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J. Nyiri , Ya. Smorodinsky

SYMMETRIES IN THE CLASSICAL THREE-BODY PROBLEM

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SYMMETRIES IN THE CLASSICAL THREE-BODY PROBLEM

Объединенный институт ядерных исследований БИБЛИОТЕКА

* On leave of absence from the Central Research Institute for Physics, Budapest.

§1. Introduction

Several authors (see e.g. [1-8]) have noticed that the three-body systems in quantum mechanics possess the symmetry of the motion of the five dimensional sphere. This symmetry can be observed both in the case of the free motion and in the case of elastic forces.

Obviously, the quantum mechanical problem has the same symmetry as the classical one. Thus it seems to be worthwile to consider the classical equations of motion from this point of view.

Arbitrary motions of the three-body system can be described as rotations and deformations of a triangle formed by the three particles. The equations of motion of the triangle turn out to be very similar to the equations of two coupled tops; one of them reflects the hidden (non-geometrical) symmetry of the deformational motion of the triangle.

In the present paper we collected different types of equations and formulae connected with the classical three-body problem. In calculating, we have used along with the special coordinate system described in [6] the ingenious (but not well-known) method of quasi-coordinates developed by Boltzmann [9].

§2. Non-Rotating Triangles (Examples)

Dealing with a three-particle system, let's first remind the system of coordinates introduced in /6/. The radius-vectors of the three particles $\vec{x}_i$ ($i = 1,2,3$) are fixed by the condition

$$\vec{x}_1 + \vec{x}_2 + \vec{x}_3 = 0. \qquad (1)$$

The Jacobi coordinates $\vec{\xi}$ and $\vec{\eta}$ are given in the case of equal masses in the form x/

$$\vec{\xi} = -\sqrt{\frac{3}{2}}\,(\vec{x}_1 + \vec{x}_2)$$

$$\vec{\eta} = \frac{1}{\sqrt{2}}\,(\vec{x}_1 - \vec{x}_2) \qquad (2)$$

$$\xi^2 + \eta^2 = x_1^2 + x_2^2 + x_3^2 = \rho^2,$$

where ρ is the radius of the five-dimensional sphere. Further, we introduce the complex vector

$$\vec{z} = \vec{\xi} + i\,\vec{\eta}$$

$$\vec{z}^{\,*} = \vec{\xi} - i\,\vec{\eta}\,. \qquad (3)$$

Consider now a triangle, with vertices x_1, x_2, x_3. The position of this triangle in space is characterized by the vectors $\vec{\ell}_1$ and $\vec{\ell}_2$, which form together with the vector $\vec{\ell} = \vec{\ell}_1 \times \vec{\ell}_2$ the moving coordinate system. They are connected with the vectors z and $\vec{z}^{\,*}$ in the following way:

$$\vec{z} = \frac{\rho}{\sqrt{2}}\, e^{-i\frac{\lambda}{2}} \left(e^{i\frac{a}{2}}\, \vec{\ell}_1 + i\, e^{-i\frac{a}{2}}\, \vec{\ell}_2 \right)$$

$$\vec{z}^{\,*} = \frac{\rho}{\sqrt{2}}\, e^{i\frac{\lambda}{2}} \left(e^{-i\frac{a}{2}}\, \vec{\ell}_1 - i\, e^{i\frac{a}{2}}\, \vec{\ell}_2 \right). \qquad (4)$$

x/ For different masses we have

$$\vec{\xi} = m_3 \vec{x}_3 \left(\frac{m_3 (m_1 + m_2 + m_3)}{m_1 + m_2} \right)^{1/2}; \quad \vec{\eta} = \left(\frac{m_1 m_2}{m_1 + m_3} \right)^{1/2} (\vec{x}_1 - \vec{x}_2).$$

The variables λ and a determine the form of the triangle. Using expressions (4) we can write $\vec{\xi}$ and $\vec{\eta}$ in the form

$$\vec{\xi} = \frac{\rho}{\sqrt{2}}\left(\cos\frac{a-\lambda}{2}\vec{\ell}_1 + \sin\frac{a+\lambda}{2}\vec{\ell}_2\right)$$

$$\vec{\eta} = \frac{\rho}{\sqrt{2}}\left(\sin\frac{a-\lambda}{2}\vec{\ell}_1 + \cos\frac{a+\lambda}{2}\vec{\ell}_2\right). \qquad (5)$$

That means, that we can consider $\vec{\xi}$ and $\vec{\eta}$ as a result of two transformations

$$\begin{pmatrix}\vec{\xi}\\ \vec{\eta}\end{pmatrix} = \frac{\rho}{\sqrt{2}}\begin{pmatrix}\cos\frac{\lambda}{2} & \sin\frac{\lambda}{2}\\ -\sin\frac{\lambda}{2} & \cos\frac{\lambda}{2}\end{pmatrix}\begin{pmatrix}\cos\frac{a}{2} & \sin\frac{a}{2}\\ \sin\frac{a}{2} & \cos\frac{a}{2}\end{pmatrix}\begin{pmatrix}\vec{\ell}_1\\ \vec{\ell}_2\end{pmatrix}. \qquad (6)$$

To make the picture clearer, consider the case of a non-rotating triangle. We need for that purpose the expressions

$$\xi^2 = \frac{\rho^2}{2}(1 + \sin a \sin\lambda), \qquad (7)$$

$$\eta^2 = \frac{\rho^2}{2}(1 - \sin a \sin\lambda), \quad \vec{\xi}\vec{\eta} = \frac{\rho^2}{2}\sin a \cos\lambda . \qquad (8)$$

The angle Θ between vectors $\vec{\xi}$ and $\vec{\eta}$

$$\vec{\xi}\cdot\vec{\eta} = |\xi|\,|\eta|\cos\Theta \qquad (9)$$

can be written in terms of our variables as

$$\cos\Theta = \frac{\cos\lambda \sin a}{\sqrt{1-\sin^2\lambda \sin^2 a}}. \qquad (10)$$

Note, that the components of the moment of inertia are

$$\sin^2\left(\frac{a}{2}-\frac{\pi}{4}\right),\ \cos^2\left(\frac{a}{2}-\frac{\pi}{4}\right),\ 1. \qquad (11)$$

Thus it is obvious, that, if a =const, the variations of λ lead to such deformations of the triangle, which do not affect the values of momenta of inertia.

If a is constant, we can write

$$|\xi| = \frac{\rho}{\sqrt{2}} \sqrt{1 + C\sin\lambda}\,,$$

$$|\eta| = \frac{\rho}{\sqrt{2}} \sqrt{1 - C\sin\lambda}\,, \tag{12}$$

$$\cos\Theta = \frac{C\cos\lambda}{\sqrt{1 - C^2\sin^2\lambda}}\,,$$

where $C = \sin a$.

For example, if $a=0$,i.e. $C = 0$, we have

$$|\xi| = \frac{\rho}{\sqrt{2}}\,,\ |\eta| = \frac{\rho}{\sqrt{2}}\,,\ \cos\Theta = 0. \tag{13}$$

In this case the vectors $\vec{\xi}$ and $\vec{\eta}$ are orthogonal independently from the value of λ , and only similarity transformations of the triangle are possible. On the other hand, if $a = \frac{\pi}{2}$, then

$$|\xi| = \frac{\rho}{\sqrt{2}} \sqrt{1 + \sin\lambda}\,,$$

$$|\eta| = \frac{\rho}{\sqrt{2}} \sqrt{1 - \sin\lambda}\,, \tag{14}$$

$$\cos\Theta = 1$$

i.e. the system is linear, and the ends of the vectors $\vec{\xi}$ and $\vec{\eta}$ are oscillating about the point $\frac{\rho}{\sqrt{2}}$.

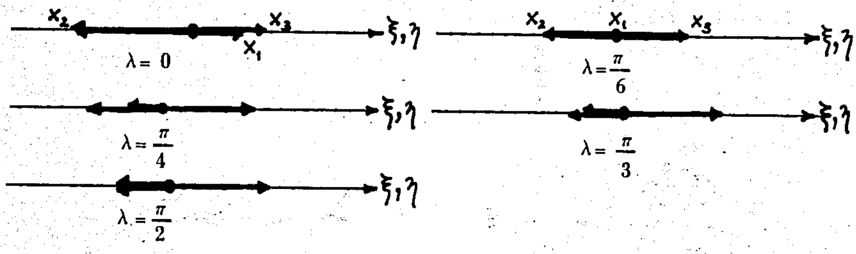
Expressing the positions of all the three particles in the c.m. system in terms of $\vec{\xi}$ and $\vec{\eta}$:

$$\vec{x}_1 = -\frac{1}{\sqrt{6}}\vec{\xi} + \frac{1}{\sqrt{2}}\vec{\eta} = \sqrt{\frac{2}{3}}\left(\cos\frac{2\pi}{3}\vec{\xi} + \sin\frac{2\pi}{3}\vec{\eta}\right)$$

$$\vec{x}_2 = -\frac{1}{\sqrt{6}}\vec{\xi} - \frac{1}{\sqrt{2}}\vec{\eta} = \sqrt{\frac{2}{3}}\left(\cos\frac{4\pi}{3}\vec{\xi} + \sin\frac{4\pi}{3}\vec{\eta}\right) \tag{15}$$

$$\vec{x}_3 = \sqrt{\frac{2}{3}}\,\vec{\xi}$$

it will be easy to represent the position of the particles by their radius-vectors. As an illustration, we consider the case $a = \frac{\pi}{2}$ with different values of λ :



Consider now those deformations, which are connected with the change of a , i.e. which don't leave the moment of inertia unaltered.

