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HIGHER HYPERCOMPLEX NUMBERS AND QUANTUM MECHANICS

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1. Complex numbers and quaternions play an important role in quantum theory. Now octonions [1-11,22,25] are also popular in Grand Unification models based on exceptional groups [12-20]. May further hypercomplex number algebras [10,23,24] be useful? These sets are not only noncommutative (like quaternions), nonassociative (like octonions), but also nonalternative (unlike octonions), being therefore not division algebras. They do not permit the composition of quadratic forms in the Hurwitz sense [21]. However, they all have unit element, the usual involution operation (conjugation), remain to be flexible and power-associative ($a^m a^n = a^{m+n}$). But we shall see that algebras considered are in fact reducible except for the octonion one.

We call as the hypercomplex number an element $a = a_0 e_0 + a_j e_j$ of algebra A_q ($q=1,2,3,\ldots$), where e_0 is the real unit element and e_j ($j=1,2,3,\ldots p-1$) are imaginary unit elements ($p=2^q=2,4,8,16,\ldots$) with the multiplication table

$$e_0^2 = e_0\,,\ e_j e_k = -\delta_{jk} e_0 + \varepsilon_{jkl} e_l \qquad (j,k,l = 1,2,\ldots,p-1). \qquad (1)$$

The "tensor" ε_{jkl} is totally antisymmetric. Its components equal ± 1 or 0. Each pair jk defines uniquely some value l (the converse is not true). For complex numbers (algebra A_I) $\varepsilon_{jkl}=0$. For quaternions (algebra A_2) there is only one independent component $\varepsilon_{123}=1$. For octonions (algebra A_3) there are 7 independent components (see Appendix A). It is well known that they can be represented by "lines" (in terms of the projective geometry) of the Freudental triangle (fig. 1), that contains 7 points and 7 lines (one of the lines is represented by the circle 123, $\varepsilon_{123}=1$).

For the next algebra, A_4, elements of which we name sedenions (16-nions), 35 independent components ε_{jkl} can be represented by lines of the tetrahedron of fig. 2, that contains 15 points and 35 lines (not all lines are drawn, in particular, not all circles), 4 points are placed at vertices, 6 at the midpoints of edges, 4 in the centres of faces, and 1 in the center of the tetrahedron. The tetrahedron can be decomposed into triangles, each containing 7 points and 7 lines (fig. 5), Besides the faces (the triangles 1-4 in fig. 5), there are "inner" triangles formed by one of the edges and the midpoint of the opposite edge (there are 6 such triangles, the triangles 5-10 in fig. 5). 4 triangles (11-14 in fig. 5) in terms of the tetrahedron are represented by conii inscribed into tetrahedron with their vertices at vertices of tetrahedron. For example, triangle 12 is represented by the cone drawn in fig. 3. "Lines" on the cone are 3 generatrices of the cone and 4 conics: 3 ellipses (one of them is represented by the dot-line in fig. 3) and the circle-base. At last the triangle 15 of fig. 5 is represented by the sphere (fig. 4) touching the edges of the tetrahedron at the points 1,2,3,9,10,11 and with point 8 as its center. Lines at the sphere are 3 diameters and 4 cicles of the faces of the tetrahedron (sections of the sphere by the faces). The tetrahedron can be rebuilt in many ways. Any point can be made a vertex. Any inner triangle can be converted into a face.

Similarly for any of the further algebras A_q ($q=5,6,\dots$) ε_{jkl} is represented by $\frac{(p-1)(p-2)}{2\cdot 3}$ lines of a body in the (q-1)-dimensional space. However, these bodies can be decomposed into sets of tetrahedrons in the 3-dimensional space and further into triangles.

Giving directions to lines by arrows we fix values of ε_{jkl} (choose +1 or -1). Thus, one can form different algebras A_q for the same q.* One of the ways to choose ε_{jkl} is to find them using the Dickson doubling process /1,10,23,24/. Starting with an algebra A_q (with elements $a = a_0 e_0 + a_j e_j$, $j=1,\dots,p-1$) and introducing a new unit element e_p ($p=2^q$), we pass to the next algebra A_{q+1} with the elements

$$z = a + a' e_p = a_0 e_0 + a_J e_J \qquad (J = 1,2,\dots,2p-1), \tag{2}$$

where $a, a' \in A_q$, $e_{p+m} = e_m e_p$, $a_{p+m} = a'_m$ ($m = 0,\dots,p-1$) (all a_0, a_J are real) imposing the condition

$$z_1 z_2 = (a + a' e_p)(b + b' e_p) = ab - \bar{b}' a' + (b' a + a' \bar{b}) e_p \tag{3}$$

that defines a new multiplication table. This process leads from real numbers to complex ones, from complex numbers to quaternions, from quaternions to octonions, from octonions to sedenions, etc.

The arrows in figs.1-5 (values of ε_{jkl}) correspond to the doubling process (3). Tables of the values of ε_{jkl} thus obtained are given in Appendix A.

2. From A_3 all the algebras A_q are nonassociative, i.e. the <u>associators</u>

$$(a, b, c) = (ab)c - a(bc) \tag{4}$$

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* For q=3 one can also form algebras distinct from the octonion one. They are not only nonassociative but also nonalternative (see, e.g., triangles 2-4, 11-14 in fig. 5).

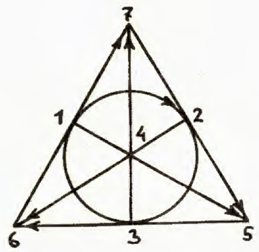

Fig. 1

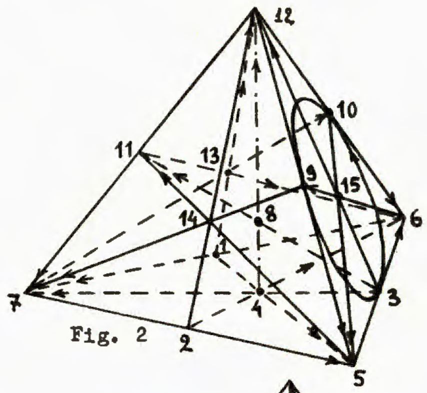

Fig. 2

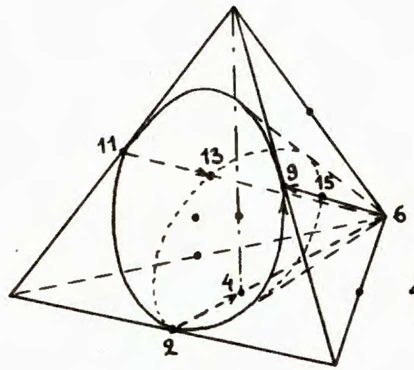

Fig. 3

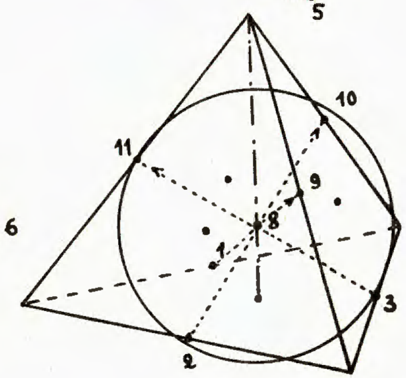

Fig. 4

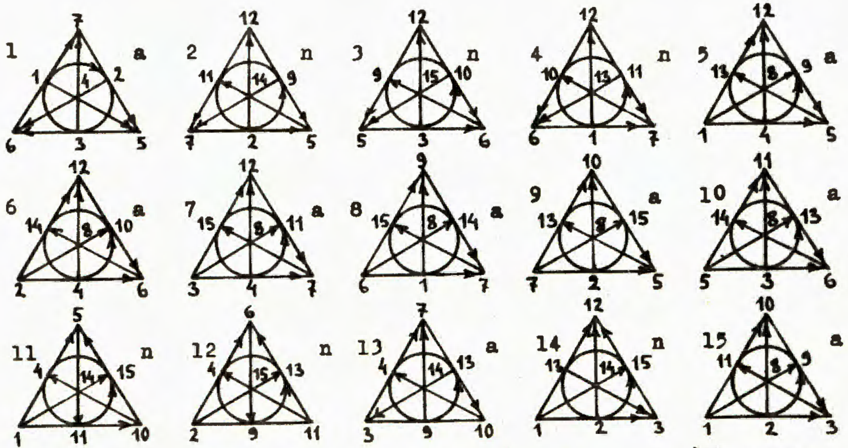

Fig. 5. a (n) means an alternative (nonalternative) triangle (according to process (3), see Table 3 of Appendix A).

are in general nonzero. Using eq.(1) we find for the basis elements

$$(e_j,e_k,e_l)=(e_je_k)e_l-e_j(e_ke_l)=\alpha_{jklm}e_m, \tag{5}$$

$$\alpha_{jklm}=-\delta_{jk}\delta_{lm}+\delta_{kl}\delta_{jm}+\varepsilon_{jkn}\varepsilon_{nlm}+\varepsilon_{kln}\varepsilon_{njm}=$$
$$=\beta_{jklm}+\beta_{kljm}. \tag{6}$$

Hence there follow the properties:

1) any associator is pure imaginary,

2) α_{jklm} is antisymmetric in the 1st and 3rd indices, antisymmetric in the 2nd and 4th indices, and each cyclic permutation of all four indices changes its sign

$$\alpha_{jklm}=-\alpha_{lkjm}=-\alpha_{jmlk}=-\alpha_{mjkl}=\alpha_{lmjk}=-\alpha_{klmj}, \tag{7}$$

3) $\alpha_{jjlm}=\alpha_{jkll}=0$ (no summation), (8)

4) $\alpha_{jklm}\neq 0$ only with no coincident indices,

5) in the general case all the 4! components of α_{jklm} with fixed set $jklm$ are expressed via three of them differing in cyclic permutation of three indices, e.g., via α_{jklm}, α_{ljkm}, α_{kljm},

6) for all A_q the flexible law is held

$$(e_j,e_k,e_l)=-(e_l,e_k,e_j),\qquad (a,b,c)=-(c,b,a), \tag{9}$$

$$(e_j,e_k,e_j)=0 \ \text{(no summation)} \qquad (a,b,a)=0, \tag{10}$$

7) for quaternions (p=4) $\alpha_{jklm}=0$ (associativity). For octonions

$$\alpha_{jklm}=\alpha_{ljkm}=\alpha_{kljm} \tag{11}$$

and hence α_{jklm} is totally antisymmetric and therefore associators (a,b,c) are totally antisymmetric under permutations of a,b,c (alternativity)[x]. In particular,

$$(a,a,b)=(a,b,b)=0. \tag{12}$$

In the general case for $p\geq 16$ the alternative law is no longer valid, i.e.,in general

$$(a,a,b)\neq 0,\ (a,b,b)\neq 0 \tag{13}$$

and hence there is no division, in general, (since $\bar a(ab)\neq(\bar a a)b$). However, for the basis units the equations

$$(e_j,e_j,e_l)=0,\qquad (e_j,e_k,e_k)=0 \qquad \text{(no summation)} \tag{14}$$

("quasialternativity") remain valid for all A_q together with division: $\bar e_j(e_j b)=(\bar e_j e_j)b=b$.

8) In each algebra A_{q+1} after the octonionic one (i.e. with $q+1>3$) formed by the doubling process (3) the unit e_p plays a privi-

[x]For octonions $\alpha_{jklm}=-\frac{1}{3}\varepsilon_{jklmnrs}\varepsilon_{nrs}$, where $\varepsilon_{jklmnrs}$ is the totally antisymmetric tensor of R^7 ($\varepsilon_{1234567}=1$).

leged role: all the associators (a,b,e_p) $(a,b\in A_{q+1})$ are alternative, i.e. totally antisymmetric, as all components α_{jklm} with any of indices equal to p (α_{jklp}) . This fact can be proved as follows. With no condition (3) we have identically

$$z_1z_2=(a+a'e_p)(b+b'e_p)=ab+(a'e_p)(b'e_p)+(a'e_p)b+a(b'e_p). \tag{15}$$

Further, one can show that (see Appendix B)

$$a(b'e_p)=(b'a)e_p+(e_p,\bar a,\bar b')-(a,e_p,\bar b'), \tag{16}$$

$$(a'e_p)b=(a'\bar b)e_p-(a',\bar b,e_p)+(a',e_p,b), \tag{17}$$

$$(a'e_p)(b'e_p)=-\bar b'a'-e_p(a',e_p,\bar b')+e_p(a',b',e_p)-$$
$$-(a'e_p,e_p,\bar b')-(e_p,a'e_p,\bar b'). \tag{18}$$

Therefore, the doubling process (3) imposes the conditions

$$(e_p,\bar a,\bar b')=(a,e_p,\bar b'),\qquad (a',\bar b,e_p)=(a',e_p,b), \tag{19,20}$$

$$(a'e_p,e_p,\bar b')+(e_p,a'e_p,\bar b')=0 \qquad (a,a',b,b'\in A_q). \tag{21}$$

Hence one can deduce the alternativity of all associators with e_p, i.e., (a,b,e_p), where $a,b\in A_{q+1}$ (see Appendix B)[x].

There are other sets $jklm$ for which equalities (11) and the alternativity remain valid. However, for some sets a new particular type of the nonassociativity (asymmetricity) is valid: two of the quantities $\alpha_{jklm},\alpha_{ljkm},\alpha_{kljm}$ are equal to zero and the third one to +2 or -2 (see tables 4 and 6 of Appendix A and ref.[24] for some details).

