contribution to the Cumulative Production of Pions in the Statistical Bootstrap Model. Yad.Fiz. 27 (1978) 1329
29. M.A. Braun, V.V Vechernin, A.K. Vlasnikov. Effect of Rescattering in Electroproduction of Cumulative Pions. Yad.Fiz. 28 (1978) 142
30. M.A. Braun, V.V Vechernin. Quasielastic and Inelastic Rescattering as a Source of Cumulative Particle Production on Deuteron. Yad.Fiz. 28 (1978) 1466
31. M.A. Braun, V.V. Vechernin. On The Importance Of Pion Rescattering In Cumulative Proton Production Of Deuterons. Yad.Fiz. 40 (1984) 1588
32. M.A. Braun, V.V. Vechernin. Pion Rescattering Contribution To Proton Cumulative Production On Deuteron. Yad.Fiz. 43 (1986) 1579
33. M.A. Braun, V.V. Vechernin. On interference of cumulative proton production mechanisms. J.Phys. G19 (1993) 517
34. M.A. Braun, V.V. Vechernin. Separation of the contributions originating from short-range and long-range nuclear phenomena for cumulative proton production. J.Phys. G19 (1993) 531
35. M.A. Braun, V.V. Vechernin, B. Vlahovic. Leptoproduction of cumulative nucleons. Fizika B13 (1997) 337
36. V.B. Kopeliovich. Long and Intermediate Range Phenomena: Selected Topics In High-energy Nuclear Physics. Phys.Rept. 139 (1986) 51
37. V.B. Kopeliovich. Mixing Effect And The Ratio Of Neutron And Proton Yields From Nuclei In The Kinematically Forbidden Region. Sov.J.Nucl.Phys. 39 (1984) 383 [Yad.Fiz. 39 (1984) 606]
38. V.B. Kopeliovich and V.B. Radomanov. A-dependence Of Inclusive Cross-sections For Particles Production On Nuclei In KFR And Multiple Interaction Processes. Sov.J.Nucl.Phys. 40 (1984) 686 [Yad.Fiz. 40 (1984) 1078]
39. A.A. Sibirtsev, N.V. Stepanov. Multiple Rescattering In The Intranuclear Cascade Model. 1985. 23 pp. ITEP-38-1985

40. A.A. Sibirtsev, N.V. Stepanov, Yu.V. Trebukhovsky. Proton Production In Kinematically Forbidden Range. Yad.Fiz. 42 (1985) 482

41. L.G. Dakhno, V.A. Nikonov. Application Of The Multiple Scattering Theory to Calculation Of The Cumulative Proton Production In The Reaction $PD \to PPN$. Nucl.Phys. A491 (1989) 652

42. S.G. Mashnik. Neutron induced particle production in the cumulative and noncumulative regions at intermediate-energies. Nucl.Phys. A568 (1994) 703; JINR-E2-92-320

43. P. Astier et al. NOMAD Collaboration. A Study of backward going p and pi in neutrino (muon) CC interactions with the NOMAD detector. Nucl.Phys. B609 (2001) 255

44. M. Veltri for NOMAD Collaboration. A Study of nuclear effects in neutrino interactions with the NOMAD detector. Nucl.Phys.Proc.Suppl. 112 (2002) 124; e-Print: hep-ex/0202038

45. A.V. Stavinsky (Moscow, ITEP) et al., CLAS Collaboration. Proton source size measurements in the $eA \to e'ppX$ reaction. Phys.Rev.Lett. 93 (2004) 192301

46. A.M. Baldin, V.K. Bondarev, L.B. Golovanov et al, Cumulation of Light Nuclei. JINR-P1-11168, 1977

47. V.S. Stavinsky. Limiting Fragmentation Of Nuclei - The Cumulative Effect. Fiz. Elem. Chast .Atom. Yadra 10 (1979) 949

48. Yu.D. Bayukov, V.B. Gavrilov, N.A. Goryainov et al. Angular Dependence Of Yield And Spectrum Of Protons Produced In pA Interactions At 7.5 GeV/c. 10 pp. ITEP-90-1981

49. L.S. Vorob'ev, V.B. Gavrilov, N.A. Goryainov et al. Angular Distribution Of P, D And Pi From The Reaction $\pi(5Gev/c)A(C, Pb) \to \alpha(p, D, \pi)X$ Near 180-degrees In The Lab System. 1984. 20 pp. ITEP-24-1984