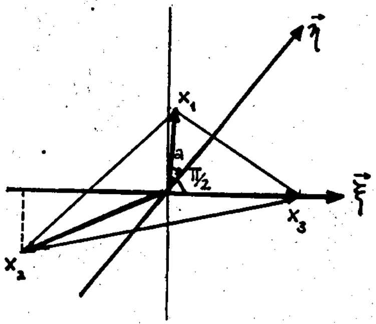
Let $\lambda = 0$, then

$$|\xi| = \frac{\rho}{\sqrt{2}}, \quad |\eta| = \frac{\rho}{\sqrt{2}}, \quad \sin a = \cos \Theta \ . \tag{16}$$

The angle between $\vec{\xi}$ and $\vec{\eta}$ is

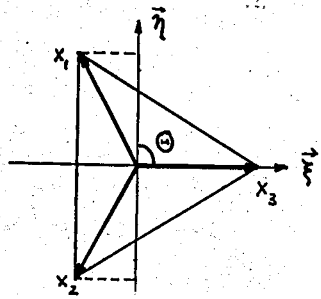
$$\Theta = \frac{\pi}{2} - a$$

and we have

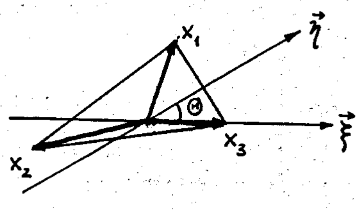


We list here a few particular cases:

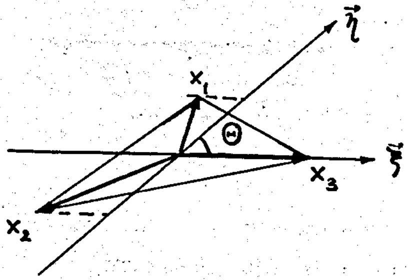
$a = 0, \ \Theta = \frac{\pi}{2}$



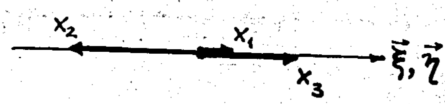
$a = \frac{\pi}{4}, \ \Theta = \frac{\pi}{4}$



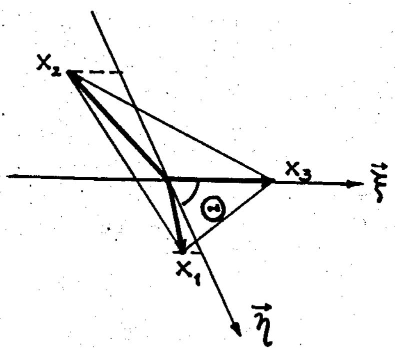
$a = \frac{\pi}{3}$, $\Theta = \frac{\pi}{6}$



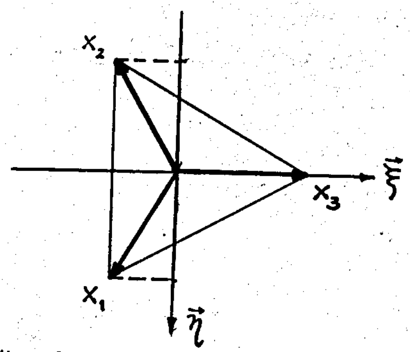
$a = \frac{\pi}{2}$, $\Theta = 0$



$a = \frac{3}{4}\pi$, $\Theta = -\frac{\pi}{4}$



$a = \pi$, $\Theta = -\frac{\pi}{2}$

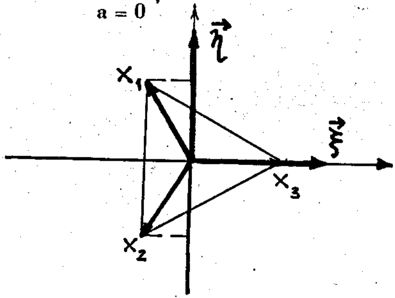


Considering the case $\lambda = \frac{\pi}{2}$, from the formulae

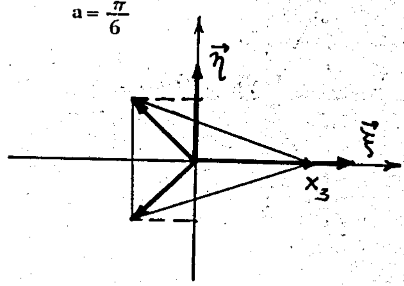
$$
\begin{aligned}
|\xi| &= \frac{\rho}{\sqrt{2}} \sqrt{1 + \sin a} \\
|\eta| &= \frac{\rho}{\sqrt{2}} \sqrt{1 - \sin a} \\
\cos \Theta &= 0
\end{aligned}
\tag{17}
$$

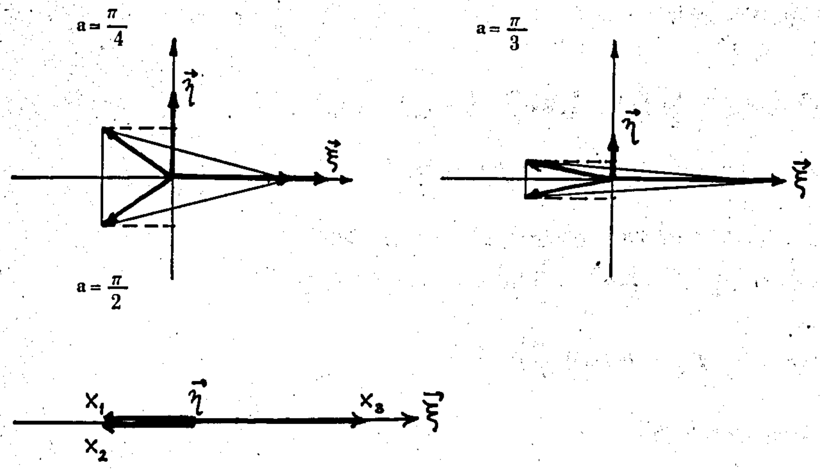
it can be easily seen, that $\vec{\xi}$ and $\vec{\eta}$ are orthogonal , and their lengths can oscillate between zero and ρ.

$a = 0$



$a = \frac{\pi}{6}$





§3. The Free Lagrangian

In this paragraph we present the Euler equations. First of all, we have to construct the Lagrangian $L = T - U$. For free particles we have

$$L = T = \frac{1}{2}\left(\frac{ds}{dt}\right)^2. \tag{18}$$

Let's begin with the expression

$$dz = \frac{1}{\rho} z\, d\rho - i z\, d\lambda + \frac{1}{2} e^{-i\lambda} (\ell\ x z^*)\, da - (d\omega\, x z), \tag{19}$$

where $d\omega$ is the infinitesimal rotation with projections $d\omega_i$ onto the fixed axes. The rotations about the moving axes are defined as

$$d\Omega_i = \ell_i\, d\omega . \tag{20}$$

They can be expressed in terms of the Euler angles in the form

$$\begin{aligned} d\Omega_1 &= -\cos\phi_1 \sin\theta\, d\phi_2 + \sin\phi_1 d\theta , \\ d\Omega_2 &= \sin\phi_1 \sin\theta\, d\phi_2 + \cos\phi_1 d\theta , \\ d\Omega_3 &= -d\phi_1 - \cos\theta\, d\phi_2 . \end{aligned} \tag{21}$$

From (19) we get

$$ds^2 = |dz|^2 =$$

$$= \rho^2 [\frac{1}{4} da^2 + \frac{1}{4} d\lambda^2 + \frac{1}{2} d\Omega_1^2 + \frac{1}{2} d\Omega_2^2 + d\Omega_3^2 -$$

$$- \sin a \, d\Omega_1 d\Omega_2 - \cos a \, d\Omega_3 \, d\lambda \,] + d\rho^2 .$$

Obviously, the wanted expression will be

$$T = \frac{1}{2} \rho^2 [\frac{1}{4} \dot{a}^2 + \frac{1}{4} \dot{\lambda}^2 + \frac{1}{2} \dot{\Omega}_1^2 + \frac{1}{2} \dot{\Omega}_2^2 + \dot{\Omega}_3^2 -$$

$$- \sin a \, \dot{\Omega}_1 \dot{\Omega}_2 - \cos a \, \dot{\Omega}_3 \dot{\lambda} \,] + \frac{1}{2} \dot{\rho}^2. \tag{23}$$