For the double commutator

$$[a[bc]]=4a_kb_lc_m\varepsilon_{jkn}\varepsilon_{nlm}e_j \tag{22}$$

from the identity

$$[a[bc]]=\{c\{ab\}\}-\{b\{ca\}\}-(a,b,c)+(c,a,b)-(b,c,a)+$$
$$+(c,b,a)-(b,a,c)+(a,c,b), \tag{23}$$

valid in any associative and nonassociative cases (in the case of the hypercomplex numbers of algebras A_q these associators are reduced to $-2(a,b,c)+2(c,a,b)-2(b,c,a)$ due to the flexible law (9)), it follows that

$$\varepsilon_{jkn}\varepsilon_{nlm}=\delta_{jl}\delta_{km}-\delta_{jm}\delta_{kl}+\beta_{jklm}, \tag{24}$$

where

$$\beta_{jklm}=\frac{1}{2}(\alpha_{jklm}+\alpha_{ljkm}-\alpha_{kljm}). \tag{25}$$

[x]For reduction of associators of A_{q+1} to A_q terms see Appendix C.

It is clear from eq.(24) (or from eq.(25)) that β_{jklm} is antisymmetric in the 1st and 2nd indices, antisymmetric in the 3rd and 4th indices, and does not alter by interchange of the first pair of indices and the second one (like R_{jklm}):

$$\beta_{jklm} = -\beta_{kjlm} = -\beta_{jkml} = \beta_{lmjk}. \tag{26}$$

All the 4! components of β_{jklm} with a fixed set $jklm$ can be expressed via three of them, e.g., via β_{jklm}, β_{ljkm}, β_{kljm}. Relation (24) (and eq.(23) too) is a generalization of the well-known rule of the vector calculus $[\vec{a}[\vec{b}\vec{c}]] = \vec{b}(\vec{a}\vec{c}) - \vec{c}(\vec{a}\vec{b})$ (for quaternions $\beta = 0$)/24/

Trace of a hypercomplex number is defined to be its doubled real part

$$\mathrm{tr}\, a = a + \bar{a} = 2\,\mathrm{Re}\, a. \tag{27}$$

It obeys the properties

$$\mathrm{tr}(ab) = \mathrm{tr}(ba), \qquad \mathrm{tr}(a(bc)) = \mathrm{tr}((ab)c). \tag{28,29}$$

The latter is due to the associator being always a pure imaginary quantity.

Derivation algebra of A_q and automorphism group. A derivation of a nonassociative algebra A is a linear transformation with the property

$$D(xy) = (Dx)y + x(Dy). \tag{30}$$

Any infinitesimal automorphism* is a derivation.

The derivation Lie algebra $D(A_q)$ of any algebra A_q obtained from the octonion algebra A_3 by iteration of the doubling process (3), is isomorphic to the derivation Lie algebra $D(A_3)$ of octonions, $D(A_q) \cong D(A_3)$, i.e., to the 14-parameter Lie algebra G_2/10/. Infinitesimal G_2 transformations (derivations) of octonions are known to be/11/

$$\delta x = Dx = [[\alpha\beta]x] - 3(\alpha, \beta, x) \qquad (\alpha, \beta, x \in A_3) \tag{31}$$

and any finite G_2 transformation for octonions can be symbolically written as follows:

$$x' = e^D x. \tag{32}$$

* The transformation $x \to x'$ is an automorphism of an algebra, if $(xy)' = x'y'$, and, therefore, $\delta(xy) = (\delta x)y + x(\delta y)$ infinitesimally.

By the doubling process any hypercomplex number can be represented as

$$x = x_1 + x_2 e_8 + (x_3 + x_4 e_8)e_{16} + (x_5 + x_6 e_8 + (x_7 + x_8 e_8)e_{16})e_{32} + \cdots \tag{33}$$

where $x_1, x_2, x_3, \ldots$ are octonions. Then, in accord with the Schafer theorem/10/ we can write infinitesimal and final G_2 transformations of x as follows

$$\delta x = Dx = [[\alpha\beta]x_1] - 3(\alpha,\beta,x_1) + ([[\alpha\beta]x_2] - 3(\alpha,\beta,x_2))e_8 +$$
$$+ ([[\alpha\beta]x_3] - 3(\alpha,\beta,x_3) + ([[\alpha\beta]x_4] - 3(\alpha,\beta,x_4))e_8)e_{16} + \cdots \tag{34}$$

$$x' = e^D x. \tag{35}$$

The group (algebra) G_2 is an automorphism group (algebra) of A_q. The trace is invariant under automorphism transformations G_2:

$$\mathrm{tr}\, x' = \mathrm{tr}\, x, \qquad \mathrm{tr}\, \delta x = 0. \tag{36}$$

3. Quantum mechanics with A_q as algebras of observables. In quantum mechanics observables (operators) are usually presented by some matrices or differential operators with the property of associativity. We shall consider now quantum mechanics with observables being imaginary hypercomplex numbers $a_m e_m$ (or better $i a_m e_m$, the imaginary unit i making observables "Hermitian", i.e. eigenvalues and expectation values real) of algebra A_q. The particular case of the quaternionic quantum mechanics (algebra A_2 of observables) - the quantum mechanics of spin $\frac{1}{2}$ - is well-known. It uses the Pauli matrices σ_m, i.e. quaternions multiplied by i as observables. By analogy with the quantum mechanics of spin we can construct a (nonassociative) octonionic quantum mechanics (algebra A_3 of observables)/2,22/ * and quantum mechanics for higher hypercomplex numbers (algebras A_q, q=4,5,..., of observables) (see also/23,24/).

Eigenvalues and eigenstates. Let us solve the eigenvalue problem for e_1 (the quaternion, octonion or higher hypercomplex number)

$$e_1 \rho = \lambda_1 \rho, \qquad \rho e_1 = \mu_1 \rho, \tag{37}$$

where $\rho = \rho_0 e_0 + \rho_m e_m$ is unknown, and λ_1 and μ_1 are eigenvalues. We have as solutions

* One more quantum mechanics using octonions is well known: the famous exceptional quantum mechanics by Jordan, von Neumann and Wigner/3/ (see also/12,15-17,25/).

$$\lambda_1=\mu_1=\pm i \quad \frac{1}{2}(e_0\mp e_1), \left(= \begin{cases} |\lambda_1=i\rangle\langle\mu_1=i| \\ |\lambda_1=-i\rangle\langle\mu_1=-i| \end{cases}\right.$$

$$\lambda_1=-\mu_1=\pm i \begin{cases} \frac{1}{2}(e_2\mp ie_3), \left. = \begin{cases} |\lambda_1=i\rangle\langle\mu_1=-i| \\ |\lambda_1=-i\rangle\langle\mu_1=i| \end{cases} \right) \text{for quaternions} \\ \frac{1}{2}(e_4\mp ie_5),\ \frac{1}{2}(e_6\pm ie_7),\ \frac{1}{2}(e_8\mp ie_9), \\ \frac{1}{2}(e_{10}\pm ie_{11}),\ \frac{1}{2}(e_{12}\pm ie_{13}),\ \frac{1}{2}(e_{14}\mp ie_{15}) \end{cases} \tag{38}$$

for sedenions (by table 3 of Appendix A). There exist only four first solutions in the case of quaternions, and eight first solutions in the case of octonions. However, in all the cases only first two of them are density operators ("matrices") orthogonal to each other. As to other solutions, for the quaternions they represent the nondiagonal direct products of bras and kets (see expressions in parentheses in eqs.(38)). In the case of an arbitrary axis $\vec{c}$ density operators (eigenstates of $c_m e_m$) are written as

$$\rho(\lambda_c=\pm i)=\frac{1}{2}(e_0\mp ic_m e_m), \qquad c_m c_m=1\,. \tag{39}$$

The set of operators (38) is complete and orthogonal (if the operation $tr=2Re$ is used to form a scalar product). The set of density operators $\frac{1}{2}(e_0\pm ie_k)$ (k=1,2,...,p-1) is complete too but not orthogonal.

<u>Probability for finding one state in another</u> is defined as

$$w(\lambda_b=i,\lambda_a=i)=tr\left[\tfrac{1}{2}(e_0-i\vec{b}\vec{e})\tfrac{1}{2}(e_0-i\vec{a}\vec{e})\right]=\tfrac{1}{2}(1+\vec{a}\vec{b}), \tag{40}$$

$$w(\lambda_b=i,\lambda_a=i)+w(\lambda_b=-i,\lambda_a=i)=\tfrac{1}{2}(1+\vec{a}\vec{b})+\tfrac{1}{2}(1-\vec{a}\vec{b})=1 \tag{41}$$

for all the algebras A_q. In the quaternion case eq.(40) corresponds to the well-known Pauli result. Equations (40) and (41) prove that the probabilities w are always positive and their sum is equal to 1. These properties are conserved in the course of time evolution (see below) since any transformation of the automorphism group preserves the form (39) ($\vec{c}\to\vec{c}'$ with $\vec{c}'\vec{c}'=1$).

<u>Expectation value of an operator F</u> is defined as usual

$$tr(\rho F)\,. \tag{42}$$

For example, for $F=e_1$ and $\rho=\frac{1}{2}(e_0-ie_1)$

$$tr(e_1\tfrac{1}{2}(e_0-ie_1))=i\,. \tag{43}$$

Therefore, the quantities ie_m can serve as the Hermitian operators.

<u>Equations of motion for the density operator and observables</u>. In the case of quaternions equations of motion for the density matrix and the observables F, which do not depend explicitly on time, in the Schrödinger and Heisenberg pictures are the Neumann (Liouville) and

the Heisenberg-Born-Jordan-Dirac equations

$$\frac{d}{dt}\rho(t)=-[\gamma,\rho(t)], \qquad \frac{d}{dt}F=0, \tag{44}$$

$$\frac{d}{dt}F(t)=[\gamma,F(t)], \qquad \frac{d}{dt}\rho=0, \tag{45}$$

respectively, with the formal solutions

$$\rho(t)=e^{-\gamma t}\rho(0)e^{\gamma t}, \tag{46}$$

$$F(t)=e^{\gamma t}F(0)e^{-\gamma t}\,. \tag{47}$$

The Hamiltonian γ is a pure imaginary quaternion. The evolution laws (46) and (47) are transformations of an 1-parameter subgroup of the automorphism group of the quaternion algebra A_2.

Analogously, for other algebras A_q we also assume 1-parameter subgroups of the automorphism group G_2 to be evolution laws

$$\rho(t)=e^{-tD}\rho(0), \tag{48}$$

$$F(t)=e^{tD}F(0), \tag{49}$$

where the derivation symbol D is defined by eqs.(31) or (34). As equations of motion in the octonion case we get the Lie group equations

$$\frac{d}{dt}\rho(t)=-[[\alpha\beta]\rho(t)]+3(\alpha,\beta,\rho(t)), \tag{50}$$

$$\frac{d}{dt}F(t)=[[\alpha\beta]\rho(t)]-3(\alpha,\beta,F(t)), \tag{51}$$

where α and β are imaginary octonions, that together play the role of Hamiltonian. These equations generalize the Neumann (Liouville) and the Heisenberg-Born-Jordan-Dirac equations to the octonion case. Note the condition of conservation in time

$$[[\alpha\beta]F]=3(\alpha,\beta,F)\,. \tag{52}$$

For other algebras A_q formed by the Dickson process (3) we can represent ρ (and F) as

$$\rho=\rho_1+\rho_2e_8+(\rho_3+\rho_4e_8)e_{16}+\cdots \tag{53}$$

where $\rho_1,\rho_2,\rho_3,\rho_4,\cdots$ are octonions. Then, according to eq.(34) equations of motion can be written as follows:

$$\frac{d}{dt}\rho=-[[\alpha\beta]\rho_1]+3(\alpha,\beta,\rho_1)-([[\alpha\beta]\rho_2]-3(\alpha,\beta,\rho_2))e_8- \\ -([[\alpha\beta]\rho_3]-3(\alpha,\beta,\rho_3)+([[\alpha\beta]\rho_4]-3(\alpha,\beta,\rho_4))e_8)e_{16}-\cdots, \tag{54}$$

$$\frac{d}{dt}F=[[\alpha\beta]F_1]-3(\alpha,\beta,F_1)+([[\alpha\beta]F_2]-3(\alpha,\beta,F_2))e_8+ \\ +([[\alpha\beta]F_3]-3(\alpha,\beta,F_3)+([[\alpha\beta]F_4]-3(\alpha,\beta,F_4))e_8)e_{16}+\cdots\,. \tag{55}$$

For sedenions only ρ_1 and ρ_2 and F_1 and F_1 are nonzero. Let us give the law of conservation in time

$$[[\alpha\beta]F_1]-3(\alpha,\beta,F_1)+([[\alpha\beta]F_2]-3(\alpha,\beta,F_2))e_8+$$
$$+([[\alpha\beta]F_3]-3(\alpha,\beta,F_3)+([[\alpha\beta]F_4]-3(\alpha,\beta,F_4))e_8)e_{16}+\dots=0. \qquad (56)$$

All the above evolution laws obey the usual property

$$tr(F_1(0)\cdots F_n(0)\rho(t)) = tr(F_1(t)\cdots F_n(t)\rho(0)) \qquad (57)$$

with any arrangement of brackets that fix an order of multiplications.

