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L.Hlavatý

SOLUTION TO THE YANG-BAXTER EQUATION CORRESPONDING TO THE XXZ MODELS IN AN EXTERNAL MAGNETIC FIELD

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Two-dimensional statistical model of antiferroelectrics was solved by Lieb [1]. He observed that the transfer-matrix has the same eigenvectors as the Hamiltonian for the Heisenberg's XXZ model of the one-dimensional ferromagnetics given by the Hamiltonian

$$H = \mathrm{const} \sum_{n=1}^{N} \left(\sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y + J \sigma_n^z \sigma_{n+1}^z \right). \qquad (1)$$

Soon afterwards, a generalization of Lieb's result, namely the model of ferroelectrics in an arbitrary external field, was presented by C. P. Yang [2]. An attempt at corresponding generalization of (1) was made in [3] but the Hamiltonian turned non-hermitian and thus physically unattractive. The transfer-matrix only with horizontal external field was considered in [3] .

In this paper we point out that if a more general transfer-matrix with both horizontal and vertical electric field is used then the Hamiltonian

$$H = \mathrm{const} \sum_{n=1}^{N} \left(\sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y + J \sigma_n^z \sigma_{n+1}^z \right) + h \sum_{n=1}^{N} \sigma_n^z , \qquad (2)$$

that represents the XXZ (or XXX) model in an external magnetic field, is obtained.

The most important point consist in search for a solution of the Yang-Baxter equation (see e.g. [4])

$$R_{\alpha_1\alpha_2}^{\beta_1\beta_2}(\lambda,\mu)\, R_{\beta_1\alpha_3}^{\gamma_1\beta_3}(\lambda,\nu)\, R_{\beta_2\beta_3}^{\gamma_2\gamma_3}(\mu,\nu) = \qquad (3)$$

$$= R_{\alpha_2\alpha_3}^{\beta_2\beta_3}(\mu,\nu)\, R_{\alpha_1\beta_3}^{\beta_1\gamma_3}(\lambda,\nu)\, R_{\beta_1\beta_2}^{\gamma_1\gamma_2}(\lambda,\mu)$$

that would be a true function of two variables, not just a function of the difference $\lambda-\mu$. Such a solution is

$$R(\lambda,\mu) = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & c & 0 \\ 0 & c & b' & 0 \\ 0 & 0 & 0 & a' \end{pmatrix}, \tag{4}$$

where

$$a = D(\mu)D(\lambda)^{-1}F(\mu-\lambda+\eta), \tag{5a}$$

$$a' = D(\lambda)D(\mu)^{-1}F(\mu-\lambda+\eta), \tag{5a'}$$

$$b = D(\lambda)D(\mu)F(-\lambda+\mu), \tag{5b}$$

$$b' = D(\lambda)^{-1}D(\mu)^{-1}F(\mu-\lambda), \tag{5b'}$$

$$c = F(\eta), \tag{5c}$$

D is an arbitrary function, η is a constant, and

$$F(x) = \sin x \quad \text{or} \quad F(x) = x. \tag{6}$$

This asymmetric 6-vertex solution corresponds to the two-dimensional statistical model [2] of ferroelectrics in an external electric field having both horizontal and vertical components and also to the Ising model in two dimensions [5], [6] with both first and second neighbour interaction.

Having the solution of (3) we can proceed to look for the Hamiltonian constructing the transfer-matrix $T(\lambda,\nu)$ such that

$$[T(\lambda,\nu),\ T(\mu,\nu)] = 0 \tag{7}$$

and define

$$H(\nu) := \frac{d}{d\lambda} \ln T(\lambda,\nu)\Big|_{\lambda=\nu}. \tag{8}$$

For $R(\lambda,\nu)$ such that

$$R^{\gamma\delta}_{\alpha\beta}(\nu,\nu) = f(\nu)\,\delta^{\delta}_{\alpha}\delta^{\gamma}_{\beta} \tag{9}$$

one gets (see e.g. [7])

$$H(\nu) = f(\nu)^{-1}\sum_{n=1}^{N}\sum_{k,l=0}^{3} J_{kl}(\nu)\,\sigma^k_n\sigma^l_{n+1}, \tag{10a}$$

where σ^k_n are Pauli matrices and

$$J_{kl} := \partial_\lambda \rho_{kl}(\lambda,\nu)\Big|_{\lambda=\nu}, \tag{10b}$$

$$R^{\beta_1\beta_2}_{\alpha_1\alpha_2}(\lambda,\nu) = \rho_{kl}(\lambda,\nu)\,\sigma^k_{\alpha_1\beta_2}\sigma^l_{\alpha_2\beta_1}. \tag{10c}$$

