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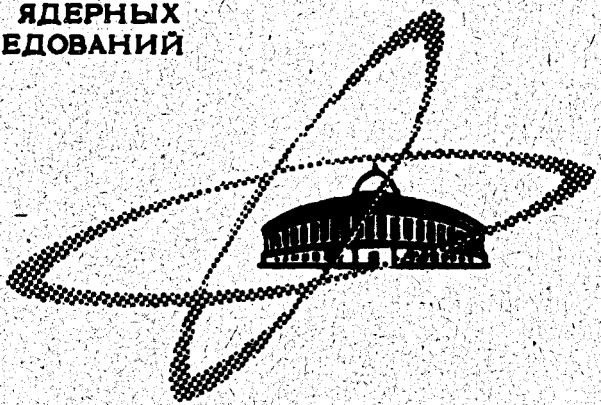

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GENERALIZED ROTONS IN AN ELLIOTT TYPE MODEL

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1. Introduction

In ref.[1] an approach to the nuclear rotation was developed based upon the assumption that the Hamiltonian of a nucleus can be split into two parts - the Hamiltonian of a rigid rotor (h) and an operator (H_i) describing the intrinsic excitations. The spectra in real nuclei differ from those of the rigid rotor in two points:

1). In the spectra of a nucleus one finds many levels with identical quantum numbers of the angular momentum. These states belong to different rotational bands and in general correspond to different values of the moment of inertia.

2). The bands in nuclei contain only a certain part of the rigid rotor states. E.g. the $K=0$ bands of even-even nuclei are composed of I even states, and the values of I in any band do not exceed some maximum value I_{max}

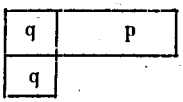
In this paper we take the Elliott model[2] to examine the theory formulated in ref.[1]. In the model the properties 1,2 of nuclear states are maintained, but all the moments of inertia are equal and the energies of states with a fixed I coincide. We show that all essential features of the theory may be left unchan-

ged while describing the Elliott model along the lines in ref.[1].

We show also the way in which the Elliott model may be generalized to account for the coupling between rotation and intrinsic excitations.

2. Multiplets of Broken SU_3 Symmetry

Suppose that a number of nuclear eigenstates $\{\Omega\}$ may be labelled by the eigenvalues of SU_3, representation in such a way that the energies of states are given by a simple function of the eigenvalues. We define the latter by a Young pattern



(1)

where the two integers p and q are such that

$$\Omega = (1+p)(1+q)\,[\,1+1/2\,(p+q)\,]$$

and by the two eigenvalues I, M of the O_3 subgroup of SU_3. Following ref.[2] we identify I, M with the quantum numbers of the angular momentum of a nuclear eigenstate. We distinguish between different states with the same I, M by an additional quantum number ω the definition of which is given in ref.[3] and is discussed in Appendix 1. In the following the two indices p, q are often represented by the symbol μ. The other indices marking states within the representation μ are represented by the symbol ν.

We come to the definition of a nuclear model assuming that the Hamiltonian H may be written as in the formula

$$H = H_1 + h(\hat{\mu}, \hat{\nu}) \equiv H_1 + E(\mu \to \hat{\mu}, \nu \to \hat{\nu}), \qquad (2)$$

where the quantity h depends on the operators $(\hat{\mu}, \hat{\nu})$ corresponding to the eigenvalues (μ, ν) precisely in the same way as the energy of nuclear eigenstates E depends on the quantities (μ, ν).

The structure of operators $(\hat{\mu}, \hat{\nu})$ in terms of nucleon fields ψ, ψ^+ should be regarded as unknown at this point and the aim of the theory is to find the nature of variables entering the collective Hamiltonian h. Having this in mind we are setting up a scheme, where all essential information on h and on the eigenstates is represented as in ref.[1], by the transition operators

$$E^{\nu_2}_{\nu_1}(\mu_0) \equiv |\mu_0, \nu_2\rangle\langle\mu_0, \nu_1| \qquad (3)$$

operating within the irreducible representation of SU_3.