Note also the law of conservation of the total probability

$$tr\,\dot\rho(0)=0, \qquad tr\,\rho(t)=tr\,\rho(0). \qquad (58)$$

4.Matrix representations. The set of traces $tr(\bar e_\mu F)$, arranged into colomn,

$$\tilde F = [tr(\bar e_\mu F)] \quad (\mu = 0,1,2,\dots,p-1), \quad F = e_\mu tr(\bar e_\mu F) \qquad (59,60)$$

can serve as a representative of an arbitrary hypercomplex number F (an observable, a density operator). Two more representatives can be introduced as follows. Multiplications of F by a hypercomplex number α from the left or right may be represented as operators, $p\times p$ matrices, acting on the representative $\tilde F$ (59)

$$tr(\bar e_\mu(\alpha F)) = (\alpha^l)_{\mu\nu} tr(\bar e_\nu F) \equiv \alpha^l tr(\bar e_\mu F), \qquad (61)$$

$$tr(\bar e_\mu(F\alpha)) = (\alpha^r)_{\mu\nu} tr(\bar e_\nu F) \equiv \alpha^r tr(\bar e_\mu F), \qquad (62)$$

α^l being the left and α^r the right matrix representatives of α. In particular, using eq.(1) one can easily find

$$e_j^{\substack{l\\r}} = \|(e_j^{\substack{l\\r}})_{\mu\nu}\|, \quad (e_j^{\substack{l\\r}})_{\mu\nu} = \delta_{\mu j}\delta_{\nu 0} - \delta_{\mu 0}\delta_{\nu j} \pm \epsilon_{0\mu j\nu},$$

$$e_0^l = e_0^r = 1 \quad \text{(unit pxp matrix)}, \qquad (63)$$

$$e_j^r = -\eta e_j^l \eta, \quad \eta = \tfrac{1}{2}(e_0^l + e_j^l e_j^r), \quad \eta^2 = e_0^l, \qquad (64)$$

$$\alpha^l = \alpha_0 e_0^l + \alpha_m e_m^l, \quad \alpha^r = \alpha_0 e_0^r + \alpha_m e_m^r. \qquad (65)$$

Introducing the matrix representations /22,24/ we follow the spirit of the Dirac representation theory/26/. Notation adopted for the left and right representatives seems more natural than the symbols L_α and R_α used in algebra (so-called left and right multiplications/11/) having the similar sense but introduced in a different way*.

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* In these terms, e.g., representative $-i\hbar\frac{\partial}{\partial x^l_k}$ of the momentum $\hat p_k$ in quantum mechanics should be denoted not by p_k (or p^l_k to distinguish from $p^r_k = i\hbar\frac{\partial}{\partial x^r_k}$) but by $L_{\hat p_k}$ what seems to be inappropriate.

For the quaternions the left and right representatives commute mutually

$$[e_j^l e_k^r] = 0, \qquad [\alpha^l \beta^r] = 0, \qquad (66)$$

as in all associative algebras. However, they do not commute in all nonassociative algebras, in particular, in the octonion one and subsequent A_q

$$[e_j^l e_k^r] = [e_j^r e_k^l] \neq 0, \qquad [\alpha^l \beta^r] = [\alpha^r \beta^l] \neq 0, \qquad (67)$$

$$[e_j^l e_j^r] = 0 \qquad \text{(no summation)}, \qquad (68)$$

where the equalities follow from the flexible law (9). Consider now various situations with repeated multiplications

$$tr(\bar e_\mu(\alpha(\beta F))) = \alpha^l \beta^l tr(\bar e_\mu F), \qquad (69)$$

$$tr(\bar e_\mu(\alpha(F\beta))) = \alpha^l \beta^r tr(\bar e_\mu F), \qquad (70)$$

$$tr(\bar e_\mu((F\alpha)\beta)) = \beta^r \alpha^r tr(\bar e_\mu F), \qquad (71)$$

$$tr(\bar e_\mu(\alpha,F,\beta)) = -[\alpha^l \beta^r] tr(\bar e_\mu F), \qquad (72)$$

$$tr(\bar e_\mu(\alpha,\beta,F)) = -tr((\bar e_\mu,\alpha,\beta)F) = -\alpha_j\beta_k tr((\bar e_\mu,e_j,e_k)F) =$$
$$= -\alpha_j\beta_k d_{\mu jk\nu} tr(\bar e_\nu F), \qquad (73)$$

$$tr(\bar e_\mu(\alpha\beta)F) = (\alpha\beta)^l tr(\bar e_\mu F) = tr(\bar e_\mu((\alpha,\beta,F) + \alpha(\beta F))) =$$
$$= ((\alpha^l\beta^l)_{\mu\nu} - \alpha_j\beta_k d_{\mu jk\nu}) tr(\bar e_\nu F), \qquad (74)$$

$$tr(\bar e_\mu(F(\alpha\beta))) = (\alpha\beta)^r tr(\bar e_\mu F) = ((\beta^r\alpha^r)_{\mu\nu} - \alpha_k\beta_j d_{\mu jk\nu}) tr(\bar e_\nu F). \qquad (75)$$

Hence, representatives of the product of two hypercomplex numbers are given by

$$(\alpha\beta)^l = \alpha^l\beta^l - \alpha_j\beta_k d_{jk}, \text{ i.e. } (\alpha\beta)^l_{\mu\nu} = (\alpha^l\beta^l)_{\mu\nu} - \alpha_j\beta_k d_{\mu jk\nu}, \qquad (76)$$

$$(\alpha\beta)^r = \beta^r\alpha^r - \alpha_k\beta_j d_{jk}, \text{ i.e. } (\alpha\beta)^r_{\mu\nu} = (\beta^r\alpha^r)_{\mu\nu} - \alpha_k\beta_j d_{\mu jk\nu}, \qquad (77)$$

$d_{\mu jk\nu}$ equals zero, if any subscript of $\mu jk\nu$ equals zero. Note that for the octonions $\alpha_j\beta_k d_{jk} = -[\alpha^l\beta^r] = -[\alpha^r\beta^l]$ due to the alternativity.

The derivation can also be represented by $p\times p$ matrix $\mathfrak{D}$

$$tr(\bar e_\mu(DF)) = \mathfrak{D}\, tr(\bar e_\mu F). \qquad (78)$$

For quaternions $\mathfrak{D}_4 = \gamma^l - \gamma^r$, $\qquad (79)$

for octonions $\qquad \mathcal{D}_8=[d^{\ell}\beta^{\ell}]+[d^{r}\beta^{r}]+[d^{\ell}\beta^{r}]\,,$ (80)

for other A_q, formed by the Dickson process (3),

$$\mathcal{D}=\left[\begin{array}{ccc:c} \mathcal{D}_8 & 0 & 0 & \\ 0 & \mathcal{D}_8 & 0 & \\ 0 & 0 & \mathcal{D}_8 & \\ \hdashline & & & \end{array}\right], \tag{81}$$

where all the entries are 8x8 matrices.

All relations can be translated into terms of the representatives introduced. For example, equations of motion (54) and (55) (in particular, eqs.(50) and (51))and their formal solutions take the form

$$\frac{d}{dt}\tilde{\rho}(t)=-\mathcal{D}\tilde{\rho}(t), \qquad \tilde{\rho}(t)=e^{-t\mathcal{D}}\tilde{\rho}(0), \tag{82}$$

$$\frac{d}{dt}\tilde{F}(t)=\mathcal{D}\tilde{F}(t), \qquad \tilde{F}(t)=e^{t\mathcal{D}}\tilde{F}(0), \tag{83}$$

$$\frac{d}{dt}\rho^{\overset{\ell}{r}}(t)=-[\mathcal{D}\,\rho^{\overset{\ell}{r}}(t)], \qquad \rho^{\overset{\ell}{r}}(t)=e^{-t\mathcal{D}}\rho^{\overset{\ell}{r}}(0)\,e^{t\mathcal{D}}, \tag{84}$$

$$\frac{d}{dt}F^{\overset{\ell}{r}}(t)=[\mathcal{D}\,F^{\overset{\ell}{r}}(t)], \qquad F^{\overset{\ell}{r}}(t)=e^{t\mathcal{D}}F^{\overset{\ell}{r}}(0)\,e^{-t\mathcal{D}}, \tag{85}$$

where the last two equations were obtained using property (30). Note that equations (82) and (83) look like the Schrödinger equation, being, however, equations for ρ and F. Equations (84) and (85) are similar to the standard quantum-mechanical evolution equations by Neumann (Liouville) and Heisenberg, Born and Jordan (unlike eqs.(50), (51), (54) and (55) written in terms of hypercomplex numbers), $\mathcal{D}$ playing the role of a Hamiltonian. This fact comes from the automorphism property of the underlying group* (for eqs.(50), (51), (54) and (55)), being a general rule**.

Law (30) of conservation of the total probability can be written as

$$\dot{\tilde{\rho}}_0(t)=0, \qquad \tilde{\rho}_0(t)=\tilde{\rho}_0(0) \qquad \text{(zero component)}, \tag{86}$$

$$\mathrm{tr}\,\dot{\rho}^{\overset{\ell}{r}}(t)=0, \qquad \mathrm{tr}\,\rho^{\overset{\ell}{r}}(t)=\mathrm{tr}\,\rho^{\overset{\ell}{r}}(0). \tag{87}$$

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* The automorphism property is directly used in deriving eqs.(84) and (85) from eqs.(54) and (55) (in particular, from eqs.(50) and (51)):

$$\mathrm{tr}(\bar{e}_\mu((D\rho)F))=\mathrm{tr}(\bar{e}_\mu(D(\rho F)-\rho(DF)))=$$
$$=\mathcal{D}\rho^{\ell}\,\mathrm{tr}(\bar{e}_\mu F)-\rho^{\ell}\mathcal{D}\,\mathrm{tr}(\bar{e}_\mu F).$$

The same reasoning shows that the matrices $\mathcal{D}$ form real matrix (reducible) representations of the G_2 algebra.

** For example, the same situation occurs in the exceptional quantum mechanics (see ref.[25]).

5. Let us give some identities and first of all the following one*

$$\begin{aligned}(\bar{a}b)(\bar{b}a)&=(\bar{a}a)(\bar{b}b)+\tfrac{1}{2}(\bar{a},b,\bar{b})a-\tfrac{1}{2}\bar{a}(b,\bar{b},a)=\\ &=(\bar{a}a)(\bar{b}b)+\tfrac{1}{2}(a,b,b)a+\tfrac{1}{2}\bar{a}(b,b,a)=\\ &=(\bar{a}a)(\bar{b}b)+\tfrac{1}{2}\{(a,b,b),\mathrm{Im}\,a\}=\\ &=(\bar{a}a)(\bar{b}b)+d_{jk\ell m}a_j a_k b_\ell b_m=\\ &=(\bar{a}a)(\bar{b}b)+\beta_{jk\ell m}a_j b_k a_\ell b_m\,.\end{aligned} \tag{88}$$

For complex numbers and quaternions when $d=\beta=0$ and for octonions when $d_{jk\ell m}$ and $\beta_{jk\ell m}=\frac{1}{2}d_{jk\ell m}$ are totally antisymmetric, identity (88) is reduced to

$$(\bar{a}b)(\bar{b}a)=(\bar{a}a)(\bar{b}b).$$

the norm of the product equals the product of the norms, the composition of quadratic forms in the Hurwitz sense[21] (the 2,4 and 8 square identities). By Hurwitz[21] the composition of quadratic forms for p=2,4,8 follows from the existence of sets of p-1 real antisymmetric pxp matrices with the property

$$B_jB_k+B_kB_j=-2\delta_{jk}\mathbb{1} \qquad (j,k=1,2,\ldots,p-1) \tag{89}$$

(such are the sets of matrices e_j^{ℓ} or e_j^{r} for complex numbers, quaternions and octonions) and the lack of the necessary number of such matrices for $p\geqslant 16$ makes impossible the Hurwitz composition of quadratic forms[21]. Indeed, although we find for $p\geqslant 16$ the sets of p-1 real antisymmetric pxp matrices e_j^{ℓ} (or e_j^{r}) they are however subjected (according to eqs.(1) and (76)) to the property

$$e_j^{\ell}e_k^{\ell}+e_k^{\ell}e_j^{\ell}=-2\delta_{jk}e_0^{\ell}+d_{jk}+d_{kj}. \tag{90}$$

It is due to this fact the law of composition is replaced by eq.(88).

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* To derive it, we represent $(\bar{a}b)(\bar{b}a)$ in two forms

$$\begin{aligned}(\bar{a}b)(\bar{b}a)&=((\bar{a}b)\bar{b})a-(\bar{a}b,\bar{b},a)=\bar{a}(b\bar{b})a+(\bar{a},b,\bar{b})a-(\bar{a}b,\bar{b},a)=\\ &=\bar{a}(b(\bar{b}a))+(\bar{a},b,\bar{b}a)=\bar{a}(b\bar{b})a-\bar{a}(b,\bar{b},a)+(\bar{a},b,\bar{b}a)\end{aligned}$$

and use the equality $(\bar{a},b,\bar{b}a)=(\bar{a}b,\bar{b},a)$. In passing we find that

$$(\bar{a}b,\bar{b},a)=(\bar{a},b,\bar{b}a)=\tfrac{1}{2}(\bar{a},b,\bar{b})a+\tfrac{1}{2}\bar{a}(b,\bar{b},a)=\tfrac{1}{2}(a,b,b)a-\tfrac{1}{2}\bar{a}(b,b,a).$$

It is clear that for octonions $(\bar{a}b,\bar{b},a)=(\bar{a},b,\bar{b}a)=0$. Hence, it follows that any pair of octonions forms an associative subalgebra and that higher algebras A_q lose this property.