Due to the formula

$$p_i = \frac{\partial T}{\partial \dot{q}_i} \tag{24}$$

we can write down the momenta

$$p_a = \frac{1}{4} \rho^2 \dot{a}$$

$$p_\lambda = \frac{1}{2} \rho^2 (\frac{1}{2} \dot{\lambda} - \cos a \, \dot{\Omega}_3)$$

$$p_{\Omega_1} = \frac{1}{2} \rho^2 (\dot{\Omega}_1 - \sin a \, \dot{\Omega}_2)$$

$$p_{\Omega_2} = \frac{1}{2} \rho^2 (\dot{\Omega}_2 - \sin a \, \dot{\Omega}_1) \tag{25}$$

$$p_{\Omega_3} = \frac{1}{2} \rho^2 (2\dot{\Omega}_3 - \cos a \, \dot{\lambda})$$

$$p_\rho = \dot{\rho}$$

and the corresponding $\dot{p}_i$:

$$\dot{p}_a = \frac{1}{2} \rho \dot{\rho} \dot{a} + \frac{1}{4} \rho^2 \ddot{a}$$

$$\dot{p}_\lambda = \frac{1}{2} \rho^2 (\frac{1}{2} \ddot{\lambda} + \sin a \, \dot{a} \, \dot{\Omega}_3 - \cos a \, \ddot{\Omega}_3) + \rho \, (\frac{1}{2} \dot{\lambda} \dot{\rho} - \cos a \, \dot{\Omega}_3 \dot{\rho})$$

$$\dot{p}_{\Omega_1} = \frac{1}{2}\rho^2(\ddot{\Omega}_1 - \cos a\,\dot{a}\dot{\Omega}_2 - \sin a\,\ddot{\Omega}_2) + \rho\,(\dot{\rho}\dot{\Omega}_1 - \sin a\,\dot{\rho}\dot{\Omega}_2)$$

$$\dot{p}_{\Omega_2} = \frac{1}{2}\rho^2(\ddot{\Omega}_2 - \cos a\,\dot{a}\,\dot{\Omega}_1 - \sin a\,\ddot{\Omega}_1) + \rho\,(\dot{\rho}\,\dot{\Omega}_2 - \sin a\,\dot{\rho}\,\dot{\Omega}_1)$$

$$\dot{p}_{\Omega_3} = \frac{1}{2}\rho^2(2\,\ddot{\Omega}_3 + \sin a\,\dot{a}\,\dot{\lambda} - \cos a\,\ddot{\lambda}) + \rho\,(2\dot{\rho}\,\dot{\Omega}_3 - \cos a\,\dot{\rho}\dot{\lambda}) \tag{26}$$

$$\dot{p}_{\rho} = \ddot{\rho}\,.$$

Now we can construct the equations of motion

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} = 0. \tag{27}$$

To obtain the equations explicitly, we have to return to the Euler angles. Indeed, $\dot{\Omega}_i$ are not derivatives of any angles Ω_i (that's why they are called quasi-coordinates), and the Euler equation in terms of these quasi-coordinates will be written in another form.

Thus, instead of (23) we have to take the Lagrangian expressed in terms of the Euler angles:

$$\begin{aligned} T = \frac{1}{2}\rho^2 [&\frac{1}{4}\dot{a}^2 + \frac{1}{4}\dot{\lambda}^2 + \dot{\phi}_1^2 + \frac{1}{2}\dot{\theta}^2 + \frac{1}{2}\dot{\phi}_2^2 + \\ &+ \frac{1}{2}\cos^2\theta\dot{\phi}_2^2 + 2\cos\theta\,\dot{\phi}_1\dot{\phi}_2 + \\ &+ \sin a\,(\frac{1}{2}\sin 2\phi_1\,\sin^2\theta\,\dot{\phi}_2^2 + \cos 2\phi_1\,\sin\theta\dot{\phi}_2\dot{\theta} - \frac{1}{2}\sin 2\phi_1\dot{\theta}^2) + \\ &+ \cos a\,(\dot{\phi}_1\dot{\lambda} + \cos\theta\dot{\phi}_2\dot{\lambda})] + \frac{1}{2}\dot{\rho}^2. \end{aligned} \tag{28}$$

The equations of free motion are, as follows:

$$\frac{1}{2}\ddot{a} + \sin a\,(\dot{\phi}_1\dot{\lambda} + \cos\theta\,\dot{\phi}_2\dot{\lambda}) -$$
$$- \cos a\,(\frac{1}{2}\sin 2\phi_1 \sin^2\theta\,\dot{\phi}_2^{\,2} + \cos 2\phi\ \sin\theta\,\dot{\phi}_2\dot{\theta} -$$
$$- \frac{1}{2}\sin 2\phi_1\ \dot{\theta}^2) + \frac{1}{\rho}\dot{\rho}\,\dot{a} = 0, \qquad (29)$$

$$\frac{1}{2}\ddot{\lambda} - \sin a\,(\dot{a}\dot{\phi}_1 + \cos\theta\,\dot{a}\dot{\phi}_2) +$$
$$+ \cos a\,(\ddot{\phi}_1 - \sin\theta\,\dot{\theta}\dot{\phi}_2 + \cos\theta\,\ddot{\phi}_2) +$$
$$+ \frac{1}{\rho}\dot{\rho}\dot{\lambda} + \frac{2}{\rho}\cos a\,(\dot{\phi}_1\dot{\rho} + \cos\theta\ \dot{\phi}_2\dot{\rho}) = 0, \qquad (30)$$

$$\ddot{\phi}_1 - \sin\theta\,\dot{\theta}\,\dot{\phi}_2 + \cos\theta\,\ddot{\phi}_2 + \frac{1}{2}\cos a\,\ddot{\lambda} -$$
$$- \frac{1}{2}\sin a\,(\dot{a}\dot{\lambda} + \cos 2\phi_1\ \sin^2\theta\,\dot{\phi}_2^{\,2} -$$
$$-2\sin 2\phi_1\ \sin\theta\ \dot{\phi}_2\dot{\theta} - \cos 2\phi_1\ \dot{\theta}^2) +$$
$$+ \frac{1}{\rho}(2\dot{\phi}_1\ \dot{\rho}_1 + 2\cos\theta\,\dot{\phi}_2\dot{\rho} + \cos a\ \dot{\lambda}\dot{\rho}) = 0, \qquad (31)$$

$$\frac{1}{2}\sin a\cos 2\phi_1\ddot{\theta} + \frac{1}{2}\sin\theta\,(1 + \sin a\sin 2\phi_1)\ddot{\phi}_2 -$$
$$- (1 + \sin a\sin 2\phi_1)\dot{\phi}_1\ \dot{\theta} + \sin a\,(\cos 2\phi_1\ \sin\theta\,\dot{\phi}_1\dot{\phi}_2 +$$
$$+ \frac{1}{2}\operatorname{ctg}\theta\cos 2\phi_1\ \dot{\theta}^2 + \sin 2\phi_1\ \cos\theta\,\dot{\phi}_2\ \dot{\theta}) +$$
$$+ \cos a\,(\frac{1}{2}\sin 2\phi_1\ \sin\ \theta\,\dot{a}\dot{\phi}_2 + \frac{1}{2}\cos 2\phi_1\dot{\theta}\dot{a} - \frac{1}{2}\dot{\lambda}\dot{\theta}) + \qquad (32)$$
$$+ \frac{1}{\rho}[\sin\theta\,\dot{\phi}_2\,\dot{\rho}\,(1 + \sin a\ \sin 2\phi_1) + \sin a\ \cos 2\phi_1\ \dot{\theta}\dot{\rho}] = 0,$$

$$(1-\sin a \sin 2\phi_1)\ddot{\theta} + \frac{1}{2}\sin 2\theta\,\dot{\phi}_2^2 + 2\sin\theta\,\dot{\phi}_1\dot{\phi}_2 + \sin a\,(\cos 2\phi_1 \sin\theta\,\ddot{\phi}_2 -$$

$$-2\cos 2\phi_1\,\dot{\phi}_1\,\dot{\theta} - 2\sin 2\phi_1 \sin\theta\,\dot{\phi}_1\,\dot{\phi}_2 - \frac{1}{2}\sin 2\phi_1\,\dot{a}\,\dot{\theta} + \sin\theta\,\dot{\phi}_2\,\dot{\lambda}) + \tag{33}$$

$$+ \cos a\,(\cos 2\phi_1 \sin\theta\,\dot{a}\,\dot{\phi}_2 - \sin 2\phi_1\,\dot{a}\,\dot{\theta} + \sin\theta\,\dot{\phi}_2\dot{\lambda}) +$$

$$+\frac{2}{\rho}[\,\dot{\theta}\dot{\rho} + \sin a\,(\cos 2\phi_1 \sin\theta\,\dot{\rho}\,\dot{\phi}_2 - \sin 2\phi_1\,\dot{\rho}\,\dot{\theta})] = 0,$$

and finally,

$$\ddot{\rho} - \rho\,[\frac{1}{4}\dot{a}^2 + \frac{1}{4}\dot{\lambda}^2 + \dot{\phi}_1^2 + \frac{1}{2}\dot{\theta}^2 + \frac{1}{2}\dot{\phi}_2^2 + \frac{1}{2}\cos^2\theta\,\dot{\phi}_2^2 + 2\cos\theta\,\dot{\phi}_1\,\dot{\phi}_2 +$$

$$+ \sin a\,(\frac{1}{2}\sin 2\phi_1 \sin^2\theta\,\dot{\phi}_2^2 + \cos 2\phi_1 \sin\theta\,\dot{\phi}_2\,\dot{\theta} - \frac{1}{2}\sin 2\phi_1\,\dot{\theta}^2) + \tag{34}$$

$$+ \cos a\,(\dot{\phi}_1\dot{\lambda} + \cos\theta\,\dot{\phi}_2\,\dot{\lambda})\,] = 0.$$

In the following we will consider a few particular cases.