The first problem in formulating the theory is to define a small number of elementary transition operators by means of which one could eventually find all the operators in eq. (3). This is solved by taking linear combinations of $E^{\nu'}_{\nu}$ which transform as basic states of irreps of SU_3

$$R^{+}_{\mu,\nu} = \sum_{\nu_1\nu_2} c^{\nu}_{\nu_1\nu_2}(\mu) E^{\nu_2}_{\nu_1}. \qquad (4)$$

The commutators of $R^{+}_{\mu,\nu}$ with the group generators entering h and defining the result of infinitesimal transformations are linear combinations of $R^{+}_{\mu,\nu'}$ with the same μ, and the weights of various ν' are specified uniquelly by the group indices ν, ν'.

Assuming, as in ref.[1], the commutators $[H, R^{+}_{\mu\nu}]$ being identical with $[h, R^{+}_{\mu\nu}]$ one finds that the different values of μ do not mix in the formula

$$[H, R^{+}_{\mu\nu}] = [h, R^{+}_{\mu\nu}] \equiv \sum_{\nu'} \tilde{J}_{\nu\nu'} R^{+}_{\mu\nu'} . \qquad (5)$$

Since the operators $R^{+}_{\mu\gamma}$ are linear combinations of the skew-projecting operators (3) in the space of eigenstates of an irrep of SU_3 they are determined completely by the projecting operator

$$P_{\mu_0} = \sum_{\nu} E^{\nu}_{\nu} \quad (\mu_0) \qquad (6)$$

and the eight generators A_{ij} $(i, j = 1, 2, 3;\ \sum_i A_{ii} = 0)$, satisfying the commutation relations

$$[A_{kl}, A_{\ell j}] = \delta_{\ell i} A_{kj} - \delta_{kj} A_{\ell i} . \qquad (7)$$

The operator in eq. (6) being obviously invariant under any unitary transformation of states within the irrep μ_0 is just one of the $R^{+}_{\mu\nu}$ operators:

$$R^{+}(N=1) = P_{\mu_0} \qquad (8)$$

(here and in eq. (9) N being the dimension of the representation μ). The operators

$$R^{+}_{ij} \quad (N=8) = \sqrt{\frac{3}{2g}}\, A_{ij} \qquad (9)$$

span the simplest nontrivial irrep of SU_3 (1,1). To complete the definition of the elementary transition operators $R^{+}_{\mu\nu}$ we

write down the value of the normalization constant g in eq. (9), chosen so that

$$\sum_{ij} R^{+}_{ij} R^{+}_{ji} = R^{+}(N=1). \tag{10}$$

The right value of g is

$$g = p^2 + pq + q^2 + 3(p+q). \tag{11}$$

Eqs. (7), (9) together with the formula for the model Hamiltonian h determine the r.h.s. of eq. (5) which in the present approach substitutes the dynamic equation for the roton operators in the earlier formulation (see eqs. (3.5) and (2.2), (2.4)) in the second paper referred to in[1]). We analyse this equation in the next two sections. A comment on the nature of the transition operators $R^{+}_{\mu,\nu}$ should be made here. Some of the operators R^{+}_{ij} are raising and lowering operators in SU_3. But in the basis where I is a quantum number they couple many states of various rotational bands. Thus the concept of R^{+}_{ij} is a generalization of the notion of a roton operator similar to that given in the last section of the second paper in ref.[1].

3. Transition Operators in the Elliott Model

The Elliott model corresponds to the simplest way of breaking the SU_3 invariance of the Hamiltonian. If one associates with the generators A_{ij} in eq. (7) certain one-body operators then any two-body scalar which operates within the representation of SU_3 is equivalent to the total angular momentum. That is, the simplest form of h is given by

$$h = \frac{a}{2} \vec{I}^{\,2} , \qquad (12)$$

where a is a constant and the components of $\vec{I}$ are determined by the group generators

$$A_{ij} - A_{ji} = i \sum_k \epsilon_{ijk} I_k \qquad (13)$$

ϵ_{ijk} being the completely antisymmetric third-rank tensor.