Identity (88) (for A_q) can be written via the hypercomplex numbers of half-dimensionality $(a, a', b, b' \in A_{q-1},\ p = 2^{q-1})$ as follows:

$$(\bar{a}a+\bar{a}'a')(\bar{b}b+\bar{b}'b') = |\bar{a}b-\bar{b}'a'|^2 + |b'\bar{a}+a'\bar{b}|^2 + Y, \qquad (91.a)$$

$$(\bar{a}a+\bar{a}'a')(\bar{b}b+\bar{b}'b') = |b\bar{a}-a'\bar{b}'|^2 + |\bar{a}b'+\bar{b}a'|^2 + Y, \qquad (91.b)$$

where

$$Y = \begin{cases} 0 & \text{for } p = 1,2,4 \\ -4\beta_{jklm}a_j b_k a'_l b'_m & \text{for } p = 8 \\ -\beta_{jklm}\left[(a_j b'_k - a'_j b_k)(a_l b'_m - a'_l b_m) + (a_j b_k + a'_j b'_k)(a_l b_m + a'_l b'_m)\right] & \text{for } p \geqslant 16. \end{cases} \qquad (92)$$

Identifying $(a, a') = (b, b')$ in eqs.(91) we get the identities

$$(\bar{a}a+\bar{a}'a')^2 = \bar{x}x + x_p^2 + X, \qquad (93)$$

where

$$x = 2a'\bar{a} \quad (\text{or } 2\bar{a}a'),\quad x_p = \bar{a}a - \bar{a}'a', \qquad (94)$$

$$X = \begin{cases} 0 & \text{for } p = 1,2,4,8 \\ -4\beta_{jklm}a_j a'_k a_l a'_m & \text{for } p \geqslant 16 \end{cases} \qquad (95)$$

In the cases p=1,2,4,8, when $X=0$, change of variables (94) realizes mapping $R^{2p} \to R^{p+1}$ which maps the sphere S_ρ^{2p-1} with the radius $\rho = \sqrt{\bar{a}a+\bar{a}'a'}$ in the space R^{2p} onto the sphere S_r^p with the radius $r = \sqrt{\bar{x}x + x_p^2} = \rho^2$ (due to eq.(93)) in the space R^{p+1} :*

A) $R^2 \to R^2$ such that $S_\rho^1 \to RP^1 = S_r^1$ (fiber is Z_2, a pair of antipodal points (a,a') and (-a,-a')), double covering;

B) $R^4 \to R^3$ such that $S_\rho^3 \to CP^1 = S_r^2$ (fiber is $S^1 = SO(2) = U(1)$);

C) $R^8 \to R^5$ such that $S_\rho^7 \to QP^1 = S_r^4$ (fiber is $S^3 = SU(2) = Sp(1)$);

D) $R^{16} \to R^9$ such that $S_\rho^{15} \to OP^1 = S_r^8$ (fiber is S^7).

Mappings B),C) and D) are the Hopf fiber bundles with the Hopf invariant (linking number) H, equal to 1. Some of these fiber bundles were used in classical and quantum mechanics[27,39,23] and in field theory[40-54]. For the cases C) and D) (the latter corresponds to the sedenions) and treatment of all the cases A)-D) in terms of the standard (in physics) spinor language see refs.[23,38,39]. For $p > 16$ the connection with maps of spheres onto spheres disappears.

Note also the identity

$$\varepsilon_{ijk}\varepsilon_{klm}\varepsilon_{mnr} = \delta_{il}\varepsilon_{jnr} - \delta_{jl}\varepsilon_{inr} - \delta_{ln}\varepsilon_{ijr} + \delta_{lr}\varepsilon_{ijn} - \delta_{in}\varepsilon_{jlr} + \delta_{ir}\varepsilon_{jln} + \delta_{jn}\varepsilon_{ilr} - \delta_{jr}\varepsilon_{iln} + \gamma_{ijlnr}, \qquad (96)$$

* RP^1, CP^1, QP^1 and OP^1 mean real, complex, quaternion and octonion projective spaces (lines, planes).

where γ corresponds to some set of terms with associators (see ref.[24]). For quaternions and octonions $\gamma=0$.

6. Since for algebras A_q $(q > 3)$ formed by the Dickson process (3) representations of their automorphism group G_2 are reducible (see eqs. (34) or (84) with $\mathcal{D}$ of form (81)) these algebras split into sets of octonions, i.e., are "reducible". Thus, octonions (algebra A_3) remain the last irreducible hypercomplex system (in addition to the Hurwitz and generalized Frobenius theorems).

As to algebras A_q, formed otherwise, the majority of the above formulas hold, with the exception of the automorphism transformations and equations of motion.

7. A.B.Govorkov[55] has proposed to connect the existence of the lepton and quark generations with the sequence of algebras A_q formed by the doubling process.

8. Note that Hermitian 3x3 matrices over any hypercomplex number algebra A_q form a Jordan algebra. Indeed, when choosing the basis of these matrices

$$\mu_0 = \lambda_0 e_0, \qquad \mu_a = \sqrt{\tfrac{3}{2}}\lambda_a e_0 \qquad (a = 1,3,4,6,8),$$

$$\mu_{fi} = \sqrt{\tfrac{3}{2}}\varkappa_f e_i \quad (\varkappa_f = -i\lambda_f,\ f = 2,5,7;\ i = 1,2,\dots,p-1), \qquad (97)$$

where λ_a and λ_f are the Gell-Mann 3x3 matrices, we obtain the multiplication table

$$\mu_a\cdot\mu_b = \delta_{ab}\mu_0 + \sqrt{\tfrac{3}{2}}d_{abc}\mu_c \qquad (a,b,c=1,3,4,6,8),$$

$$\mu_a\cdot\mu_{fi} = \sqrt{\tfrac{3}{2}}d_{afg}\mu_{gi} \qquad (f,g=2,5,7), \qquad (98)$$

$$\mu_{fi}\cdot\mu_{gj} = \delta_{ij}\left(\delta_{fg}\mu_0 + \sqrt{\tfrac{3}{2}}d_{fgc}\mu_c\right) + \sqrt{\tfrac{3}{2}}f_{fgh}\varepsilon_{ijk}\mu_{hk} \quad (f,g,h=2,5,7)$$

(as for octonions, see ref.[25], pp.2 and 7) where d_{abc}, d_{afg} and f_{fgh} are the Gell-Mann structure constants and ε_{ijk} is defined as above (n.1), dot means the Jordan product $a\cdot b = \frac{1}{2}(ab+ba)$.

Renumbering the basis the multiplication table can be written as

$$\mu_0\cdot\mu_0 = \mu_0, \qquad \mu_0\cdot\mu_A = \mu_A$$

$$\mu_A\cdot\mu_B = \delta_{AB}\mu_0 + s_{ABC}\mu_C \qquad (A,B,C = 1,2,\dots,(2+3p);\ p = 2^q) \qquad (99)$$

with totally symmetric s_{ABC} as it was proposed in ref.[56] for the cases of real numbers, complex numbers, quaternions, and octonions (p=1,2,4,8).

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Appendix A. The tables below correspond to the doubling process (3). The author is deeply grateful to A.I.Polubarinova for the program generating both these tables and the tables for further algebras A_q.

Table 1. $\varepsilon_{jk\ell}$ for octonions.

I,2,3	I
I,4,5	I
I,6,7	-I
2,4,6	I
2,5,7	I
3,4,7	I
3,5,6	-I

Table 2. $\alpha_{jk\ell m}$ and $\beta_{jk\ell m}$ for octonions.

I,2,4,7	2	I
I,2,5,6	-2	-I
I,3,4,6	-2	-I
I,3,5,7	-2	-I
2,3,4,5	2	I
2,3,6,7	-2	-I
4,5,6,7	-2	-I

Table 3. $\varepsilon_{jk\ell}$ for sedenions.

I, 2, 3	I
I, 4, 5	I
I, 6, 7	-I
I, 8, 9	I
I,IO,II	-I
I,I2,I3	-I
I,I4,I5	I
2, 4, 6	I
2, 5, 7	I
2, 8,IO	I

2, 9,II	I
2,I2,I4	-I
2,I3,I5	-I
3, 4, 7	I
3, 5, 6	-I
3, 8,II	I
3, 9,IO	-I
3,I2,I5	-I
3,I3,I4	I

4, 8,I2	I
4, 9,I3	I
4,IO,I4	I
4,II,I5	I
5, 8,I3	I
5, 9,I2	-I
5,IO,I5	I
5,II,I4	-I

6, 8,I4	I
6, 9,I5	-I
6,IO,I2	-I
6,II,I3	I
7, 8,I5	I
7, 9,I4	I
7,IO,I3	-I
7,II,I2	-I

Table 4. $\alpha_{jk\ell m}$, $\alpha_{\ell jkm}$ and $\alpha_{k\ell jm}$ for sedenions *.

I, 2, 4, 7	2		
I, 2, 5, 6	-2		
I, 2, 8,II	2		
I, 2, 9,IO	-2		
I, 2,I2,I5	0	0	2
I, 2,I3,I4	0	0	-2
I, 3, 4, 6	-2		
I, 3, 5, 7	-2		
I, 3, 8,IO	-2		
I, 3, 9,II	-2		
I, 3,I2,I4	0	0	-2
I, 3,I3,I5	0	0	-2
I, 4, 8,I3	2		
I, 4, 9,I2	-2		
I, 4,IO,I5	0	0	-2
I, 4,II,I4	0	0	2
I, 5, 8,I2	-2		
I, 5, 9,I3	-2		
I, 5,IO,I4	0	0	2
I, 5,II,I5	0	0	2
I, 6, 8,I5	-2		
I, 6, 9,I4	-2		
I, 6,IO,I3	0	0	-2
I, 6,II,I2	0	0	-2
I, 7, 8,I4	2		
I, 7, 9,I5	-2		
I, 7,IO,I2	0	0	2
I, 7,II,I3	0	0	-2
2, 3, 4, 5	2		
2, 3, 6, 7	-2		
2, 3, 8, 9	2		
2, 3,IO,II	-2		
2, 3,I2,I3	0	0	2
2, 3,I4,I5	0	0	-2
2, 4, 8,I4	2		
2, 4, 9,I5	0	0	2
2, 4,IO,I2	-2		
2, 4,II,I3	0	0	-2

2, 5, 8,I5	2		
2, 5, 9,I4	0	0	-2
2, 5,IO,I3	-2		
2, 5,II,I2	0	0	2
2, 6, 8,I2	-2		
2, 6, 9,I3	0	0	2
2, 6,IO,I4	-2		
2, 6,II,I5	0	0	2
2, 7, 8,I3	-2		
2, 7, 9,I2	0	0	-2
2, 7,IO,I5	-2		
2, 7,II,I4	0	0	-2
3, 4, 8,I5	2		
3, 4, 9,I4	0	0	-2
3, 4,IO,I3	0	0	2
3, 4,II,I2	-2		
3, 5, 8,I4	-2		
3, 5, 9,I5	0	0	-2
3, 5,IO,I2	0	0	-2
3, 5,II,I3	-2		
3, 6, 8,I3	2		
3, 6, 9,I2	0	0	2
3, 6,IO,I5	0	0	-2
3, 6,II,I4	-2		
3, 7, 8,I2	-2		
3, 7, 9,I3	0	0	2
3, 7,IO,I4	0	0	2
3, 7,II,I5	-2		
4, 5, 6, 7	-2		
4, 5, 8, 9	2		
4, 5,IO,II	0	0	2
4, 5,I2,I3	-2		
4, 5,I4,I5	0	0	-2
4, 6, 8,IO	2		
4, 6, 9,II	0	0	-2
4, 6,I2,I4	-2		
4, 6,I3,I5	0	0	2

4, 7, 8,II	2		
4, 7, 9,IO	0	0	2
4, 7,I2,I5	-2		
4, 7,I3,I4	0	0	-2
5, 6, 8,II	-2		
5, 6, 9,IO	0	0	-2
5, 6,I2,I5	0	0	-2
5, 6,I3,I4	-2		
5, 7, 8,IO	2		
5, 7, 9,II	0	0	-2
5, 7,I2,I4	0	0	2
5, 7,I3,I5	-2		
6, 7, 8, 9	-2		
6, 7,IO,II	0	0	-2
6, 7,I2,I3	0	0	-2
6, 7,I4,I5	-2		
8, 9,IO,II	-2		
8, 9,I2,I3	-2		
8, 9,I4,I5	2		
8,IO,I2,I4	-2		
8,IO,I3,I5	-2		
8,II,I2,I5	-2		
8,II,I3,I4	2		
9,IO,I2,I5	2		
9,IO,I3,I4	-2		
9,II,I2,I4	-2		
9,II,I3,I5	-2		
IO,II,I2,I3	2		
IO,II,I4,I5	-2		
I2,I3,I4,I5	-2		

*Only one value $\alpha_{jk\ell m}$ is given, when $\alpha_{jk\ell m}=\alpha_{\ell jkm}=\alpha_{k\ell jm}$.

Table 5. $\varepsilon_{jk\ell}$ for 32-nions.