1) Motion of the triangle in the plane

$\dot{\theta} = \dot{\phi}_2 = 0.$

In this case the free Lagrangian can be written in the form

$$T = \frac{1}{2}\rho^2[\frac{1}{4}\dot{a}^2 + \frac{1}{4}\dot{\lambda}^2 + \cos a\,\dot{\phi}_1\dot{\lambda} + \dot{\phi}_1^2] + \frac{1}{2}\dot{\rho}^2, \tag{35}$$

or remembering, that

$$p_{\phi_1} = \frac{1}{2}\rho^2\,(2\dot{\phi}_1 + 2\cos\theta\,\dot{\phi}_2 + \cos a\,\dot{\lambda})$$

$$p_{\phi_2} = \frac{1}{2}\rho^2\,[\dot{\phi}_2\,(1 + \cos^2\theta + \sin a \sin 2\phi_1 \sin^2\theta) + 2\cos\theta\,\dot{\phi}_1 +$$

$$+ \sin a \cos 2\phi_1 \sin\theta\,\dot{\theta} + \cos a \cos\theta\,\dot{\lambda}\,] \tag{36}$$

$$p_\theta = \frac{1}{2}\rho^2\,[\,\dot{\theta} + \sin a\,(\cos 2\phi_1 \sin\theta\,\dot{\phi}_2 - \sin 2\phi_1\,\dot{\theta})],$$

$$T = \frac{1}{\rho^2}\,[2 p_a^2 + \frac{1}{\sin^2 a}\,(\frac{1}{2} p_{\phi_1}^2 - 2 p_{\phi_1} p_\lambda \cos a + 2 p_\lambda^2)] + \frac{1}{2} p_\rho^2 . \qquad (37)$$

The equations of motion are in this case the following:

$$\frac{1}{2}\ddot{a} + \frac{1}{\rho}\dot{\rho}\,\dot{a} + \sin a\,\dot{\phi}_1\,\dot{\lambda} = 0,$$

$$\frac{1}{2}\ddot{\lambda} + \frac{1}{\rho}\dot{\rho}\dot{\lambda} - \sin a\,\dot{a}\,\dot{\phi}_1 + \cos a\,\ddot{\phi}_1 + \frac{2}{\rho}\cos a\,\dot{\phi}_1\,\dot{\rho} = 0, \qquad (38)$$

$$\ddot{\phi}_1 + \frac{1}{2}\cos a\,\ddot{\lambda} - \frac{1}{2}\sin a\,\dot{a}\,\dot{\lambda} + \frac{1}{\rho}(2\dot{\phi}_1\,\dot{\rho} + \cos a\,\dot{\lambda}\dot{\rho}) = 0,$$

$$\ddot{\rho} - \rho\,(\frac{1}{4}\dot{a}^2 + \frac{1}{4}\dot{\lambda}^2 + \dot{\phi}_1^2 + \cos a\,\dot{\phi}_1\,\dot{\lambda}) = 0.$$

2) Deforming triangle.

$$\dot{\theta} = \dot{\phi}_2 = 0, \quad p_{\phi_1} = 0.$$

The free Lagrangian obtains the form

$$T = \frac{1}{2}\rho^2\,(\frac{1}{4}\dot{a}^2 + \frac{1}{4}\dot{\lambda}^2 \sin^2 a) + \frac{1}{2}\dot{\rho}^2 =$$

$$= \frac{2}{\rho^2}\,(p_a^2 + \frac{1}{\sin^2 a} p_\lambda^2) + \frac{1}{2} p_\rho^2 . \qquad (39)$$

Let's note here, that for $p_\rho = 0$ this expression has the same form as the Lagrangian of the rotator. We see here an example of the hidden symmetry, which can be generalized to the case of a deforming rotator.

The equations of motion corresponding to the Lagrange function (39), are

$$\frac{1}{2}\ddot{a} + \frac{1}{\rho}\dot{\rho}\,\dot{a} - \frac{1}{2}\sin a\cos a\,\dot{\lambda}^2 = 0$$

$$\sin a\,(\frac{1}{2}\ddot{\lambda} + \frac{1}{\rho}\dot{\rho}\,\dot{\lambda}) + \cos a\,\dot{a}\,\dot{\lambda} = 0 \qquad (40)$$

$$\ddot{\rho} - \rho\,(\frac{1}{4}\dot{a}^2 + \frac{1}{4}\sin^2 a\,\dot{\lambda}^2) = 0.$$

If we add on the right hand side forces depending only on ρ , we get the equations of a non rigid rotator.

§4. Potentials for Three-Body Systems.

Let's investigate two examples of interacting particles.

1). The harmonic oscillator potential.

The equilibrium state of the three-particle system is an equilateral triangle with a side ρ_0 . It can be described by the vectors $\vec{\eta}_0$ and $\vec{\xi}_0$

$$\vec{\xi}_0 = \begin{pmatrix} 0 \\ \frac{\rho_0}{\sqrt{2}} \end{pmatrix} \qquad \vec{\eta}_0 = \begin{pmatrix} \frac{\rho_0}{\sqrt{2}} \\ 0 \end{pmatrix}$$

$$\vec{\xi}_0 \vec{\eta}_0 = 0 , \quad \xi_0^2 - \eta_0^2 = 0 . \tag{41}$$

The parameters a_0 and λ_0 corresponding to the equilibrium state will have the values

$$a_0 = \pi , \quad \lambda_0 = 0, .$$

and consequently

$$\vec{z}_0 = \frac{\rho_0}{\sqrt{2}} (e^{i\frac{\pi}{2}} \vec{\ell}_1 + e^{-i\frac{\pi}{2}} \vec{\ell}_2). \tag{42}$$

Consider the motion of the three particles in the potential:

$$U = \frac{1}{2} [(\xi - \xi_0)^2 + (\eta - \eta_0)^2] =$$

$$+ \frac{1}{2} (\rho^2 + \rho_0^2 - 2\rho\rho_0 \sin \frac{a}{2} \cos \frac{\lambda}{2}). \tag{43}$$

From the formula

$$F_i = -\frac{\partial U}{\partial q_i} \tag{44}$$

we obtain

$$\begin{aligned} F_a &= \frac{1}{2}\rho\rho_0 \cos\frac{a}{2}\cos\frac{\lambda}{2} \\ F_\lambda &= -\frac{1}{2}\rho\rho_0 \sin\frac{a}{2}\sin\frac{\lambda}{2} \\ F_\rho &= -\rho+\rho_0 \sin\frac{a}{2}\cos\frac{\lambda}{2} \\ F_{\phi_1} &= F_{\phi_2} = F_\theta = 0. \end{aligned} \tag{45}$$

Constructing $L = T - U$, it is easy to get now the equations of motion

$$\frac{d}{dt}\frac{\partial L}{\partial \dot q_i} - \frac{\partial L}{\partial q_i} = 0. \tag{46}$$

The equations $\frac{d}{dt}\frac{\partial L}{\partial \dot\phi_i} - \frac{\partial L}{\partial \phi_i} = 0$ (where $\phi_3 \equiv \theta$) stay unchanged; instead of the equations (29), (30) and (34) we obtain

$$\begin{aligned} &\frac{1}{2}\ddot a - \cos a\left(\frac{1}{2}\sin 2\phi_1 \sin^2\theta\dot\phi_2^2 + \cos 2\phi_1 \sin\theta\,\dot\phi_2\dot\theta - \frac{1}{2}\sin 2\phi_1\dot\theta^2\right) + \\ &+ \sin a\,(\dot\phi_1\dot\lambda + \cos\theta\,\dot\phi_2\dot\lambda) + \frac{1}{\rho}\dot\rho\dot a - \frac{\rho_0}{\rho}\cos\frac{a}{2}\cos\frac{\lambda}{2} = 0, \end{aligned} \tag{47}$$

$$\begin{aligned} &\frac{1}{2}\ddot\lambda - \sin a\,(\dot a\dot\phi_1 + \cos\theta\,\dot a\dot\phi_2) + \cos a\,(\ddot\phi_1 - \sin\theta\,\dot\theta\dot\phi_2 + \cos\theta\,\ddot\phi_2) + \\ &+ \frac{1}{\rho}\dot\rho\dot\lambda + \frac{2}{\rho}\cos a\,(\dot\phi_1\dot\rho + \cos\theta\,\dot\phi_2\dot\rho) + \frac{\rho_0}{\rho}\sin\frac{a}{2}\sin\frac{\lambda}{2} = 0, \end{aligned} \tag{48}$$

and

$$\begin{aligned} &\ddot\rho - \rho\left[\frac{1}{4}\dot a^2 + \frac{1}{4}\dot\lambda^2 + \dot\phi_1^2 + \frac{1}{2}\dot\theta^2 + \frac{1}{2}\dot\phi_2^2 + \frac{1}{2}\cos^2\theta\,\dot\phi_2^2 + 2\cos\theta\,\dot\phi_1\dot\phi_2 + \right. \\ &+ \sin a\left(\frac{1}{2}\sin 2\phi_1 \sin^2\theta\,\dot\phi_2^2 + \cos 2\phi_1 \sin\theta\,\dot\phi_2\dot\theta - \frac{1}{2}\sin 2\phi_1\dot\theta^2\right) + \\ &\left.+ \cos a\,(\dot\phi_1\dot\lambda + \cos\theta\,\dot\phi_2\dot\lambda)\right] + \rho - \rho_0 \sin\frac{a}{2}\cos\frac{\lambda}{2} = 0. \end{aligned} \tag{49}$$

Consider now once more the case of a non-rotating triangle. $\dot{\theta} = \dot{\phi}_2 = 0$ and $p_{\phi_1} = 0$, i.e. $\dot{\phi}_1 = -\frac{1}{2}\cos a \dot{\lambda}$.