Another way of writing eq. (12) is

$$h = a (\Gamma - Q^2) = a \Gamma + h' , \qquad (12')$$

where $\Gamma = \Sigma_{ij} A_{ij} A_{ji}$ is the second-order Casimir operator of the group and $Q^2 = \Sigma_{ij} Q_{ij}^2$ is a scalar composed of the components of the tensor

$$Q_{ij} = \frac{1}{2} (A_{ij} + A_{ji}) . \qquad (13)$$

The commutation relations (7) imply definite commutation relations between Q_{ij} operators. The QQ interaction of eq. (12') in the many-body Hamiltonian and the commutation relations between the operators Q_{ij} lie in the foundation of the Elliott model. This makes the commutation relation (5) an operator identity in this case. It is easy to see that eq. (5) reduces to the dynamic equation of ref.[1] for the roton operators $R^+_{\ell=2,m}$. Indeed, the three of the operators R^+_{ij} are proportional to the angular momentum operator

$$\vec{R}^{+} = \sqrt{\frac{3}{2g}} P_{\mu_0} \vec{I} \qquad (14)$$

and therefore commute with the model Hamiltonian h. If the many-body Hamiltonian is a scalar, this part of eq. (5) is trivial. The r.h.s. of the remaining equations are determined uniquelly by the tensorial properties in O_3 of the operators

$$R^{+}_{\ell=2,m} = \sqrt{\frac{3}{2g}}\, P_{\mu 0}\, Q_m , \qquad (15)$$

where Q_m are the components of the quadrupole operators Q_{ij} in eq. (13) written in the spherical basis:

$$Q_0 = -\sqrt{\frac{3}{2}}\, Q_{33} , \quad Q_{\pm 1} = \pm (Q_{13} \pm i Q_{23}) ,$$

$$Q_{\pm 2} = -\frac{1}{2}(Q_{11} - Q_{22} \pm i 2 Q_{12}) .$$

Introducing the operators I_p $(p = 0, \pm 1)$ so that $I_0 = I_3$ $I_{\pm 1} = \mp 1/\sqrt{2}\,(I_1 \pm i I_2)$ one obtains

$$[H , R^{+}_{\ell=2,m}] = \frac{a}{2}\{ 2\sqrt{6} \sum_p (2 \mid m, -p \mid 2m-p) I_p R^{+}_{\ell=2,m-p} - 6 R^{+}_{\ell=2,m} \} \equiv a \Sigma_m , J_{mm} , R^{+}_{\ell=2,m} , \qquad (16)$$

This is precisely the same equation as in ref.[1] for the roton operator $R^{+}_{\ell=2,m}$.

Eq. (16) are not sufficient to define the operators Q and R^{+}. This is related to the fact that the substitution

$$R^{+}_m \rightarrow R^{+}_m + c_m P_{\mu 0} \qquad (17)$$

does not affect eq. (16). The latter may be supplemented by subsidiary conditions

$$[I_0, Q_m] = m Q_m \qquad (18)$$

as it is suggested in ref.[1].

Following[1] it is easy to see that the microscopic structure of the one-body operators Q_m and the value of the moment of inertia $\mathcal{F} = 1/a$ are given by the equations[x]

$$\langle - | [a_i^+ a_j, \{[H, Q_m] - a \sum_m, J_{mm}, Q_m, \}] | -> = 0,$$
$$\langle - | \{ [I_0, Q_m] - m Q_m \} | -> = 0 \; . \qquad (19)$$

In eq. (18) the state $|->$ may be chosen arbitrarily. This is because the moments of inertia of different bands coincide in the Elliott model. One cannot expect this feature of the theory to be maintained in the more realistic cases.

From the above formulae we conclude that the Elliott model may indeed be studied by using the roton operators. It is interesting to note that the well-known Thouless formula for the moment of inertia follows from eq. (19) if one takes the Hartree-Fock state for $|->$ and substitutes

$$\sum_m, J_{mm}, Q_m, |-> \quad \text{by} \quad J_{m0} |-><-| Q_0 |-> .$$

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[x] Strictly speaking, these equations determine only the product $Q_m|->$ which is sufficient to evaluate a. The procedure may be extended to obtain information on the other blocks of Q_m by taking into account more detailed subsidiary conditions.