I, 2, 3	I	3,25,26	I	7, 9,I4	I	II,2I,30	-I
I, 4, 5	I	3,28,3I	I	7,IO,I3	-I	II,22,29	I
I, 6, 7	-I	3,29,30	-I	7,II,I2	-I	II,23,28	-I
I, 8, 9	I	4, 8,I2	I	7,I6,23	I	I2,I6,28	I
I,IO,II	-I	4, 9,I3	I	7,I7,22	I	I2,I7,29	-I
I,I2,I3	-I	4,IO,I4	I	7,I8,2I	-I	I2,I8,30	-I
I,I4,I5	I	4,II,I5	I	7,I9,20	-I	I2,I9,3I	-I
I,I6,I7	I	4,I6,20	I	7,24,3I	-I	I2,20,24	-I
I,I8,I9	-I	4,I7,2I	I	7,25,30	-I	I2,2I,25	I
I,20,2I	-I	4,I8,22	I	7,26,29	I	I2,22,26	I
I,22,23	I	4,I9,23	I	7,27,28	I	I2,23,27	I
I,24,25	-I	4,24,28	-I	8,I6,24	I	I3,I6,29	I
I,26,27	I	4,25,29	-I	8,I7,25	I	I3,I7,28	I
I,28,29	I	4,26,30	-I	8,I8,26	I	I3,I8,3I	-I
I,30,3I	-I	4,27,3I	-I	8,I9,27	I	I3,I9,30	I
2, 4, 6	I	5, 8,I3	I	8,20,28	I	I3,20,25	-I
2, 5, 7	I	5, 9,I2	-I	8,2I,29	I	I3,2I,24	-I
2, 8,IO	I	5,IO,I5	I	8,22,30	I	I3,22,27	-I
2, 9,II	I	5,II,I4	-I	8,23,3I	I	I3,23,26	I
2,I2,I4	-I	5,I6,2I	I	9,I6,25	I	I4,I6,30	I
2,I3,I5	-I	5,I7,20	-I	9,I7,24	-I	I4,I7,3I	I
2,I6,I8	I	5,I8,23	I	9,I8,27	I	I4,I8,28	I
2,I7,I9	I	5,I9,22	-I	9,I9,26	-I	I4,I9,29	-I
2,20,22	-I	5,24,29	-I	9,20,29	I	I4,20,26	-I
2,2I,23	-I	5,25,28	I	9,2I,28	-I	I4,2I,27	I
2,24,26	-I	5,26,3I	-I	9,22,3I	-I	I4,22,24	-I
2,25,27	-I	5,27,30	I	9,23,30	I	I4,23,25	-I
2,28,30	I	6, 8,I4	I	IO,I6,26	I	I5,I6,3I	I
2,29,3I	I	6, 9,I5	-I	IO,I7,27	-I	I5,I7,30	-I
3, 4, 7	I	6,IO,I2	-I	IO,I8,24	-I	I5,I8,29	I
3, 5, 6	-I	6,II,I3	I	IO,I9,25	I	I5,I9,28	I
3, 8,II	I	6,I6,22	I	IO,20,30	I	I5,20,27	-I
3, 9,IO	-I	6,I7,23	-I	IO,2I,3I	I	I5,2I,26	-I
3,I2,I5	-I	6,I8,20	-I	IO,22,28	-I	I5,22,25	I
3,I3,I4	I	6,I9,2I	I	IO,23,29	-I	I5,23,24	-I
3,I6,I9	I	6,24,30	-I	II,I6,27	I		
3,I7,I8	-I	6,25,3I	I	II,I7,26	I		
3,20,23	-I	6,26,28	I	II,I8,25	-I		
3,2I,22	I	6,27,29	-I	II,I9,24	-I		
3,24,27	-I	7, 8,I5	I	II,20,3I	I		

Table 6. d_{jklm}, d_{ljkm} and d_{kljm} for 32-nions*.

I, 2, 4, 7	2		
I, 2, 5, 6	-2		
I, 2, 8,II	2		
I, 2, 9,I0	-2		
I, 2,I2,I5	0	0	2
I, 2,I3,I4	0	0	-2
I, 2,I6,I9	2		
I, 2,I7,I8	-2		
I, 2,20,23	0	0	2
I, 2,2I,22	0	0	-2
I, 2,24,27	0	0	2
I, 2,25,26	0	0	-2
I, 2,28,3I	2		
I, 2,29,30	-2		
I, 3, 4, 6	-2		
I, 3, 5, 7	-2		
I, 3, 8,I0	-2		
I, 3, 9,II	-2		
I, 3,I2,I4	0	0	-2
I, 3,I3,I5	0	0	-2
I, 3,I6,I8	-2		
I, 3,I7,I9	-2		
I, 3,20,22	0	0	-2
I, 3,2I,23	0	0	-2
I, 3,24,26	0	0	-2
I, 3,25,27	0	0	-2
I, 3,28,30	-2		
I, 3,29,3I	-2		
I, 4, 8,I3	2		
I, 4, 9,I2	-2		
I, 4,I0,I5	0	0	-2
I, 4,II,I4	0	0	2
I, 4,I6,2I	2		
I, 4,I7,20	-2		
I, 4,I8,23	0	0	-2
I, 4,I9,22	0	0	2
I, 4,24,29	0	0	2
I, 4,25,28	0	0	-2
I, 4,26,3I	-2		
I, 4,27,30	2		
I, 5, 8,I2	-2		
I, 5, 9,I3	-2		
I, 5,I0,I4	0	0	2
I, 5,II,I5	0	0	2
I, 5,I6,20	-2		
I, 5,I7,2I	-2		
I, 5,I8,22	0	0	2
I, 5,I9,23	0	0	2
I, 5,24,28	0	0	-2
I, 5,25,29	0	0	-2
I, 5,26,30	2		
I, 5,27,3I	2		
I, 6, 8,I5	-2		
I, 6, 9,I4	-2		
I, 6,I0,I3	0	0	-2
I, 6,II,I2	0	0	-2
I, 6,I6,23	-2		
I, 6,I7,22	-2		
I, 6,I8,2I	0	0	-2
I, 6,I9,20	0	0	-2
I, 6,24,3I	0	0	-2
I, 6,25,30	0	0	-2
I, 6,26,29	-2		
I, 6,27,28	-2		
I, 7, 8,I4	2		
I, 7, 9,I5	-2		
I, 7,I0,I2	0	0	2
I, 7,II,I3	0	0	-2
I, 7,I6,22	2		
I, 7,I7,23	-2		
I, 7,I8,20	0	0	2
I, 7,I9,2I	0	0	-2
I, 7,24,30	0	0	2
I, 7,25,3I	0	0	-2
I, 7,26,28	2		
I, 7,27,29	-2		
I, 8,I6,25	2		
I, 8,I7,24	-2		
I, 8,I8,27	0	0	-2
I, 8,I9,26	0	0	2
I, 8,20,29	0	0	-2
I, 8,2I,28	0	0	2
I, 8,22,3I	0	0	2
I, 8,23,30	0	0	-2
I, 9,I6,24	-2		
I, 9,I7,25	-2		
I, 9,I8,26	0	0	2
I, 9,I9,27	0	0	2
I, 9,20,28	0	0	2
I, 9,2I,29	0	0	2
I, 9,22,30	0	0	2
I, 9,23,3I	0	0	2
I,I0,I6,27	-2		
I,I0,I7,26	-2		
I,I0,I8,25	0	0	-2
I,I0,I9,24	0	0	-2
I,I0,20,3I	0	-2	0
I,I0,2I,30	0	2	0
I,I0,22,29	0	-2	0
I,I0,23,28	0	2	0
I,II,I6,26	2		
I,II,I7,27	-2		
I,II,I8,24	0	0	2
I,II,I9,25	0	0	-2
I,II,20,30	0	2	0
I,II,2I,3I	0	2	0
I,II,22,28	0	-2	0
I,II,23,29	0	-2	0
I,I2,I6,29	-2		
I,I2,I7,28	-2		
I,I2,I8,3I	0	2	0
I,I2,I9,30	0	-2	0
I,I2,20,25	0	0	-2
I,I2,2I,24	0	0	-2
I,I2,22,27	0	2	0
I,I2,23,26	0	-2	0
I,I3,I6,28	2		
I,I3,I7,29	-2		
I,I3,I8,30	0	-2	0
I,I3,I9,3I	0	-2	0
I,I3,20,24	0	0	2
I,I3,2I,25	0	0	-2
I,I3,22,26	0	2	0
I,I3,23,27	0	2	0
I,I4,I6,3I	2		
I,I4,I7,30	-2		
I,I4,I8,29	0	2	0
I,I4,I9,28	0	2	0
I,I4,20,27	0	-2	0
I,I4,2I,26	0	-2	0
I,I4,22,25	0	0	-2
I,I4,23,24	0	0	2
I,I5,I6,30	-2		
I,I5,I7,3I	-2		
I,I5,I8,28	0	-2	0
I,I5,I9,29	0	2	0
I,I5,20,26	0	2	0
I,I5,2I,27	0	-2	0
I,I5,22,24	0	0	-2
I,I5,23,25	0	0	-2
2, 3, 4, 5	2		
2, 3, 6, 7	-2		
2, 3, 8, 9	2		
2, 3,I0,II	-2		
2, 3,I2,I3	0	0	2
2, 3,I4,I5	0	0	-2
2, 3,I6,I7	2		
2, 3,I8,I9	-2		
2, 3,20,2I	0	0	2
2, 3,22,23	0	0	-2
2, 3,24,25	0	0	2
2, 3,26,27	0	0	-2
2, 3,28,29	2		
2, 3,30,3I	-2		
2, 4, 8,I4	2		
2, 4, 9,I5	0	0	2
2, 4,I0,I2	-2		
2, 4,II,I3	0	0	-2
2, 4,I6,22	2		
2, 4,I7,23	0	0	2
2, 4,I8,20	-2		
2, 4,I9,2I	0	0	-2
2, 4,24,30	0	0	2
2, 4,25,3I	2		
2, 4,26,28	0	0	-2
2, 4,27,29	-2		
2, 5, 8,I5	2		
2, 5, 9,I4	0	0	-2
2, 5,I0,I3	-2		
2, 5,II,I2	0	0	2
2, 5,I6,23	2		
2, 5,I7,22	0	0	-2
2, 5,I8,2I	-2		
2, 5,I9,20	0	0	2
2, 5,24,3I	0	0	2
2, 5,25,30	-2		
2, 5,26,29	0	0	-2
2, 5,27,28	2		
2, 6, 8,I2	-2		
2, 6, 9,I3	0	0	2
2, 6,I0,I4	-2		
2, 6,II,I5	0	0	2
2, 6,I6,20	-2		
2, 6,I7,2I	0	0	2
2, 6,I8,22	-2		
2, 6,I9,23	0	0	2
2, 6,24,28	0	0	-2
2, 6,25,29	2		
2, 6,26,30	0	0	-2
2, 6,27,3I	2		
2, 7, 8,I3	-2		
2, 7, 9,I2	0	0	-2
2, 7,I0,I5	-2		
2, 7,II,I4	0	0	-2
2, 7,I6,2I	-2		
2, 7,I7,20	0	0	-2
2, 7,I8,23	-2		
2, 7,I9,22	0	0	-2
2, 7,24,29	0	0	-2
2, 7,25,28	-2		
2, 7,26,3I	0	0	-2
2, 7,27,30	-2		
2, 8,I6,26	2		
2, 8,I7,27	0	0	2
2, 8,I8,24	-2		
2, 8,I9,25	0	0	-2
2, 8,20,30	0	0	-2
2, 8,2I,3I	0	0	-2
2, 8,22,28	0	0	2
2, 8,23,29	0	0	2
2, 9,I6,27	2		
2, 9,I7,26	0	0	-2
2, 9,I8,25	-2		
2, 9,I9,24	0	0	2
2, 9,20,3I	0	2	0
2, 9,2I,30	0	-2	0
2, 9,22,29	0	2	0
2, 9,23,28	0	-2	0
2,I0,I6,24	-2		
2,I0,I7,25	0	0	2
2,I0,I8,26	-2		
2,I0,I9,27	0	0	2
2,I0,20,28	0	0	2
2,I0,2I,29	0	0	2
2,I0,22,30	0	0	2
2,I0,23,3I	0	0	2
2,II,I6,25	-2		
2,II,I7,24	0	0	-2
2,II,I8,27	-2		
2,II,I9,26	0	0	-2
2,II,20,29	0	-2	0
2,II,2I,28	0	2	0
2,II,22,3I	0	2	0
2,II,23,30	0	-2	0
2,I2,I6,30	-2		
2,I2,I7,3I	0	-2	0
2,I2,I8,28	-2		
2,I2,I9,29	0	2	0
2,I2,20,26	0	0	-2
2,I2,2I,27	0	-2	0
2,I2,22,24	0	0	-2
2,I2,23,25	0	2	0
2,I3,I6,3I	-2		
2,I3,I7,30	0	2	0
2,I3,I8,29	-2		
2,I3,I9,28	0	-2	0
2,I3,20,27	0	2	0
2,I3,2I,26	0	0	-2
2,I3,22,25	0	-2	0
2,I3,23,24	0	0	-2
2,I4,I6,28	2		
2,I4,I7,29	0	-2	0
2,I4,I8,30	-2		
2,I4,I9,3I	0	-2	0
2,I4,20,24	0	0	2
2,I4,2I,25	0	2	0
2,I4,22,26	0	0	-2
2,I4,23,27	0	2	0
2,I5,I6,29	2		
2,I5,I7,28	0	2	0
2,I5,I8,3I	-2		
2,I5,I9,30	0	2	0
2,I5,20,25	0	-2	0
2,I5,2I,24	0	0	2
2,I5,22,27	0	-2	0
2,I5,23,26	0	0	-2
3, 4, 8,I5	2		
3, 4, 9,I4	0	0	-2
3, 4,I0,I3	0	0	2
3, 4,II,I2	-2		
3, 4,I6,23	2		
3, 4,I7,22	0	0	-2
3, 4,I8,2I	0	0	2
3, 4,I9,20	-2		
3, 4,24,3I	0	0	2
3, 4,25,30	-2		
3, 4,26,29	2		
3, 4,27,28	0	0	-2
3, 5, 8,I4	-2		
3, 5, 9,I5	0	0	-2
3, 5,I0,I2	0	0	-2
3, 5,II,I3	-2		
3, 5,I6,22	-2		
3, 5,I7,23	0	0	-2
3, 5,I8,20	0	0	-2
3, 5,I9,2I	-2		
3, 5,24,30	0	0	-2
3, 5,25,3I	-2		
3, 5,26,28	-2		
3, 5,27,29	0	0	-2
3, 6, 8,I3	2		
3, 6, 9,I2	0	0	2
3, 6,I0,I5	0	0	-2
3, 6,II,I4	-2		
3, 6,I6,2I	2		
3, 6,I7,20	0	0	2
3, 6,I8,23	0	0	-2
3, 6,I9,22	-2		
3, 6,24,29	0	0	2
3, 6,25,28	2		
3, 6,26,3I	-2		
3, 6,27,30	0	0	-2
3, 7, 8,I2	-2		
3, 7, 9,I3	0	0	2
3, 7,I0,I4	0	0	2
3, 7,II,I5	-2		
3, 7,I6,20	-2		
3, 7,I7,2I	0	0	2
3, 7,I8,22	0	0	2
3, 7,I9,23	-2		
3, 7,24,28	0	0	-2
3, 7,25,29	2		
3, 7,26,30	2		
3, 7,27,3I	0	0	-2
3, 8,I6,27	2		
3, 8,I7,26	0	0	-2
3, 8,I8,25	0	0	2
3, 8,I9,24	-2		
3, 8,20,3I	0	0	-2
3, 8,2I,30	0	0	2
3, 8,22,29	0	0	-2
3, 8,23,28	0	0	2
3, 9,I6,26	-2		
3, 9,I7,27	0	0	-2
3, 9,I8,24	0	0	-2
3, 9,I9,25	-2		
3, 9,20,30	0	-2	0
3, 9,2I,3I	0	-2	0
3, 9,22,28	0	2	0
3, 9,23,29	0	2	0
3,I0,I6,25	2		
3,I0,I7,24	0	0	2
3,I0,I8,27	0	0	-2
3,I0,I9,26	-2		
3,I0,20,29	0	2	0
3,I0,2I,28	0	-2	0
3,I0,22,3I	0	-2	0
3,I0,23,30	0	2	0
3,II,I6,24	-2		
3,II,I7,25	0	0	2
3,II,I8,26	0	0	2
3,II,I9,27	-2		
3,II,20,28	0	0	2
3,II,2I,29	0	0	2
3,II,22,30	0	0	2
3,II,23,3I	0	0	2
3,I2,I6,3I	-2		
3,I2,I7,30	0	2	0
3,I2,I8,29	0	-2	0
3,I2,I9,28	-2		
3,I2,20,27	0	0	-2
3,I2,2I,26	0	2	0
3,I2,22,25	0	-2	0
3,I2,23,24	0	0	-2
3,I3,I6,30	2		
3,I3,I7,3I	0	2	0
3,I3,I8,28	0	2	0
3,I3,I9,29	-2		
3,I3,20,26	0	-2	0
3,I3,2I,27	0	0	-2

*Only one value d_{jklm} is given, when $d_{jklm}=d_{ljkm}=d_{kljm}$.