The equations of motion obtain the form

$$\frac{1}{2}\ddot{a} + \frac{1}{\rho}\dot{\rho}\dot{a} - \frac{1}{2}\sin a \cos a \dot{\lambda}^2 - \frac{\rho_0}{\rho}\cos\frac{a}{2}\cos\frac{\lambda}{2} = 0$$

$$\sin^2 a\left(\frac{1}{2}\ddot{\lambda} + \frac{1}{\rho}\dot{\rho}\dot{\lambda}\right) + \sin a \cos a \dot{a}\dot{\lambda} + \frac{\rho_0}{\rho}\sin\frac{a}{2}\sin\frac{\lambda}{2} = 0 \qquad (50)$$

$$\ddot{\rho} - \rho\left(\frac{1}{4}\dot{a}^2 + \frac{1}{4}\sin^2 a \dot{\lambda}^2\right) + \rho - \rho_0 \sin\frac{a}{2}\cos\frac{\lambda}{2} = 0.$$

If in this case we take $\rho_0 = 0$, we will have

$$\frac{1}{2}\ddot{a} + \frac{1}{\rho}\dot{\rho}\dot{a} - \frac{1}{2}\sin a \cos a \dot{\lambda}^2 = 0$$

$$\sin a\left(\frac{1}{2}\ddot{\lambda} + \frac{1}{\rho}\dot{\rho}\dot{\lambda}\right) + \cos a \dot{a}\dot{\lambda} = 0 \qquad (51)$$

$$\ddot{\rho} - \rho\left(\frac{1}{4}\dot{a}^2 + \frac{1}{4}\sin^2 a \dot{\lambda}^2\right) + \rho = 0.$$

As an example, we consider solutions with constant λ and a and illustrate a few cases of the deformed triangle in details. The projections of the radius-vectors $\vec{x}_i$ onto the axes $\vec{\ell}_1$ and $\vec{\ell}_2$ are as follows

$$x_1^{(1)} = \frac{\rho}{2}\left(\sin\frac{a-\lambda}{2} - \frac{1}{\sqrt{3}}\cos\frac{a-\lambda}{2}\right)$$

$$x_2^{(1)} = -\frac{\rho}{2}\left(\sin\frac{a-\lambda}{2} + \frac{1}{\sqrt{3}}\cos\frac{a-\lambda}{2}\right) \qquad (52)$$

$$x_3^{(1)} = \frac{\rho}{\sqrt{3}}\cos\frac{a-\lambda}{2}$$

and

$$x_1^{(2)} = \frac{\rho}{2}\left(\cos\frac{a+\lambda}{2} - \frac{1}{\sqrt{3}}\sin\frac{a+\lambda}{2}\right)$$

$$x_2^{(2)} = -\frac{\rho}{2}\left(\cos\frac{a+\lambda}{2} + \frac{1}{\sqrt{3}}\sin\frac{a+\lambda}{2}\right)$$

$$x_3^{(2)} = \frac{\rho}{\sqrt{3}}\sin\frac{a+\lambda}{2}.$$

If a = const. and λ = const, and $\rho_0 \neq 0$, we obtain from (50)

$$\frac{\rho_0}{\rho}\cos\frac{a}{2}\cos\frac{\lambda}{2} = 0$$

$$\frac{\rho_0}{\rho}\sin\frac{a}{2}\sin\frac{\lambda}{2} = 0 \qquad (53)$$

$$\ddot{\rho} + \rho - \rho_0\sin\frac{a}{2}\cos\frac{\lambda}{2} = 0.$$

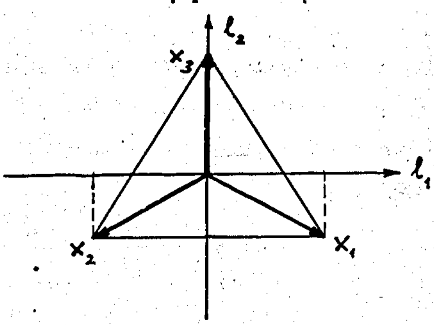
It can be easily seen, that in this case a and λ are multiples of π, and only similarity transformations are possible, for example:

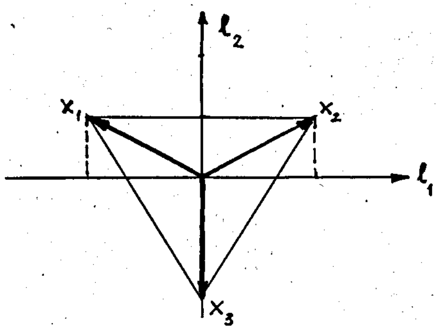
$\lambda = 0,\ a = \pi$

$x_1^{(1)} = \frac{\rho}{2}$ $\quad x_1^{(2)} = -\frac{1}{2\sqrt{3}}\rho$

$x_2^{(1)} = -\frac{\rho}{2}$ $\quad x_2^{(2)} = -\frac{1}{2\sqrt{3}}\rho$

$x_3^{(1)} = 0$ $\quad x_3^{(2)} = \frac{\rho}{\sqrt{3}}$





$\lambda = 0,\quad a = 3\pi$

$x_1^{(1)} = -\frac{\rho}{2}$ $\quad x_1^{(2)} = \frac{\rho}{2\sqrt{3}}$

$x_2^{(1)} = \frac{\rho}{2}$ $\quad x_2^{(2)} = \frac{\rho}{2\sqrt{3}}$

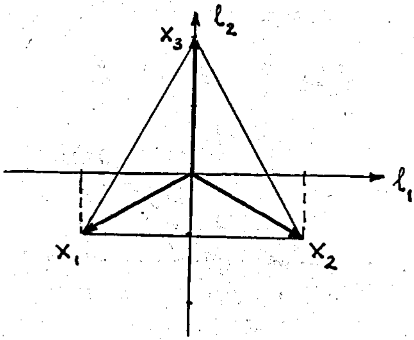
$x_3^{(1)} = 0$ $\quad x_3^{(2)} = -\frac{\rho}{\sqrt{3}}$

$\lambda = \pi \, , \, a = 0$

$x_1^{(1)} = -\frac{\rho}{2} \qquad x_1^{(2)} = -\frac{\rho}{2\sqrt{3}}$

$x_2^{(1)} = \frac{\rho}{2} \qquad x_2^{(2)} = -\frac{\rho}{2\sqrt{3}}$

$x_3^{(1)} = 0 \qquad x_3^{(2)} = \frac{\rho}{\sqrt{3}}$



If, on the contrary , $\rho_0 = 0$, then the fixed value of a still doesn't determine the value of λ , so that arbitrary deformations are possible. We give in the following a few examples:

a) $\lambda = a$

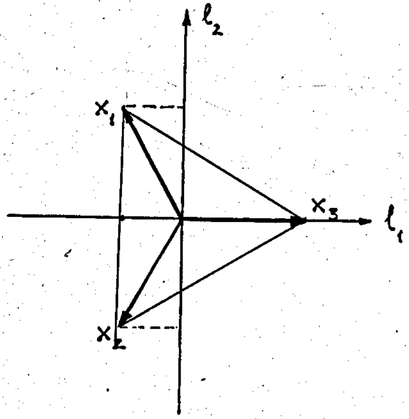
$x_1^{(1)} = -\frac{\rho}{2\sqrt{3}} \qquad x_1^{(2)} = \frac{\rho}{2} \left(\cos a - \frac{1}{\sqrt{3}} \sin a \right)$

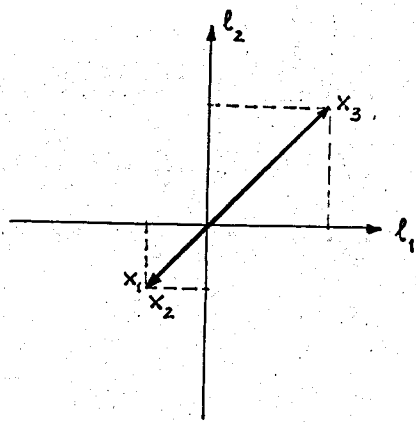
$x_2^{(1)} = -\frac{\rho}{2\sqrt{3}} \qquad x_2^{(2)} = -\frac{\rho}{2} \left(\cos a + \frac{1}{\sqrt{3}} \sin a \right)$

$x_3^{(1)} = \frac{\rho}{\sqrt{3}} \qquad x_3^{(2)} = \frac{\rho}{\sqrt{3}} \sin a$