50. M. Ermer, H. Clement, P. Grabmayer, et al, Peculiar Features of the Deuteron Scattering Potential. Phys. Lett. (1987) 17

51. L.S.Vorob'ev, V.B. Gavrilov, N.A. Goryainov, Yu.G. Grishuk, O.B. Gushchin, P.V. Degtyarenko, B.L. Druzhinin, Yu.V. Efremenko, L.N. Kondratev, G.A. Leksin et al. Irregularities In Angular Distribution Of Cumulative Particles Near 180-degrees In Lab. Sov.J.Nucl.Phys. 44 (1986) 908 [Yad.Fiz. 44 (1986) 1396]

52. L.S.Vorob'ev, V.B. Gavrilov, Yu.G. Grishuk, P.V. Degtyarenko, Yu.V. Efremenko, B.V. Zagreev, M.V. Kosov, S.V. Kuleshov, G.A. Leksin, N.A. Pivnyuk et al. Observation of an anomaly in the angular distribution of cumulative hadrons near the

backward direction. Phys.Atom.Nucl. 63 (2000) 145 [Yad.Fiz. 63 (2000) 197]

53. K.W. Ford and J.A. Wheeler. Semiclassical Description of Scattering. Ann. Phys. 7 (1959) 259
54. V.I. Komarov, H. Muller, S. Tesch. Cluster Excitation In Hadron Nucleus Collisions At Medium And High-energies. Fortsch.Phys. 33 (1985) 595
55. A. Malki, J. Alster, G. Asrian), et al. Backward emitted high-energy neutrons in hard reactions of p and pi+ on carbon. Phys.Rev. C65 (2002) 015207
56. L.D. Landau, I. Pomeranchuk. Limits of applicability of the theory of bremsstrahlung electrons and pair production at high-energies. Dokl.Akad.Nauk Ser.Fiz. 92 (1953) 535
57. N.N.Nikolaev. Quarks in High Energy Interactions of Hadrons, Photons and Leptons with Nuclei. Sov.Phys.Usp. 24 (1981) 531 [Usp.Fiz.Nauk 134, 369 (1981)]
58. V. Khare, H.M. Nussenzveig. Theory of the Glory. Phys.Rev.Lett. 38 (1977) 1279; IFUSP/P-111, Apr 1977.
59. A.V. Maiorova, D.A. Telnov, V.M. Shabaev, V.A. Zaitsev, G. Plunien, T. Stohlker. Coulomb glory effect in collisions of antiprotons with heavy nuclei: relativistic theory J.Phys. B43 (2010) 205006; arXiv:0912.5203
60. K.V. Alanakian, M.D. Amarian, R.A. Demirchyan et al. Angular Dependence Of The Yield of Cumulative Protons Produced in Gamma A Interactions at E(gamma) Max = 4.5-Gev. Sov.J.Nucl.Phys. 37 (1983) 385 [Yad.Fiz. 37 (1983) 647]
61. Yu.D. Bayukov, V.B. Gavrilov, N.A. Goryainov, Yu.G. Grishuk, O.B. Gushchin, P.V. Degtyarenko, N.L. Kornienko, G.A. Leksin, V.B. Fedorov, B.B. Shvartsman et al. Yields of Protons and Neutrons under the Angle of 90-degrees from Nuclei C, Cu, Pb, U Induced by Pi+- Mesons. Sov.J.Nucl.Phys. 35 (1982) 558 [Yad. Fiz. 35 (1982) 960]
62. Yu.D. Bayukov, A.V. Vlasov, V.B. Gavrilov, N.A. Goryainov, Yu.G. Grishuk, O.B. Gushchin, P.V. Degtyarenko, N.L. Kornienko, M.V. Kosov, G.A. Leksin et al. Proton And Neutron Yields From Nuclei Including Separated Nickel And Tin Isotopes. Sov.J.Nucl.Phys. 37 (1983) 206 [Yad. Fiz. 37 (1983) 344]
63. G.A. Leksin, A.V. Smirnitsky. Cumulative Lambda0 Hyperon Polarization. ITEP-87-1977 (Moscow, ITEP). 1977. 12 pp.
64. N.A. Burgov, M.K. Vlasov, L.S. Vorobev, S.A. Gerzon, Yu.T. Kiselev, G.A. Leksin, A.N. Martemyanov, N.A. Pivnyuk, V.L. Stolin, Yu.V. Terekhov. Polarized Cumulative Protons. JETP Lett. 31 (1980) 660 [Pisma Zh.Eksp.Teor.Fiz. 31 (1980) 700]