3,I3,22,24	0	0	2
3,I3,23,25	0	-2	0
3,I4,I6,29	-2		
3,I4,I7,28	0	-2	0
3,I4,I8,3I	0	2	0
3,I4,I9,30	-2		
3,I4,20,25	0	2	0
3,I4,2I,24	0	0	-2
3,I4,22,27	0	0	-2
3,I4,23,26	0	-2	0
3,I5,I6,28	2		
3,I5,I7,29	0	-2	0
3,I5,I8,30	0	-2	0
3,I5,I9,3I	-2		
3,I5,20,24	0	0	2
3,I5,2I,25	0	2	0
3,I5,22,26	0	2	0
3,I5,23,27	0	0	-2
4, 5, 6, 7	-2		
4, 5, 8, 9	2		
4, 5,I0,II	0	0	2
4, 5,I2,I3	-2		
4, 5,I4,I5	0	0	-2
4, 5,I6,I7	2		
4, 5,I8,I9	0	0	2
4, 5,20,2I	-2		
4, 5,22,23	0	0	-2
4, 5,24,25	0	0	2
4, 5,26,27	2		
4, 5,28,29	0	0	-2
4, 5,30,3I	-2		
4, 6, 8,I0	2		
4, 6, 9,II	0	0	-2
4, 6,I2,I4	-2		
4, 6,I3,I5	0	0	2
4, 6,I6,I8	2		
4, 6,I7,I9	0	0	-2
4, 6,20,22	-2		
4, 6,2I,23	0	0	2
4, 6,24,26	0	0	2
4, 6,25,27	-2		
4, 6,28,30	0	0	-2
4, 6,29,3I	2		
4, 7, 8,II	2		
4, 7, 9,I0	0	0	2
4, 7,I2,I5	-2		
4, 7,I3,I4	0	0	-2
4, 7,I6,I9	2		
4, 7,I7,I8	0	0	2
4, 7,20,23	-2		
4, 7,2I,22	0	0	-2
4, 7,24,27	0	0	2
4, 7,25,26	2		
4, 7,28,3I	0	0	-2
4, 7,29,30	-2		
4, 8,I6,28	2		
4, 8,I7,29	0	0	2
4, 8,I8,30	0	0	2
4, 8,I9,3I	0	0	2
4, 8,20,24	-2		

4, 8,2I,25	0	0	-2
4, 8,22,26	0	0	-2
4, 8,23,27	0	0	-2
4, 9,I6,29	2		
4, 9,I7,28	0	0	-2
4, 9,I8,3I	0	-2	0
4, 9,I9,30	0	2	0
4, 9,20,25	-2		
4, 9,2I,24	0	0	2
4, 9,22,27	0	-2	0
4, 9,23,26	0	2	0
4,I0,I6,30	2		
4,I0,I7,3I	0	2	0
4,I0,I8,28	0	0	-2
4,I0,I9,29	0	-2	0
4,I0,20,26	-2		
4,I0,2I,27	0	2	0
4,I0,22,24	0	0	2
4,I0,23,25	0	-2	0
4,II,I6,3I	2		
4,II,I7,30	0	-2	0
4,II,I8,29	0	2	0
4,II,I9,28	0	0	-2
4,II,20,27	-2		
4,II,2I,26	0	-2	0
4,II,22,25	0	2	0
4,II,23,24	0	0	2
4,I2,I6,24	-2		
4,I2,I7,25	0	0	2
4,I2,I8,26	0	0	2
4,I2,I9,27	0	0	2
4,I2,20,28	-2		
4,I2,2I,29	0	0	2
4,I2,22,30	0	0	2
4,I2,23,3I	0	0	2
4,I3,I6,25	-2		
4,I3,I7,24	0	0	-2
4,I3,I8,27	0	-2	0
4,I3,I9,26	0	2	0
4,I3,20,29	-2		
4,I3,2I,28	0	0	-2
4,I3,22,3I	0	2	0
4,I3,23,30	0	-2	0
4,I4,I6,26	-2		
4,I4,I7,27	0	2	0
4,I4,I8,24	0	0	-2
4,I4,I9,25	0	-2	0
4,I4,20,30	-2		
4,I4,2I,3I	0	-2	0
4,I4,22,28	0	0	-2
4,I4,23,29	0	2	0
4,I5,I6,27	-2		
4,I5,I7,26	0	-2	0
4,I5,I8,25	0	2	0
4,I5,I9,24	0	0	-2
4,I5,20,3I	-2		
4,I5,2I,30	0	2	0
4,I5,22,29	0	-2	0
4,I5,23,28	0	0	-2
5, 6, 8,II	-2		

5, 6, 9,I0	0	0	-2
5, 6,I2,I5	0	0	-2
5, 6,I3,I4	-2		
5, 6,I6,I9	-2		
5, 6,I7,I8	0	0	-2
5, 6,20,23	0	0	-2
5, 6,2I,22	-2		
5, 6,24,27	0	0	-2
5, 6,25,26	-2		
5, 6,28,3I	-2		
5, 6,29,30	0	0	-2
5, 7, 8,I0	2		
5, 7, 9,II	0	0	-2
5, 7,I2,I4	0	0	2
5, 7,I3,I5	-2		
5, 7,I6,I8	2		
5, 7,I7,I9	0	0	-2
5, 7,20,22	0	0	2
5, 7,2I,23	-2		
5, 7,24,26	0	0	2
5, 7,25,27	-2		
5, 7,28,30	2		
5, 7,29,3I	0	0	-2
5, 8,I6,29	2		
5, 8,I7,28	0	0	-2
5, 8,I8,3I	0	0	2
5, 8,I9,30	0	0	-2
5, 8,20,25	0	0	2
5, 8,2I,24	-2		
5, 8,22,27	0	0	2
5, 8,23,26	0	0	-2
5, 9,I6,28	-2		
5, 9,I7,29	0	0	-2
5, 9,I8,30	0	2	0
5, 9,I9,3I	0	2	0
5, 9,20,24	0	0	-2
5, 9,2I,25	-2		
5, 9,22,26	0	-2	0
5, 9,23,27	0	-2	0
5,I0,I6,3I	2		
5,I0,I7,30	0	-2	0
5,I0,I8,29	0	0	-2
5,I0,I9,28	0	2	0
5,I0,20,27	0	-2	0
5,I0,2I,26	-2		
5,I0,22,25	0	2	0
5,I0,23,24	0	0	2
5,II,I6,30	-2		
5,II,I7,3I	0	-2	0
5,II,I8,28	0	-2	0
5,II,I9,29	0	0	-2
5,II,20,26	0	2	0
5,II,2I,27	-2		
5,II,22,24	0	0	-2
5,II,23,25	0	2	0
5,I2,I6,25	2		
5,I2,I7,24	0	0	2
5,I2,I8,27	0	2	0
5,I2,I9,26	0	-2	0
5,I2,20,29	0	0	-2

5,I2,2I,28	-2		
5,I2,22,3I	0	-2	0
5,I2,23,30	0	2	0
5,I3,I6,24	-2		
5,I3,I7,25	0	0	2
5,I3,I8,26	0	0	2
5,I3,I9,27	0	0	2
5,I3,20,28	0	0	2
5,I3,2I,29	-2		
5,I3,22,30	0	0	2
5,I3,23,3I	0	0	2
5,I4,I6,27	2		
5,I4,I7,26	0	2	0
5,I4,I8,25	0	-2	0
5,I4,I9,24	0	0	2
5,I4,20,3I	0	2	0
5,I4,2I,30	-2		
5,I4,22,29	0	0	-2
5,I4,23,28	0	-2	0
5,I5,I6,26	-2		
5,I5,I7,27	0	2	0
5,I5,I8,24	0	0	-2
5,I5,I9,25	0	-2	0
5,I5,20,30	0	-2	0
5,I5,2I,3I	-2		
5,I5,22,28	0	2	0
5,I5,23,29	0	0	-2
6, 7, 8, 9	-2		
6, 7,I0,II	0	0	-2
6, 7,I2,I3	0	0	-2
6, 7,I4,I5	-2		
6, 7,I6,I7	-2		
6, 7,I8,I9	0	0	-2
6, 7,20,2I	0	0	-2
6, 7,22,23	-2		
6, 7,24,25	0	0	-2
6, 7,26,27	-2		
6, 7,28,29	-2		
6, 7,30,3I	0	0	-2
6, 8,I6,30	2		
6, 8,I7,3I	0	0	-2
6, 8,I8,28	0	0	-2
6, 8,I9,29	0	0	2
6, 8,20,26	0	0	2
6, 8,2I,27	0	0	-2
6, 8,22,24	-2		
6, 8,23,25	0	0	2
6, 9,I6,3I	-2		
6, 9,I7,30	0	0	-2
6, 9,I8,29	0	-2	0
6, 9,I9,28	0	-2	0
6, 9,20,27	0	2	0
6, 9,2I,26	0	2	0
6, 9,22,25	-2		
6, 9,23,24	0	0	-2
6,I0,I6,28	-2		
6,I0,I7,29	0	2	0
6,I0,I8,30	0	0	-2
6,I0,I9,3I	0	2	0
6,I0,20,24	0	0	-2

6,I0,2I,25	0	-2	0
6,I0,22,26	-2		
6,I0,23,27	0	-2	0
6,II,I6,29	2		
6,II,I7,28	0	2	0
6,II,I8,3I	0	-2	0
6,II,I9,30	0	0	-2
6,II,20,25	0	-2	0
6,II,2I,24	0	0	2
6,II,22,27	-2		
6,II,23,26	0	2	0
6,I2,I6,26	2		
6,I2,I7,27	0	-2	0
6,I2,I8,24	0	0	2
6,I2,I9,25	0	2	0
6,I2,20,30	0	0	-2
6,I2,2I,3I	0	2	0
6,I2,22,28	-2		
6,I2,23,29	0	-2	0
6,I3,I6,27	-2		
6,I3,I7,26	0	-2	0
6,I3,I8,25	0	2	0
6,I3,I9,24	0	0	-2
6,I3,20,3I	0	-2	0
6,I3,2I,30	0	0	-2
6,I3,22,29	-2		
6,I3,23,28	0	2	0
6,I4,I6,24	-2		
6,I4,I7,25	0	0	2
6,I4,I8,26	0	0	2
6,I4,I9,27	0	0	2
6,I4,20,28	0	0	2
6,I4,2I,29	0	0	2
6,I4,22,30	-2		
6,I4,23,3I	0	0	2
6,I5,I6,25	2		
6,I5,I7,24	0	0	2
6,I5,I8,27	0	2	0
6,I5,I9,26	0	-2	0
6,I5,20,29	0	2	0
6,I5,2I,28	0	-2	0
6,I5,22,3I	-2		
6,I5,23,30	0	0	-2
7, 8,I6,3I	2		
7, 8,I7,30	0	0	2
7, 8,I8,29	0	0	-2
7, 8,I9,28	0	0	-2
7, 8,20,27	0	0	2
7, 8,2I,26	0	0	2
7, 8,22,25	0	0	-2
7, 8,23,24	-2		
7, 9,I6,30	2		
7, 9,I7,3I	0	0	-2
7, 9,I8,28	0	2	0
7, 9,I9,29	0	-2	0
7, 9,20,26	0	-2	0
7, 9,2I,27	0	2	0
7, 9,22,24	0	0	2
7, 9,23,25	-2		
7,I0,I6,29	-2		