This may be a good approximation if the maximum value of I in the band is big[x]. But this approximation fails to reproduce the commutation relations between the R^+ operators

$$[R^+_{\ell=2,m}, R^+_{\ell=2,m'}] = \frac{-\sqrt{10}}{2g} P_{\mu_0} I_{m+m'} (22\, m m' | 1\, m+m') \qquad (20)$$

(this commutator vanishes in the approximation which leads to the Thouless formula). Eq. (20) is a particular case of the recursion formulae allowing one to find an arbitrary roton operator. Failing here the approximation may lead to errors in prescription of the transition rates (particularly between the states of different bands). In ref.[4] the three-dimensional theory of nuclear rotation is suggested accounting for the terms in $\Sigma_m, J_{mm}, Q_m, |->$ which are neglected in the above approximation.

4. Way beyond the Elliott Model. Fitting of Energies in the SU_3 Multiplets

Spectra of real nuclei deviate in many ways from the rigid rotor formula[5]. Contrary to the predictions of the Elliott model one does not find in the experiment even approximate degeneracy of states belonging to different bands but having the same value of I . The spacing between states does not follow the $I(I+1)$ rule and the coefficient before this term varies from one band to another. There exist two possibilities for the generalization of the model. The first one consists of increasing the number of transition operators appearing in the model Hamil-

[x] Our Preliminary estimates[4] show that the errors in the moment of inertia which are due to this approximation are small.

tonian, i.e. of going to a larger group than SU_3. Enlarging the group one comes to more spacious multiplets, all the members of which should be taken into account. Thus, in a way, the first possibility is just a step from one problem to a more difficult one.

We are going to explore the possibility to parametrize the energies in SU_3 multiplets. Formally this is easily achieved by using the Ω operator of ref.[3] given by the formula

$$\Omega = \Sigma_{ij}\, Q_{ij}\, I_i I_j \ . \tag{21}$$

According to ref.[3] the set of eigenvalues (ω, I, M) of the operators $(\Omega, \vec{I}^2, I_0)$ specify completely the member of a given SU_3 multiplet. It hence appears that the energies of states may be given by the finite series

$$E(K,I) = E_0 + \sum_{n,m} a_{n,m} (I(I+1))^n (\omega(K,I))^m \ , \tag{22}$$

where K distinguishes different eigenvalues of Ω corresponding to the same I. The power m of $\omega(K,I)$ may be kept smaller than the number of cases in which the same I,M quantum numbers appear in the multiplet. For the multiplets $(p,0), (0,p)$ containing only one rotational band

$$\omega(I) = -\frac{1}{3}(p + \frac{3}{2})I(I+1) \tag{23}$$

and eq. (22) reduces to the expansion in powers of $I(I+1)$. However, for other multiplets eq. (22) offers some new possibilities. For example the multiplet (p,q) with $q=2$ and p -even contains two bands[2]

$K = 0,\ I = 0, 2, 4, \ldots\ p$

$K = 2,\ I = 2, 3, \ldots, p, p+1, p+2 = I_{max}$.

In this case one has

$$\omega = -\frac{1}{6}(2I_{max}+1)I(I+1) + \delta\omega\,(K,I), \tag{24}$$

where

$$\delta\omega = \begin{cases} 4I_{max}+2 & \text{for odd } I \\ 2I_{max}+1 + (-1)^{1-\frac{K}{2}}\{[(2I_{max}+1) + \\ + I(I-1)]^2 - 4I(I-1)(I_{max}-I)\}^{\frac{1}{2}}, & \text{for even } I\,. \end{cases} \tag{25}$$

Thus, the term in eq. (22) linear in ω may account for the shift of the $K=2$ rotational band and also for the distorsion in each of the bands. Fig. 1 gives an example of the fitting of the part of ^{16}O spectrum with the formula

$$E = E_0 + aI(I+1) + b\,\delta\omega\,(K,I). \tag{26}$$

We see that the known states in ^{16}O atributed usually to the multiplet (4,2) of SU_3 follow reasonably well the simple two-parameter formula[x/]. Therefore the reasonable choice for the model Hamiltonian in this particular case corresponds to the formula

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[x/] We note that the experimental energy of the 6_+ state may be reproduced much better by the formula (26) with $I_{max}=8$ (the corresponding multiplet being (6,2)).