65. N.A. Burgov et al. Measurement Of The Polarization Of Cumulative Protons, Emitted At Angles Of 110-degrees And 135-degrees In The Laboratory System In Reactions Pi- A $\to$ P X At 1.5-gev/c Pion Momenta. Sov.J.Nucl.Phys. 39 (1984) 506 [Yad.Fiz. 39 (1984) 801]
66. L.S. Vorobev, V.B. Gavrilov, N.A. Goryainov, Yu.G. Grishuk, O.B. Gushchin, P.V. Degtyarenko, Yu.V. Efremenko, G.A. Leksin, N.A. Pivnyuk, V.B. Fedorov et al. Measurement Of Polarization Of Cumulative Protons. Yad.Fiz. 41 (1985) 1541
67. O.P. Gavrishchuk, V.F. Peresedov, N.V. Vlasov, L.S. Zolin, I.M. Belyaev. Polarization of backwardly produced protons in proton nucleus collisions at 17-GeV - 64-GeV Phys.Lett. B255 (1991) 321
68. I.M. Belyaev, N.V. Vlasov, O.P. Gavrishchuk, L.S. Zolin, V.F Peresedov. Spin Effects in Cumulative Production of Protons and Deterons in Proton - Nucleus Interactions at 16-GeV to 64-GeV. In *Bonn 1990, Proceedings, High energy spin physics, vol. 1*, 484-488; JINR-P1-89-463
69. S. Afanasev, Yu. Anisimov, M. Fujieda, S. Fukui, T. Hasegawa, N. Horikawa, A. Isupov, T. Iwata, A. Izotov, T. Kageya et al. Tensor analyzing power T(20) for cumulative pion production from deuterons in a GeV energy region. Nucl.Phys. A625 (1997) 817
70. S. Afanasev, V. Arkhipov, V. Bondarev, M. Ehara, G. Filipov, S. Fukui, S. Hasegawa, T. Hasegawa, N. Horikawa, A. Isupov et al. Fragmentation of tensor polarized deuterons into cumulative pions. Phys.Lett. B445 (1998) 14; JINR-E1-98-319
71. L.S. Zolin, S.V. Afanasev, V.V. Arkhipov, V.K. Bondarev, G. Filippov, M. Ehara, S. Fukui, S. Hasegawa, T. Hasegawa, N. Horikawa et al. Measurement of the tensor analyzing power T(20) in the fragmentation of 9-GeV tensor polarized deuterons into cumulative pions. Few Body Syst.Suppl. 10 (1999) 455
72. S. Afanasev, V. Arkhipov, V. Bondarev, I. Daito, N. Doushita, S. Fukui, N. Horikawa, T. Iwata, A. Isupov, A. Khrenov et al. Spin effects at fragmentation of polarized deuterons into cumulative pions. Prepared for Conference: C01-07-30.3, AIP Conf. Proc. 610 (2001) 395
73. L. L. Zolin for SPHERA Collaboration. Behavior of T(20), A(yy), A(y) spin observables at fragmentation of polarized deuterons into cumulative hadrons. Czech.J.Phys. 52 (2002) C313

74. S.Afanasiev, V.Arkhipov, V.Bondarev, I.Daito et al. Fragmentation of polarized deuteron into high momentum mesons as source of spin information of deuteron core structure. Nucl. Phys. A721 (2003) 645

75. B.Z. Kopeliovich, F. Niedermayer. Color Forces And The Cumulative Effect In Scattering Of Hadrons By Deuterons. Sov.Phys.JETP 60 (1984) 640 [Zh.Eksp.Teor.Fiz. 87 (1984) 1121]

76. V.B. Kopeliovich, G.K. Matushko, and I.K. Potashnikova. 'Buddha's light' of cumulative particles. J.Phys. G41 (2014) 12, 125107; e-Print: arXiv:1403.1992 [nucl-th]