7,I0,I7,28	0	-2	0
7,I0,I8,3I	0	0	-2
7,I0,I9,30	0	-2	0
7,I0,20,25	0	2	0
7,I0,2I,24	0	0	-2
7,I0,22,27	0	2	0
7,I0,23,26	-2		
7,II,I6,28	-2		
7,II,I7,29	0	2	0
7,II,I8,30	0	2	0
7,II,I9,3I	0	0	-2
7,II,20,24	0	0	-2
7,II,2I,25	0	-2	0
7,II,22,26	0	-2	0
7,II,23,27	-2		
7,I2,I6,27	2		
7,I2,I7,26	0	2	0
7,I2,I8,25	0	-2	0
7,I2,I9,24	0	0	2
7,I2,20,3I	0	0	-2
7,I2,2I,30	0	-2	0
7,I2,22,29	0	2	0
7,I2,23,28	-2		
7,I3,I6,26	2		
7,I3,I7,27	0	-2	0
7,I3,I8,24	0	0	2
7,I3,I9,25	0	2	0
7,I3,20,30	0	2	0
7,I3,2I,3I	0	0	-2
7,I3,22,28	0	-2	0
7,I3,23,29	-2		
7,I4,I6,25	-2		
7,I4,I7,24	0	0	-2
7,I4,I8,27	0	-2	0
7,I4,I9,26	0	2	0
7,I4,20,29	0	-2	0
7,I4,2I,28	0	2	0
7,I4,22,3I	0	0	-2
7,I4,23,30	-2		
7,I5,I6,24	-2		
7,I5,I7,25	0	0	2
7,I5,I8,26	0	0	2
7,I5,I9,27	0	0	2
7,I5,20,28	0	0	2
7,I5,2I,29	0	0	2
7,I5,22,30	0	0	2
7,I5,23,3I	-2		
8, 9,I0,II	-2		
8, 9,I2,I3	-2		
8, 9,I4,I5	2		
8, 9,I6,I7	2		
8, 9,I8,I9	0	0	2
8, 9,20,2I	0	0	2
8, 9,22,23	0	0	-2
8, 9,24,25	-2		
8, 9,26,27	0	0	-2
8, 9,28,29	0	0	-2
8, 9,30,3I	0	0	2
8,I0,I2,I4	-2		
8,I0,I3,I5	-2		

8,I0,I6,I8	2		
8,I0,I7,I9	0	0	-2
8,I0,20,22	0	0	2
8,I0,2I,23	0	0	2
8,I0,24,26	-2		
8,I0,25,27	0	0	2
8,I0,28,30	0	0	-2
8,I0,29,3I	0	0	-2
8,II,I2,I5	-2		
8,II,I3,I4	2		
8,II,I6,I9	2		
8,II,I7,I8	0	0	2
8,II,20,23	0	0	2
8,II,2I,22	0	0	-2
8,II,24,27	-2		
8,II,25,26	0	0	-2
8,II,28,3I	0	0	-2
8,II,29,30	0	0	2
8,I2,I6,20	2		
8,I2,I7,2I	0	0	-2
8,I2,I8,22	0	0	-2
8,I2,I9,23	0	0	-2
8,I2,24,28	-2		
8,I2,25,29	0	0	2
8,I2,26,30	0	0	2
8,I2,27,3I	0	0	2
8,I3,I6,2I	2		
8,I3,I7,20	0	0	2
8,I3,I8,23	0	0	-2
8,I3,I9,22	0	0	2
8,I3,24,29	-2		
8,I3,25,28	0	0	-2
8,I3,26,3I	0	0	2
8,I3,27,30	0	0	-2
8,I4,I6,22	2		
8,I4,I7,23	0	0	2
8,I4,I8,20	0	0	2
8,I4,I9,2I	0	0	-2
8,I4,24,30	-2		
8,I4,25,3I	0	0	-2
8,I4,26,28	0	0	-2
8,I4,27,29	0	0	2
8,I5,I6,23	2		
8,I5,I7,22	0	0	-2
8,I5,I8,2I	0	0	2
8,I5,I9,20	0	0	2
8,I5,24,3I	-2		
8,I5,25,30	0	0	2
8,I5,26,29	0	0	-2
8,I5,27,28	0	0	-2
9,I0,I2,I5	2		
9,I0,I3,I4	-2		
9,I0,I6,I9	-2		
9,I0,I7,I8	0	0	-2
9,I0,20,23	2		
9,I0,2I,22	-2		
9,I0,24,27	0	0	-2
9,I0,25,26	-2		
9,I0,28,3I	0	0	2
9,I0,29,30	0	0	-2
9,II,I2,I4	-2		
9,II,I3,I5	-2		
9,II,I6,I8	2		
9,II,I7,I9	0	0	-2
9,II,20,22	-2		
9,II,2I,23	-2		
9,II,24,26	0	0	2
9,II,25,27	-2		
9,II,28,30	0	0	-2
9,II,29,3I	0	0	-2
9,I2,I6,2I	-2		
9,I2,I7,20	0	0	-2
9,I2,I8,23	-2		
9,I2,I9,22	2		
9,I2,24,29	0	0	-2
9,I2,25,28	-2		
9,I2,26,3I	0	0	-2
9,I2,27,30	0	0	2
9,I3,I6,20	2		
9,I3,I7,2I	0	0	-2
9,I3,I8,22	2		
9,I3,I9,23	2		
9,I3,24,28	0	0	2
9,I3,25,29	-2		
9,I3,26,30	0	0	2
9,I3,27,3I	0	0	2
9,I4,I6,23	2		
9,I4,I7,22	0	0	-2
9,I4,I8,2I	-2		
9,I4,I9,20	-2		
9,I4,24,3I	0	0	2
9,I4,25,30	-2		
9,I4,26,29	0	0	-2
9,I4,27,28	0	0	-2
9,I5,I6,22	-2		
9,I5,I7,23	0	0	-2
9,I5,I8,20	2		
9,I5,I9,2I	-2		
9,I5,24,30	0	0	-2
9,I5,25,3I	-2		
9,I5,26,28	0	0	2
9,I5,27,29	0	0	-2
I0,II,I2,I3	2		
I0,II,I4,I5	-2		
I0,II,I6,I7	-2		
I0,II,I8,I9	0	0	-2
I0,II,20,2I	2		
I0,II,22,23	-2		
I0,II,24,25	0	0	-2
I0,II,26,27	-2		
I0,II,28,29	0	0	2
I0,II,30,3I	0	0	-2
I0,I2,I6,22	-2		
I0,I2,I7,23	2		
I0,I2,I8,20	0	0	-2
I0,I2,I9,2I	-2		
I0,I2,24,30	0	0	-2
I0,I2,25,3I	0	0	2
I0,I2,26,28	-2		
I0,I2,27,29	0	0	-2
I0,I3,I6,23	-2		
I0,I3,I7,22	-2		
I0,I3,I8,2I	0	0	-2
I0,I3,I9,20	2		
I0,I3,24,3I	0	0	-2
I0,I3,25,30	0	0	-2
I0,I3,26,29	-2		
I0,I3,27,28	0	0	2
I0,I4,I6,20	2		
I0,I4,I7,2I	2		
I0,I4,I8,22	0	0	-2
I0,I4,I9,23	2		
I0,I4,24,28	0	0	2
I0,I4,25,29	0	0	2
I0,I4,26,30	-2		
I0,I4,27,3I	0	0	2
I0,I5,I6,2I	2		
I0,I5,I7,20	-2		
I0,I5,I8,23	0	0	-2
I0,I5,I9,22	-2		
I0,I5,24,29	0	0	2
I0,I5,25,28	0	0	-2
I0,I5,26,3I	-2		
I0,I5,27,30	0	0	-2
II,I2,I6,23	-2		
II,I2,I7,22	-2		
II,I2,I8,2I	2		
II,I2,I9,20	0	0	-2
II,I2,24,3I	0	0	-2
II,I2,25,30	0	0	-2
II,I2,26,29	0	0	2
II,I2,27,28	-2		
II,I3,I6,22	2		
II,I3,I7,23	-2		
II,I3,I8,20	-2		
II,I3,I9,2I	0	0	-2
II,I3,24,30	0	0	2
II,I3,25,3I	0	0	-2
II,I3,26,28	0	0	-2
II,I3,27,29	-2		
II,I4,I6,2I	-2		
II,I4,I7,20	2		
II,I4,I8,23	-2		
II,I4,I9,22	0	0	-2
II,I4,24,29	0	0	-2
II,I4,25,28	0	0	2
II,I4,26,3I	0	0	-2
II,I4,27,30	-2		
II,I5,I6,20	2		
II,I5,I7,2I	2		
II,I5,I8,22	2		
II,I5,I9,23	0	0	-2
II,I5,24,28	0	0	2
II,I5,25,29	0	0	2
II,I5,26,30	0	0	2
II,I5,27,3I	-2		
I2,I3,I4,I5	-2		
I2,I3,I6,I7	-2		
I2,I3,I8,I9	2		
I2,I3,20,2I	0	0	-2

I2,I3,22,23	-2		
I2,I3,24,25	0	0	-2
I2,I3,26,27	0	0	2
I2,I3,28,29	-2		
I2,I3,30,3I	0	0	-2
I2,I4,I6,I8	-2		
I2,I4,I7,I9	-2		
I2,I4,20,22	0	0	-2
I2,I4,2I,23	2		
I2,I4,24,26	0	0	-2
I2,I4,25,27	0	0	-2
I2,I4,28,30	-2		
I2,I4,29,3I	0	0	2
I2,I5,I6,I9	-2		
I2,I5,I7,I8	2		
I2,I5,20,23	0	0	-2
I2,I5,2I,22	-2		
I2,I5,24,27	0	0	-2
I2,I5,25,26	0	0	2
I2,I5,28,3I	-2		
I2,I5,29,30	0	0	-2
I3,I4,I6,I9	2		
I3,I4,I7,I8	-2		
I3,I4,20,23	-2		
I3,I4,2I,22	0	0	-2
I3,I4,24,27	0	0	2
I3,I4,25,26	0	0	-2
I3,I4,28,3I	0	0	-2
I3,I4,29,30	-2		
I3,I5,I6,I8	-2		
I3,I5,I7,I9	-2		
I3,I5,20,22	2		
I3,I5,2I,23	0	0	-2
I3,I5,24,26	0	0	-2
I3,I5,25,27	0	0	-2
I3,I5,28,30	0	0	2
I3,I5,29,3I	-2		
I4,I5,I6,I7	2		
I4,I5,I8,I9	-2		
I4,I5,20,2I	-2		
I4,I5,22,23	0	0	-2
I4,I5,24,25	0	0	2
I4,I5,26,27	0	0	-2
I4,I5,28,29	0	0	-2
I4,I5,30,3I	-2		
I6,I7,I8,I9	-2		
I6,I7,20,2I	-2		
I6,I7,22,23	2		
I6,I7,24,25	-2		
I6,I7,26,27	2		
I6,I7,28,29	2		
I6,I7,30,3I	-2		
I6,I8,20,22	-2		
I6,I8,2I,23	-2		
I6,I8,24,26	-2		
I6,I8,25,27	-2		
I6,I8,28,30	2		
I6,I8,29,3I	2		
I6,I9,20,23	-2		
I6,I9,2I,22	2		
I6,I9,24,27	-2		
I6,I9,25,26	2		
I6,I9,28,3I	2		
I6,I9,29,30	-2		
I6,20,24,28	-2		
I6,20,25,29	-2		
I6,20,26,30	-2		
I6,20,27,3I	-2		
I6,2I,24,29	-2		
I6,2I,25,28	2		
I6,2I,26,3I	-2		
I6,2I,27,30	2		
I6,22,24,30	-2		
I6,22,25,3I	2		
I6,22,26,28	2		
I6,22,27,29	-2		
I6,23,24,3I	-2		
I6,23,25,30	-2		
I6,23,26,29	2		
I6,23,27,28	2		
I7,I8,20,23	2		
I7,I8,2I,22	-2		
I7,I8,24,27	2		
I7,I8,25,26	-2		
I7,I8,28,3I	0	0	2
I7,I8,29,30	0	0	-2
I7,I9,20,22	-2		
I7,I9,2I,23	-2		
I7,I9,24,26	-2		
I7,I9,25,27	-2		
I7,I9,28,30	0	0	-2
I7,I9,29,3I	0	0	-2
I7,20,24,29	2		
I7,20,25,28	-2		
I7,20,26,3I	0	0	-2
I7,20,27,30	0	0	2
I7,2I,24,28	-2		
I7,2I,25,29	-2		
I7,2I,26,30	0	0	2
I7,2I,27,3I	0	0	2
I7,22,24,3I	-2		
I7,22,25,30	-2		
I7,22,26,29	0	0	-2
I7,22,27,28	0	0	-2
I7,23,24,30	2		
I7,23,25,3I	-2		
I7,23,26,28	0	0	2
I7,23,27,29	0	0	-2
I8,I9,20,2I	2		
I8,I9,22,23	-2		
I8,I9,24,25	2		
I8,I9,26,27	-2		
I8,I9,28,29	0	0	2
I8,I9,30,3I	0	0	-2
I8,20,24,30	2		
I8,20,25,3I	0	0	2
I8,20,26,28	-2		
I8,20,27,29	0	0	-2
I8,2I,24,3I	2		
I8,2I,25,30	0	0	-2
I8,2I,26,29	-2		
I8,2I,27,28	0	0	2
I8,22,24,28	-2		
I8,22,25,29	0	0	2
I8,22,26,30	-2		
I8,22,27,3I	0	0	2
I8,23,24,29	-2		
I8,23,25,28	0	0	-2
I8,23,26,3I	-2		
I8,23,27,30	0	0	-2
I9,20,24,3I	2		
I9,20,25,30	0	0	-2
I9,20,26,29	0	0	2
I9,20,27,28	-2		
I9,2I,24,30	-2		
I9,2I,25,3I	0	0	-2
I9,2I,26,28	0	0	-2
I9,2I,27,29	-2		
I9,22,24,29	2		
I9,22,25,28	0	0	2
I9,22,26,3I	0	0	-2
I9,22,27,30	-2		
I9,23,24,28	-2		
I9,23,25,29	0	0	2
I9,23,26,30	0	0	2
I9,23,27,3I	-2		
20,2I,22,23	-2		
20,2I,24,25	2		
20,2I,26,27	0	0	2
20,2I,28,29	-2		
20,2I,30,3I	0	0	-2
20,22,24,26	2		
20,22,25,27	0	0	-2
20,22,28,30	-2		
20,22,29,3I	0	0	2
20,23,24,27	2		
20,23,25,26	0	0	2
20,23,28,3I	-2		
20,23,29,30	0	0	-2
2I,22,24,27	-2		
2I,22,25,26	0	0	-2
2I,22,28,3I	0	0	-2
2I,22,29,30	-2		
2I,23,24,26	2		
2I,23,25,27	0	0	-2
2I,23,28,30	0	0	2
2I,23,29,3I	-2		
22,23,24,25	-2		
22,23,26,27	0	0	-2
22,23,28,29	0	0	-2
22,23,30,3I	-2		
24,25,26,27	-2		
24,25,28,29	-2		
24,25,30,3I	2		
24,26,28,30	-2		
24,26,29,3I	-2		
24,27,28,3I	-2		
24,27,29,30	2		
25,26,28,3I	2		
25,26,29,30	-2		

25,27,28,30 -2	26,27,28,29 2	28,29,30,31 -2
25,27,29,31 -2	26,27,30,31 -2	

Note that in algebra A_q there are $N_\varepsilon = \frac{1}{3}C^2_{p-1} = \frac{(p-1)(p-2)}{2\cdot 3}$ independent nonzero components of $\varepsilon_{jk\ell}$ and $N_\alpha = \frac{1}{4}(C^3_{p-1} - \frac{1}{3}C^2_{p-1}) = \frac{(p-1)(p-2)(p-4)}{1\cdot 2\cdot 3\cdot 4}$ sets jklm with nonzero $\alpha_{jk\ell m}$ ($p=2^q$).