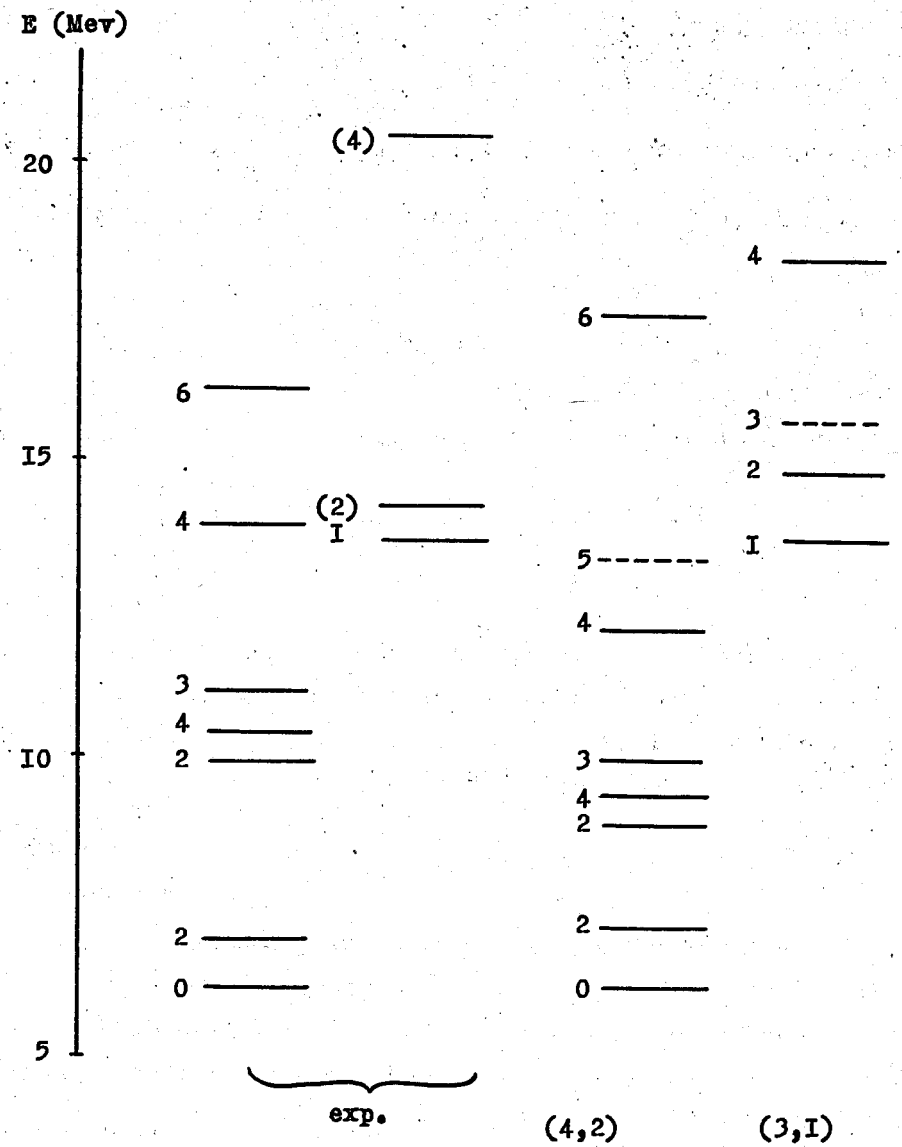

Fig. 1. Experimental data on the part of the excitation spectrum of ^{16}O belonging to the (4.2) and (3.1) multiplets of SU_3 (the two columns on the left). On the right- the same spectrum as given by the two-parameter formula (26) with a = = 0.191 MeV, b = 0.059 MeV.

$$h = a' \vec{I}^{\,2} + b' \Omega \ . \qquad (27)$$

Now we are ready to start the microscopic analysis of the structure of operators entering the Hamiltonian (27) and the earlier formulae. To do this we write

$$[H', Q_m]|-> = [h, Q_m]|-> =$$
$$= \{ a' \sum_{m'} J_{mm'} Q_{m'} + b' \sum_{m'm''} J^{(2)}_{mm'm''} Q_{m'} Q_{m''} \}|-> . \qquad (28)$$