Inserting (4) - (6) into (10) we get

$$H(\nu) = [2F(\eta)]^{-1}\sum_{n=1}^{N}\Big\{(\sigma^x_n\sigma^x_{n+1}+\sigma^y_n\sigma^y_{n+1})C(\nu) + \sigma^z_n\sigma^z_{n+1}F'(\eta) - iS(\nu)(\sigma^x_n\sigma^y_{n+1}-\sigma^y_n\sigma^x_{n+1})\Big\} + D'(\nu)D(\nu)^{-1}\sum_{n=1}^{N}\sigma^z_n + \tfrac{1}{2}NF'(\eta)F(\eta)^{-1}\mathbb{1}_N, \tag{11a}$$

where

$$C(\nu) = \tfrac{1}{2}\left(D(\nu)^2 + D(\nu)^{-2}\right), \tag{11b}$$

$$S(\nu) = \tfrac{1}{2}\left(D(\nu)^2 - D(\nu)^{-2}\right). \tag{11c}$$

Notice that for $D(\nu)$ = const we get the quadratic form obtained in [3].

The arbitrariness of D and ν can be used to secure the self-adjointness of H. For $D(\nu) = \exp(h\nu)$, $\nu = i\pi/(4h)$ we get

$$H = \text{const}\sum_{n=1}^{N}(\sigma^x_n\sigma^x_{n+1} - \sigma^y_n\sigma^x_{n+1} + J\sigma^z_n\sigma^z_{n+1}) + h\sum_{n=1}^{N}\sigma^z_n + \text{const}\cdot N\mathbb{1}_N \tag{12}$$

while for $\nu = 0$ we get the Hamiltonian (2) up to a multiple of $\mathbb{1}_N$.

The eigenvalues of the transfer-matrix $T(\lambda,\nu)$ and the Hamiltonian (11) are (by a slight modification of [7])

$$\Lambda(\lambda,\nu;\lambda_1,\lambda_2,\dots\lambda_m) =$$
$$= \left[D(\lambda)D(\nu)^{-1}F(\lambda-\nu+\eta)\right]^N \prod_{l=1}^{m} \frac{F(\lambda_l-\lambda+\eta)}{F(\lambda_l-\lambda)}$$
$$+ \left[D(\lambda)D(\nu)F(\lambda-\nu)\right]^N \prod_{l=1}^{m} \frac{F(\lambda-\lambda_l+\eta)}{F(\lambda-\lambda_l)}, \qquad (13)$$

$$E(\nu;k_1,k_2,\dots,k_m) = N\left[F'(\eta)F(\eta)^{-1} + D'(\nu)D(\nu)^{-1}\right]$$
$$+ F(\eta)^{-1}\sum_{l=1}^{m}\left[F'(\eta) - \cos k_l\right], \qquad (14)$$

where F is given by (6), λ_j are solution of the Bethe's ansatz equations.

$$\left[\frac{F(\lambda_k-\nu+\eta)}{D(\nu)^2F(\lambda_k-\nu)}\right]^N = (-1)^{m-1}\prod_{\substack{l=1\\ l\neq k}}^{m}\frac{F(\lambda_k-\lambda_l+\eta)}{F(\lambda_l-\lambda_k+\eta)}, \qquad (15)$$

and

$$\exp(ik_j) = F(\lambda_j+\tfrac{\eta}{2})\,F(\lambda_j-\tfrac{\eta}{2})^{-1}. \qquad (16)$$

All presented models are therefore solvable by the Bethe's anzatz.

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Главаты Л. E5-85-959
Решение уравнений Янга-Бакстера, отвечающее XXZ модели во внешнем магнитном поле

В работе приводится несимметричное шествершинное решение уравнений Янга-Бакстера. Отвечающие гамильтонианы включают те, которые описывают XXZ-модель в присутствии магнитного поля. Формулы для анзаца Бете выписаны в явном виде.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

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Hlavatý L. E5-85-959
Solution to the Yang-Baxter Equation Corresponding to the XXZ Models in an External Magnetic Field

Asymmetric 6-vertex solution to the Yang-Baxter equation is presented. Corresponding Hamiltonians include that of the XXZ model in a magnetic field. Formulae of Bethe's ansatz are given.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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