Appendix B. Privileged role of e_p. Let us give first, derivation of eqs. (16)-(18):

$$a(b'e_p) = a(e_p\bar b') = (ae_p)\bar b' - (a,e_p,\bar b') = (e_p\bar a)\bar b' - (a,e_p,\bar b') =$$
$$= e_p(\bar a\bar b') + (e_p,\bar a,\bar b') - (a,e_p,\bar b') =$$
$$= (b'a)e_p + (e_p,\bar a,\bar b') - (a,e_p,\bar b'); \qquad \text{(B.1)}$$

$$(a'e_p)b = a'(e_pb) + (a',e_p,b) = a'(\bar b e_p) + (a',e_p,b) =$$
$$= (a'\bar b)e_p - (a',\bar b,e_p) + (a',e_p,b), \qquad \text{(B.2)}$$

$$(a'e_p)(b'e_p) = (a'e_p)(e_p\bar b') = -(a'e_p,e_p,\bar b') + ((a'e_p)e_p)\bar b' =$$
$$= -(a'e_p,e_p,\bar b') + (e_p(e_pa'))\bar b' =$$
$$= -(a'e_p,e_p,\bar b') + (e_p,e_pa',\bar b') + e_p((e_pa')\bar b') =$$
$$= -(a'e_p,e_p,\bar b') + (e_p,e_pa',\bar b') + e_p[(\bar a',e_p,\bar b') + \bar a'(e_p\bar b')] =$$
$$= -(a'e_p,e_p,\bar b') + (e_p,e_pa',\bar b') + e_p[(\bar a',e_p,\bar b') + \bar a'(b'e_p)] =$$
$$= -(a'e_p,e_p,\bar b') + (e_p,e_pa',\bar b') + e_p[(\bar a',e_p,\bar b') - (\bar a',b',e_p) + (\bar a'b')e_p] =$$
$$= -\bar b'a' + e_p(\bar a',e_p,\bar b') - e_p(\bar a',b',e_p) - (a'e_p,e_p,\bar b') + (e_p,e_pa',\bar b'). \qquad \text{(B.3)}$$

In the last line we can make the replacements:

$\bar a' \to -a'$ in the first two associators and

$e_pa' = \bar a'e_p \to -a'e_p$ in the last one,

thus obtaining eq. (18). Therefore, according to process (3) e_p satisfies eqs. (19)-(21) and ($a,b \in A_q$)

$$a(be_p) = (ba)e_p, \qquad \text{(B.4)}$$
$$(ae_p)b = (a\bar b)e_p, \qquad \text{(B.5)}$$
$$(ae_p)(be_p) = -\bar b a. \qquad \text{(B.6)}$$

Equations (9),(19) and (20) prove alternativity of all the associators (a,b,e_p) with $a,b \in A_q$:

$$(a,e_p,b) = -(e_p,a,b) = -(b,e_p,a) = (b,a,e_p) = -(a,b,e_p) = (e_p,b,a). \qquad \text{(B.7)}$$

Further we have

$$(ae_p,e_p,b) + (ae_p,b,e_p) =$$
$$= ((ae_p)e_p)b - (ae_p)(e_pb) + ((ae_p)b)e_p - (ae_p)(be_p) =$$
$$= -ab + ba - a\bar b + \bar b a = -[a, b+\bar b] = 0, \qquad \text{(B.8)}$$

where eqs. (B.5) and (B.6) were used. Now from eqs. (9) (for A_{q+1}), (21) and (B.8) there follows alternativity of all the associators (ae_p,b,e_p) ($a,b \in A_q$):

$$(ae_p,e_p,b) = -(e_p,ae_p,b) = -(b,e_p,ae_p) = (b,ae_p,e_p) =$$
$$= -(ae_p,b,e_p) = (e_p,b,ae_p). \qquad \text{(B.9)}$$

It only remains to prove alternativity of the associators (ae_p,be_p,e_p) ($a,b \in A_q$):

$$(ae_p,e_p,be_p) + (ae_p,be_p,e_p) =$$
$$= ((ae_p)e_p)(be_p) - (ae_p)(e_p(be_p)) + ((ae_p)(be_p))e_p - (ae_p)((be_p)e_p) =$$
$$= -a(be_p) + (ae_p)\bar b - (\bar b a)e_p + (ae_p)b =$$
$$= (-ba + ab - \bar b a + a\bar b)e_p = 0. \qquad \text{(B.10)}$$

Here we make use of eqs. (B.4)-(B.6). Now

$$(ae_p,e_p,be_p) = -(ae_p,be_p,e_p) = -(be_p,e_p,ae_p) = (e_p,be_p,ae_p) =$$
$$= (be_p,ae_p,e_p) = -(e_p,ae_p,be_p). \qquad \text{(B.11)}$$

Equations (B.7),(B.9) and (B.11) mean that all the associators (a,b,e_p) with $a,b \in A_{q+1}$ are alternative.

Appendix C. Reduction of associators of A_{q+1} to A_q terms. Let us give results together with calculations ($p=2^q$)[*]

$$(e_je_p,e_k,e_\ell) = ((e_je_p)e_k)e_\ell - (e_je_p)(e_ke_\ell) =$$
$$= -((e_je_k)e_p)e_\ell - (e_j(e_\ell e_k))e_p = ((e_je_k)e_\ell)e_p - (e_j(e_\ell e_k))e_p =$$
$$= ((e_je_k)e_\ell)e_p + (e_j,e_\ell,e_k)e_p - ((e_je_\ell)e_k)e_p =$$
$$= ((e_je_k)e_\ell)e_p + (e_j,e_\ell,e_k)e_p + ((e_\ell e_j)e_k)e_p + 2\delta_{j\ell}e_ke_p =$$
$$= ((e_je_k)e_\ell)e_p + (e_j,e_\ell,e_k)e_p + (e_\ell,e_j,e_k)e_p + (e_\ell(e_je_k))e_p + 2\delta_{j\ell}e_ke_p =$$
$$= \{e_je_k,e_\ell\}e_p + 2\delta_{j\ell}e_ke_p + (e_j,e_\ell,e_k)e_p + (e_\ell,e_j,e_k)e_p =$$
$$= \{-\delta_{jk} + \varepsilon_{jkn}e_n, e_\ell\}e_p + 2\delta_{j\ell}e_ke_p + (e_j,e_\ell,e_k)e_p + (e_\ell,e_j,e_k)e_p =$$
$$= -2\varepsilon_{jk\ell}e_p + 2(\delta_{j\ell}e_k - \delta_{jk}e_\ell)e_p + (e_j,e_\ell,e_k)e_p + (e_\ell,e_j,e_k)e_p, \qquad \text{(C.1)}$$

[*]In particular, they demonstrate alternativity of octonion associators.

$$
\begin{aligned}
(e_j, e_k e_p, e_\ell) &= (e_j(e_k e_p))e_\ell - e_j((e_k e_p)e_\ell) = \\
&= ((e_k e_j)e_p)e_\ell + e_j((e_k e_\ell)e_p) = -((e_k e_j)e_\ell)e_p + ((e_k e_\ell)e_j)e_p = \\
&= 2\delta_{jk} e_\ell e_p + ((e_j e_k)e_\ell)e_p + ((e_k e_\ell)e_j)e_p = \\
&= 2\delta_{jk} e_\ell e_p + (e_j, e_k, e_\ell)e_p + (e_j(e_k e_\ell))e_p + ((e_k e_\ell)e_j)e_p = \\
&= 2\delta_{jk} e_\ell e_p + (e_j, e_k, e_\ell)e_p + \{e_j, e_k e_\ell\}e_p = \\
&= -2\varepsilon_{jk\ell} e_p + 2(\delta_{jk} e_\ell - \delta_{k\ell} e_j)e_p + (e_j, e_k, e_\ell)e_p,
\end{aligned} \tag{C.2}
$$

$$
\begin{aligned}
(e_j e_p, e_k, e_\ell e_p) &= ((e_j e_p)e_k)(e_\ell e_p) - (e_j e_p)(e_k(e_\ell e_p)) = \\
&= -((e_j e_k)e_p)(e_\ell e_p) - (e_j e_p)((e_\ell e_k)e_p) = -e_\ell(e_j e_k) + (e_k e_\ell)e_j = \\
&= e_\ell(2\delta_{jk} + e_k e_j) - (2\delta_{k\ell} + e_\ell e_k)e_j = \\
&= 2(\delta_{jk} e_\ell - \delta_{k\ell} e_j) + e_\ell(e_k e_j) - (e_\ell e_k)e_j = \\
&= 2(\delta_{jk} e_\ell - \delta_{k\ell} e_j) - (e_\ell, e_k, e_j),
\end{aligned} \tag{C.3}
$$

$$
\begin{aligned}
(e_j e_p, e_k e_p, e_\ell) &= ((e_j e_p)(e_k e_p))e_\ell - (e_j e_p)((e_k e_p)e_\ell) = \\
&= (e_k e_j)e_\ell + (e_j e_p)((e_k e_\ell)e_p) = (e_k e_j)e_\ell - (e_\ell e_k)e_j = \\
&= (e_k e_j)e_\ell + (2\delta_{k\ell} + e_k e_\ell)e_j = \\
&= (e_k e_j)e_\ell + (e_k e_\ell)e_j + 2\delta_{k\ell} e_j = \\
&= 2(\delta_{k\ell} e_j - \delta_{j\ell} e_k) + (e_k, e_j, e_\ell) + (e_k, e_\ell, e_j),
\end{aligned} \tag{C.4}
$$

$$
\begin{aligned}
(e_j e_p, e_k e_p, e_\ell e_p) &= ((e_j e_p)(e_k e_p))(e_\ell e_p) - (e_j e_p)((e_k e_p)(e_\ell e_p)) = \\
&= (e_k e_j)(e_\ell e_p) - (e_j e_p)(e_\ell e_k) = (e_\ell(e_k e_j))e_p - (e_j(e_k e_\ell))e_p = \\
&= (e_\ell(e_k e_j))e_p + (e_j, e_k, e_\ell)e_p - ((e_j e_k)e_\ell)e_p = \\
&= (e_\ell(-\delta_{kj} + \varepsilon_{kjn} e_n))e_p - ((-\delta_{jk} + \varepsilon_{jkn} e_n)e_\ell)e_p + (e_j, e_k, e_\ell)e_p = \\
&= -\varepsilon_{jkn}\{e_\ell e_n\}e_p + (e_j, e_k, e_\ell)e_p = \\
&= 2\varepsilon_{jk\ell} e_p + (e_j, e_k, e_\ell)e_p.
\end{aligned} \tag{C.5}
$$

First, we extract or cancel e_p using eqs. (B.4)-(B.6). At the last step in eq. (C.4) we use the relation

$$
(e_k e_j)e_\ell + (e_k e_\ell)e_j = -2\delta_{j\ell} e_k + (e_k, e_j, e_\ell) + (e_k, e_\ell, e_j). \tag{C.6}
$$

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Полубаринов И.В. E2-85-930
Дальнейшие системы гиперкомплексных чисел и квантовая механика

Рассмотрены алгебры гиперкомплексных чисел, следующие за алгеброй октонионов. Эти алгебры связаны с конечными проективными геометриями. Их основные свойства выведены из свойств структурных констант ϵ_{jkl}. Выведено тождество, обобщающее на случай этих алгебр закон композиции квадратичных форм. Сформулированы соответствующие квантовые механики, записаны уравнения движения. Построены матричные представления (в смысле теории представлений Дирака). В них уравнения движения принимают обычную квантово-механическую гамильтонову форму. Кроме алгебры октонионов дальнейшие алгебры, в действительности, приводимы.

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Polubarinov I.V. E2-85-930
Higher Hypercomplex Numbers and Quantum Mechanics

Algebras of hypercomplex numbers following octonion algebra are considered. These algebras are connected with finite projective geometries. Their main properties are derived from properties of structure constants ϵ_{jkl}. A derivation of the identity is given that generalizes the law of composition of quadratic forms. Quantum mechanics corresponding to these algebras are formulated, equations of motion are written. Matrix representations are constructed (in the sense of the Dirac representation theory). In these terms the equations of motion take the standard quantum-mechanical Hamiltonian form. Except for the octonion algebra, others are in fact reducible.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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