Here $J_{mm'}$ is given by eq. (16) and

$$J^{(2)}_{m,m',m''} = \sum_{\alpha\beta} \sqrt{\frac{5}{12}} \begin{pmatrix} 2 & 2 & 2 \\ m & m' & \alpha \end{pmatrix} \begin{pmatrix} 2 & 2 & 1 \\ \alpha & m'' & -\beta \end{pmatrix} (-1)^{-\beta} I_\beta . \qquad (29)$$

The state $|->$ may include any mixture of nuclear eigenstates from which the multiplet (4.2) is composed. If eq. (26) with the same values of a and b fits the energies of other multiplets the choice of the state $|->$ in eq. (28) is much less stringent[x].

Comparing eqs. (15) and (28) we see that inclusion of the operator Ω in the model Hamiltonian leads to the modification of the commutator $[h, Q_m]$ which resembles very much the modification of the random phase approximation equations for vibrations characteristic to the higher orders of this approximation. We are going to find the practical solution of eq. (28) in future.

[x] We note that the use of formula (26) with fixed values of a and b for the multiplets including states with $I > 3a/b$ brings one to inverted rotational bands with negative values of the moment of inertia.

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Appendix

Classification of States in the "Nuclear" SU_3 Multiplets[2,3,7]

The natural basis for multiplets of SU_3 (the Cartesian basis) corresponds to the reduction[6,7]

$$SU_3 \approx SU_2 \times U_1 . \qquad (A.1)$$

In this basis the states are labelled by eigenvalues of the two operators J_3 , M which are linear combinations of the generators A_{ij} in eq. (7) and also of the Casimir operator $\vec{J}^2$ of SU_2 subgroup of SU_3 . The off-diagonal operators A_{ij} are raising and lowering operators in this case and their successive application to any particular state yields all the other states of the irreducible representation.

The Cartesian basis is widely used in the elementary particle physics (see[7] for the reference) where there exists a close relatioship between the isospin quantum numbers of particles and eigenvalues of the operators of SU_2 . One may not exclude the possibility that such multiplets appear also in the nuclear physics (note that if this is the case they include states of different nuclei). However, such multiplets differ essentially from those of the nuclear shell-model and particularly from the multiplets in the Elliott scheme. The latter are associated with another reduction of SU_2 , namely

$$SU_3 \supset O_3 \supset O_2 \qquad (A\ 1.2)$$

the generators of O_3 being identified with the orbital part of the angular momentum operator $\vec{I}$ [x].

In general the irreps of SU_3 are reducible in O_3, the number of different states with the same I, M quantum numbers being given by the following rule[2]. In the multiplet (p, q) one finds different states with

$$\begin{aligned} &I = K, K+1, \ldots K + \max(p, q), \ (K \neq 0) \\ &K = \min(p, q), \ \min(p, q) - 2, \ldots 1 \ \text{or} \ 0; \\ &I = p, p-2, \ldots 1, 0 \quad (K = 0). \end{aligned} \tag{A. 3}$$

Note, that the physical meaning of the parameter K in eq. (A.3) is disputed in some papers (see, e.g., ref.[8]).

To distinguish between the states with the same values of I, M one may use the operator[3]

$$\Omega = \sum_{ij} Q_{ij} \ I_i \ I_j \ . \tag{A.4}$$

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[x] The approach in Section 2,4 of this paper takes from the Elliott model the classification of states. However, the meaning of operators is different here ($\vec{I}$ being identified with the total angular momentum of an even nucleus and the microscopic structure of the other generators of the group being an objective of the theory).

The eigenvalues of Ω ($\Omega | \omega (I,K) > = \omega (I,K) | \omega (I,K) >$) are non-degenerate and may be found from the equations

$$\det (\beta_{qq'} - \omega \delta_{qq'}) = 0, \qquad (A.5)$$

where $((\beta_{qq'}))$ is given by eqs. (66), (67) of ref.[3].

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