$\lambda = a = 0 \qquad\qquad \lambda = a = \frac{\pi}{2}$

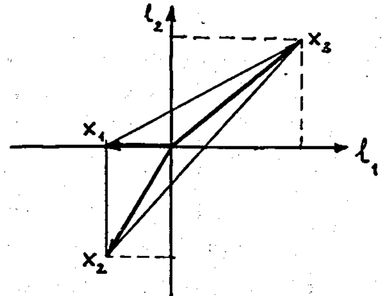
$x_1^{(2)} = \frac{\rho}{2}, \quad x_2^{(2)} = -\frac{\rho}{2}, \quad x_3^{(2)} = 0 \qquad x_1^{(2)} = -\frac{\rho}{2\sqrt{3}}, \; x_2^{(2)} = -\frac{\rho}{2\sqrt{3}}, \; x_3^{(2)} = \frac{\rho}{\sqrt{3}}$





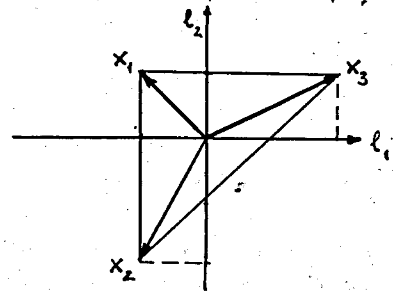
$\lambda = a = \frac{\pi}{3}$

$x_1^{(1)} = 0,\ x_2^{(2)} = -\frac{\rho}{2},\ x_3^{(2)} = \frac{\rho}{2}$



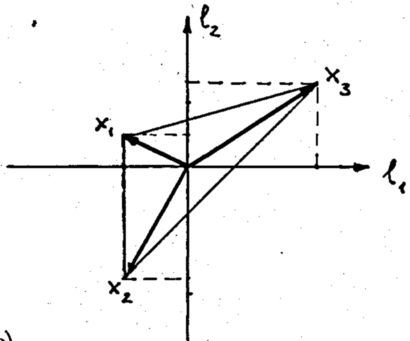
$\lambda = a = \frac{\pi}{6}$

$x_1^{(2)} = \frac{\rho}{2\sqrt{3}},\ x_2^{(2)} = -\frac{\rho}{\sqrt{3}},\ x_3^{(2)} = \frac{\rho}{2\sqrt{3}}$



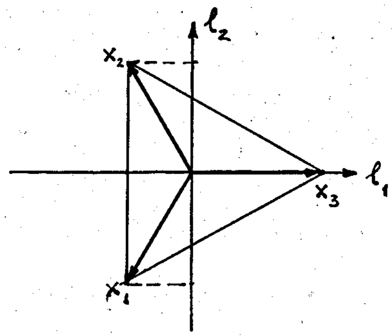
$\lambda = a = \frac{\pi}{4}$

$x_1^{(2)} = \frac{\rho}{2\sqrt{2}}\left(1 - \frac{1}{\sqrt{3}}\right),\ x_2^{(2)} = -\frac{\rho}{2\sqrt{3}}\left(1 + \frac{1}{\sqrt{3}}\right),\ x_3^{(2)} = \frac{\rho}{\sqrt{6}}$



$\lambda = a = \pi$

$x_1^{(2)} = -\frac{\rho}{2},\ x_2^{(2)} = \frac{\rho}{2},\ x_3^{(2)} = 0$



b) $a - \lambda = \frac{\pi}{3}$

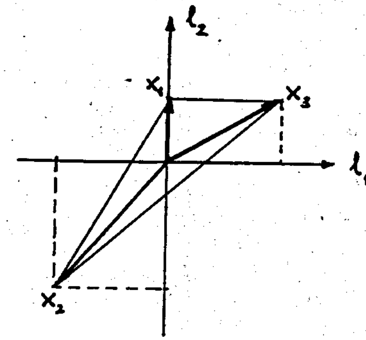
$x_1^{(1)} = 0 \qquad x_1^{(2)} = \frac{\rho}{2}\left(\cos\left(\lambda + \frac{\pi}{6}\right) - \frac{1}{\sqrt{3}} \sin\left(\lambda + \frac{\pi}{6}\right)\right)$

$x_2^{(1)} = -\frac{\rho}{2} \qquad x_2^{(2)} = -\frac{\rho}{2}\left(\cos\left(\lambda + \frac{\pi}{6}\right) + \frac{1}{\sqrt{3}} \sin\left(\lambda + \frac{\pi}{6}\right)\right)$

$x_3^{(1)} = \frac{\rho}{2} \qquad x_3^{(2)} = \frac{\rho}{\sqrt{3}} \sin\left(\lambda + \frac{\pi}{6}\right)$

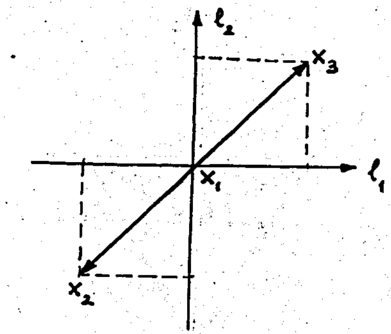
$\lambda = 0,\ a = \frac{\pi}{3}$

$x_1^{(2)} = \frac{\rho}{2\sqrt{3}},\ x_2^{(2)} = -\frac{\rho}{\sqrt{3}},\ x_3^{(2)} = \frac{\rho}{2\sqrt{3}}$



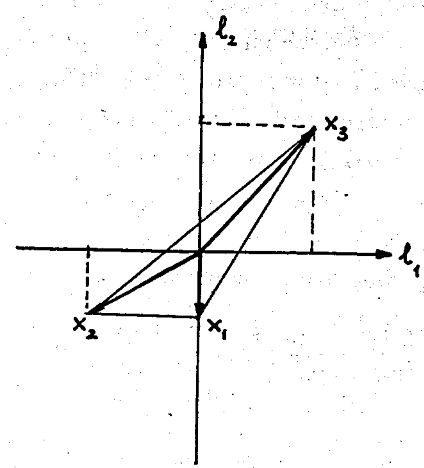
$\lambda = \frac{\pi}{6},\ a = \frac{\pi}{2}$

$x_1^{(2)} = 0,\ x_2^{(2)} = -\frac{\rho}{2},\ x_3^{(2)} = \frac{\rho}{2}$



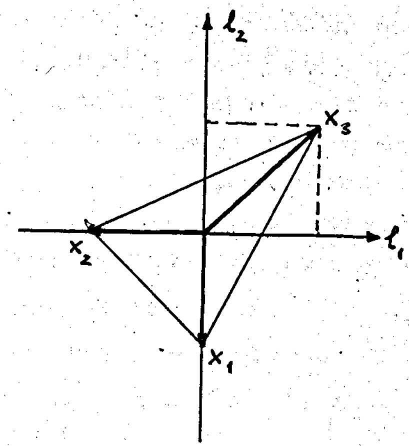
$\lambda = \frac{\pi}{3},\ a = \frac{2\pi}{3}$

$x_1^{(2)} = -\frac{\rho}{2\sqrt{3}},\ x_2^{(2)} = -\frac{\rho}{2\sqrt{3}},\ x_3^{(2)} = \frac{\rho}{\sqrt{3}}$



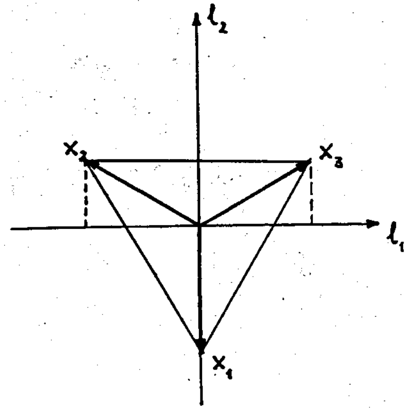
$\lambda = \frac{\pi}{2},\ a = \frac{5}{6}\pi$

$x_1^{(2)} = -\frac{\rho}{2},\ x_2^{(2)} = 0,\ x_3^{(2)} = \frac{\rho}{2}$



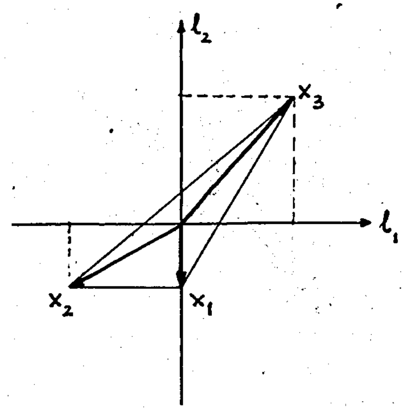
$\lambda = \frac{2}{3}\pi$, $a = \pi$

$x_1^{(2)} = -\frac{\rho}{\sqrt{3}}$, $x_2^{(2)} = \frac{\rho}{2\sqrt{3}}$, $x_3^{(2)} = \frac{\rho}{2\sqrt{3}}$



$\lambda = \frac{5}{6}\pi$, $\dot{a} = \frac{7}{6}\pi$

$x_1^{(2)} = -\frac{\rho}{2\sqrt{3}}$, $x_2^{(2)} = -\frac{\rho}{2\sqrt{3}}$, $x_3^{(2)} = \frac{\rho}{\sqrt{3}}$



2). Three-body problem in the celestial mechanics (the Laplace case [10]). Self-consistent field in classical mechanics.

Suppose, that an attractive Newtonian potential is acting between three particles. We will show, that there exists such a solution, for which all the three particles stay in the vertices of an equilateral triangle, while each particle is moving along an elliptic trajectory about the common centre-of-mass in such a way, as if there was a central body, the mass of which is equal to the sum of masses of the three particles.

Suppose $\dot{a} = \dot{\theta} = \dot{\phi}_2$ be equal to zero. If the particles form an equilateral triangle, we have $a = 0$, and the distance between the particles is ρ. The potential energy in this case is equal to $U = -\frac{3}{\rho}$, so that the equations of motion take the form

$$\ddot{\rho} - \rho\left[\frac{1}{4}\dot{\lambda}^2 + \dot{\phi}_1^2 + \dot{\phi}_1\dot{\lambda}\right] + \frac{3}{\rho^2} = 0$$

$$\frac{1}{2}\ddot{\lambda} + \ddot{\phi}_1 + \frac{\dot{\rho}}{\rho}\dot{\lambda} + \frac{2}{\rho}\dot{\phi}_1\dot{\rho} = 0. \qquad (54)$$

Introducing a new variable

$$\dot{\psi} = \frac{1}{2}\dot{\lambda} + \dot{\phi}_1 \qquad (5$$

we obtain the Kepler equations

$$\ddot{\rho} - \rho\dot{\psi}^2 + \frac{3}{\rho^2} = 0$$

$$\ddot{\psi} + \frac{2}{\rho}\dot{\rho}\dot{\psi} = 0. \qquad (5$$

If we express now $\vec{x}_1$, $\vec{x}_2$ and $\vec{x}_3$ in terms of ρ, we get fo them the equations of three ellipses. It is easy to prove, that i the case of $a \neq 0$ the equations will not lead to the Kepler equa tions. Note, that the type of the solution is independent of the for of the potential. Indeed, we use only the fact that the forces actin on any of the three particles are directed to the centre-of-mass o the triangle.

§5. Euler Equations in Terms of Quasi-Coordinates

In §3 we have obtained the Euler equations in terms of th "true" coordinates a, λ, ρ, ϕ_1, ϕ_2 and θ. Now our purpo se is to make it clear, how the form of the equations will change if we consider the components of angular velocity Ω_i (formally w will call them derivatives of quasi-coordinates) instead of the deri vatives of the Euler angles ϕ_i (we denoted $\phi_3 \equiv \theta$). Obviously Ω do not correspond to any variables Ω_i, and only $d\Omega_i$ makes sense. Still, as it was shown by Boltzmann, formally it is possible to use the quasi-coordinates.

We start from

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\phi}_\lambda} - \frac{\partial L}{\partial\phi_\lambda} = 0 \qquad (\lambda = 1,2,3). \qquad (57$$

The connection between $\dot{\Omega}_i$ and $\dot{\phi}_i$ can be written as

$$\dot{\Omega}_r = \sum_p B_{rp}\dot{\phi}_p \qquad \text{or} \qquad d\Omega_r = \sum_p B_{rp}\, d\phi_p \qquad (58)$$

and

$$\dot\phi_k = \sum_{\ell} A_{k\ell}\,\dot\Omega_\ell \qquad \text{or} \qquad d\phi_k = \sum_{\ell} A_{k\ell}\, d\Omega_\ell ,$$

where $p, r, k, \ell = 1,2,3.$

Multiplying (57) by $A_{\lambda r}$ and summarizing we get

$$\sum_\lambda A_{\lambda r}\left\{ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot\phi_\lambda}\right) - \frac{\partial L}{\partial \phi_\lambda}\right\} = 0. \tag{60}$$

Making use of (58) we can write

$$\frac{\partial L}{\partial \dot\phi_\lambda} = \sum_s \frac{\partial L}{\partial \dot\Omega_s} B_{s\lambda} \tag{61}$$

and thus

$$\sum_\lambda A_{\lambda r}\left\{ \sum_s B_{s\lambda}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\Omega_s}\right) + \sum_s \frac{\partial L}{\partial \dot\Omega_s}\frac{dB_{s\lambda}}{dt} - \frac{\partial L}{\partial \phi_\lambda}\right\} = 0. \tag{62}$$

As soon as

$$\sum_\lambda A_{\lambda r} B_{s\lambda} = \delta_{rs} \tag{63}$$

we obtain

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\Omega_r}\right) + \sum_\lambda \sum_s A_{\lambda r}\frac{dB_{s\lambda}}{dt}\frac{\partial L}{\partial \dot\Omega_s} - \sum_\lambda A_{\lambda r}\frac{\partial L}{\partial \phi_\lambda} = 0. \tag{64}$$

Making use of the conditions (58) and (59), one can express the equation in the form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\Omega_r}\right) + \sum_\lambda \sum_s \sum_m \sum_\ell A_{\lambda r} A_{m\ell}\frac{\partial L}{\partial \dot\Omega_s}\dot\Omega_\ell\left(\frac{\partial B_{s\lambda}}{\partial \phi_m} - \frac{\partial B_{sm}}{\partial \phi_\lambda}\right) - \tag{65}$$

$$-\sum_\lambda A_{\lambda r}\frac{\partial L}{\partial \phi_\lambda} = 0.$$

The expression

$$\sum_\lambda \sum_m A_{\lambda r} A_{m\ell}\left(\frac{\partial B_{s\lambda}}{\partial \phi_m} - \frac{\partial B_{sm}}{\partial \phi_\lambda}\right) \equiv \gamma_{rs\ell} \qquad (6$$

does not depend on the motion of the system, only on the connection between the derivatives of the true coordinates and of the quasi-coordinates . Thus, finally, the equations of motion obtain the following form:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\Omega_r}\right) + \sum_s \sum_\ell \dot\Omega_\ell \frac{\partial L}{\partial \dot\Omega_s}\gamma_{rs\ell} - \sum_\lambda A_{\lambda r}\frac{\partial L}{\partial \phi_\lambda} = 0. \qquad (6$$

In the last term the partial derivative is calculated at constant $\dot\Omega_i$. Formally one usually writes this term as $\frac{\partial L}{\partial \Omega_i}$.

The Euler equation written in the form (67) is invariant under a transformation group which includes the nonintegrable transformation of coordinates (58), (59). If the transformation is integrable, then the middle term in (67) vanishes, and the equation takes the usual form.

In our case $\gamma_{rs\ell}$ is a totally antisymmetric unit tensor, hence it is easy to write down the equations of motion in terms of the quasi-coordinates:

$$\frac{1}{2}\ddot a + \cos a\,\dot\Omega_1\dot\Omega_2 - \sin a\,\dot\Omega_3\,\dot\lambda + \frac{1}{\rho}\dot\rho\,\dot a = 0 \qquad (6$$

$$\frac{1}{2}\ddot\lambda + \sin a\,\dot a\,\dot\Omega_3 - \cos a\,\ddot\Omega_3 + \frac{1}{\rho}(\dot\lambda\dot\rho - 2\cos a\,\dot\Omega_3\dot\rho) = 0 \qquad ($$

$$\ddot\rho - \rho\,\left(\frac{1}{4}\dot a^2 + \frac{1}{4}\dot\lambda^2 + \frac{1}{2}\dot\Omega_1^2 + \frac{1}{2}\dot\Omega_2^2 + \dot\Omega_3^2 - \right. \qquad ($$

$$\left. - \sin a\,\dot\Omega_1\dot\Omega_2 - \cos a\,\dot\Omega_3\dot\lambda\right) = 0$$

$$\ddot\Omega_1 - \cos a\,\dot a\,\dot\Omega_2 - \sin a\,\ddot\Omega_2 - \dot\Omega_2\dot\Omega_3 - \sin a\,\dot\Omega_1\dot\Omega_3 +$$

$$+ \cos a\,\dot\lambda\,\dot\Omega_2 + \frac{2}{\rho}(\dot\rho\,\dot\Omega_1 - \sin a\,\dot\rho\,\dot\Omega_2) = 0$$

$$\ddot\Omega_2 - \cos a\,\dot a\,\dot\Omega_1 - \sin a\,\ddot\Omega_1 + \dot\Omega_1\dot\Omega_3 - \cos a\,\dot\Omega_1\dot\lambda +$$

$$+ \sin a\,\dot\Omega_3\dot\Omega_2 + \frac{2}{\rho}(\dot\rho\,\dot\Omega_2 - \sin a\,\dot\rho\,\dot\Omega_1) = 0$$

$$2\ddot{\Omega}_3 + \sin a\,\dot{a}\dot{\lambda} - \cos a\,\ddot{\lambda} + \sin a\,(\dot{\Omega}_1^2 - \dot{\Omega}_2^2) + \frac{2}{\rho}(\rho\,\dot{\Omega}_2 - \sin a\,\dot{\rho}\,\dot{\Omega}_1) = 0. \tag{73}$$

Considering the case $\dot{\Omega}_1 = \dot{\Omega}_2 = 0$, we obtain

$$\begin{aligned}
&\frac{1}{2}\ddot{a} - \sin a\,\dot{\Omega}_3\dot{\lambda} + \frac{1}{\rho}\dot{\rho}\dot{a} = 0\\
&\frac{1}{2}\ddot{\lambda} + \sin a\,\dot{a}\,\dot{\Omega}_3 - \cos a\,\ddot{\Omega}_3 + \frac{1}{\rho}(\dot{\lambda}\dot{\rho} - 2\cos a\,\dot{\Omega}_3\dot{\rho}) = 0\\
&\ddot{\rho} - \rho\left(\frac{1}{4}\dot{a}^2 + \frac{1}{4}\dot{\lambda}^2 + \dot{\Omega}_3^2 - \cos a\,\dot{\Omega}_3\dot{\lambda}\right) = 0\\
&2\ddot{\Omega}_3 + \sin a\,\dot{a}\dot{\lambda} - \cos a\,\ddot{\lambda} + \frac{2}{\rho}(2\dot{\rho}\,\dot{\Omega}_3 - \cos a\,\dot{\rho}\,\dot{\lambda}) = 0\,.
\end{aligned} \tag{74}$$

The equations of the non-rotating triangle

$$\dot{\Omega}_1 = \dot{\Omega}_2 = 0, \quad p_{\Omega_3} = 0$$

will be

$$\begin{aligned}
&\frac{1}{2}\ddot{a} - \frac{1}{2}\sin a\cos a\,\dot{\lambda}^2 + \frac{1}{\rho}\dot{\rho}\,\dot{a} = 0\\
&\frac{1}{2}\sin a\,\ddot{\lambda} + \cos a\,\dot{a}\dot{\lambda} + \frac{1}{\rho}\dot{\lambda}\dot{\rho}\sin a = 0\\
&\ddot{\rho} - \rho\left(\frac{1}{4}\dot{a}^2 + \frac{1}{4}\dot{\lambda}^2\sin^2 a\right) = 0.
\end{aligned} \tag{75}$$

If at the same time $\dot{\Omega}_3 = 0$, we have $\cos a\,\dot{\lambda} = 0$, i.e. either

$$a = \frac{\pi}{2}$$

and then

$$\begin{aligned}
&\frac{1}{2}\ddot{\lambda} + \frac{1}{\rho}\dot{\lambda}\dot{\rho} = 0\\
&\ddot{\rho} - \frac{1}{4}\rho\dot{\lambda}^2 = 0
\end{aligned} \tag{76}$$

or $\dot{\lambda} = 0$ and consequently

$$\begin{aligned}
&\frac{1}{2}\ddot{a} + \frac{1}{\rho}\dot{\rho}\dot{a} = 0\\
&\ddot{\rho} - \frac{1}{4}\rho\dot{a}^2 = 0\,.
\end{aligned} \tag{77}$$

Finally, we express the Lagrange function and the equations of motion in terms of an another system of quasi-coordinates, which describe the deformations of the triangle also.

Assume

$$\begin{aligned}
\pi_1 &= \dot{a}\sin\frac{\lambda}{2} + \dot{\Omega}_3\sin a\cos\frac{\lambda}{2}\\
\pi_2 &= \dot{a}\cos\frac{\lambda}{2} - \dot{\Omega}_3\sin a\sin\frac{\lambda}{2}\\
\pi_3 &= \frac{1}{2}\dot{\lambda} - \cos a\,\dot{\Omega}_3
\end{aligned} \tag{78}$$

and introduce

$$\begin{aligned}
\dot{\tilde{\Omega}}_1 &= \frac{1}{\sqrt{2}}(\dot{\Omega}_1 - \dot{\Omega}_2)\\
\dot{\tilde{\Omega}}_2 &= \frac{1}{\sqrt{2}}(\dot{\Omega}_1 + \dot{\Omega}_2).
\end{aligned} \tag{79}$$

In terms of these variables the free Lagrangian takes the form

$$T = \frac{\rho^2}{2}\left[\left(\frac{1}{2}\pi_1^2 + \pi_2^2 + \pi_3^2\right) + \frac{1}{2}(1 + \sin a)\,\dot{\tilde{\Omega}}_1^2 + \frac{1}{2}(1 - \sin a)\,\dot{\tilde{\Omega}}_2^2\right] + \frac{1}{2}\dot{\rho}^2\,. \tag{80}$$

or introducing $b = \frac{\pi}{2} - a$,

$$T = \frac{\rho^2}{2}\left[\left(\frac{1}{4}\pi_1^2 + \pi_2^2 + \pi_3^2\right) + \cos^2\frac{b}{2}\,\dot{\tilde{\Omega}}_1^2 + \sin^2\frac{b}{2}\,\dot{\tilde{\Omega}}_2^2\right] + \frac{1}{2}\dot{\rho}^2\,. \tag{81}$$

The Lagrangian written in the form (81) corresponds to the motion of two coupled tops. If the triangle is either only rotating or only deforming, (81) leads to the usual equation of motion of a top.

Finally, we will list the equations of motion in terms of the variables (78) and (79). The general expression is

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \pi_r}\right) + \sum_s\sum_\ell \pi_\ell \frac{\partial T}{\partial \pi_s}\gamma'_{rs\ell} - \sum_\lambda A'_{\lambda r}\frac{\partial T}{\partial q_\lambda} = 0. \tag{82}$$

We introduced here the notations $q_1 = a$, $q_2 = \lambda$, $q_3 = \Omega_3$,

$$\begin{aligned}
\pi_r &= \sum_p B'_{rp}\dot{q}_p\\
\dot{q}_k &= \sum_\ell A'_{k\ell}\pi_\ell\,.
\end{aligned} \tag{83}$$

and, correspondingly,

$$\sum_{\lambda}\sum_{m} A'_{\lambda r} A'_{m\ell} \left(\frac{\partial B'_{s\lambda}}{\partial q_m} - \frac{\partial B'_{sm}}{\partial q_\lambda} \right) = \gamma'_{rs\ell} \qquad (84)$$

In our case $\gamma'_{rs\ell} = -\gamma_{rs\ell}$, hence we obtain

$$\frac{1}{4}\frac{d}{dt}(\rho^2 \pi_1) - \frac{1}{4}\rho^2 \sin\frac{\lambda}{2}\cos a\,(\dot{\tilde{\tilde{\Omega}}}{}_1^2 - \dot{\tilde{\tilde{\Omega}}}{}_2^2) = 0$$

$$\frac{d}{dt}(\rho^2 \pi_2) - \frac{3}{4}\rho^2 \pi_1 \pi_3 - \frac{1}{4}\rho^2 \cos\frac{\lambda}{2}\cos a\,(\dot{\tilde{\tilde{\Omega}}}{}_1^2 - \dot{\tilde{\tilde{\Omega}}}{}_2^2) = 0$$

$$\frac{d}{dt}(\rho^2 \pi_3) + \frac{3}{4}\rho^2 \pi_1 \pi_2 = 0$$

$$\ddot{\rho} - \rho\left(\frac{1}{4}\pi_1^2 + \pi_2^2 + \pi_3^2\right) + \dot{\tilde{\tilde{\Omega}}}{}_1^2 \cos^2\frac{b}{2} + \dot{\tilde{\tilde{\Omega}}}{}_2^2 \sin^2\frac{b}{2} = 0$$

$$\ddot{\tilde{\tilde{\Omega}}}_2 + \frac{2}{\rho}\dot{\rho}\,\dot{\tilde{\tilde{\Omega}}}_2 + \frac{1}{\sin a}\left(\cos\frac{\lambda}{2}\pi_1 - \sin\frac{\lambda}{2}\pi_2\right)\dot{\tilde{\tilde{\Omega}}}_1 -$$

$$- \frac{\cos a}{1 - \sin a}\left[\left(\sin\frac{\lambda}{2}\pi_1 + \cos\frac{\lambda}{2}\pi_2\right)\dot{\tilde{\tilde{\Omega}}}_2 +\right.$$

$$\left. + 2\,\mathrm{ctg}\,a\left(\cos\frac{\lambda}{2}\pi_1 - \sin\frac{\lambda}{2}\pi_2\right)\dot{\tilde{\tilde{\Omega}}}_1 + 2\pi_3\dot{\tilde{\tilde{\Omega}}}_1\right] = 0$$

$$\ddot{\tilde{\tilde{\Omega}}}_1 + \frac{2}{\rho}\dot{\rho}\,\dot{\tilde{\tilde{\Omega}}}_1 - \frac{1}{\sin a}\left(\cos\frac{\lambda}{2}\pi_1 - \sin\frac{\lambda}{2}\pi_2\right)\dot{\tilde{\tilde{\Omega}}}_2 +$$

$$+ \frac{\cos a}{1 + \sin a}\left[\left(\sin\frac{\lambda}{2}\pi_1 + \cos\frac{\lambda}{2}\pi_2\right)\dot{\tilde{\tilde{\Omega}}}_1 +\right.$$

$$\left. + 2\,\mathrm{ctg}\,a\left(\cos\frac{\lambda}{2}\pi_1 - \sin\frac{\lambda}{2}\pi_2\right)\dot{\tilde{\tilde{\Omega}}}_2 + 2\pi_3\dot{\tilde{\tilde{\Omega}}}_2\right] = 0.$$

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