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DUALITY AND FINITE ENERGY SUM RULES

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Дуальность и правила сумм при конечных энергиях

В работе рассматривается проблема вывода представления типа Венециано на основе вспомогательного интегрального параметрического представления амплитуды рассеяния, удовлетворяющей безвычитательным дисперсионным соотношениям.

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Duality and Finite Energy Sum Rules

The problem of the derivation of the Veneziano type representation is considered on the basis of an auxiliary integral parametric representation of scattering amplitudes satisfying unsubtracted dispersion relations.

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After having established dispersion relations[1,2] an important factor of the development of strong interaction theory was the Mandelstam hypothesis on the existence of a unique analytic function of the variables s , t and u which coincides in the physical domains of these variables with the scattering amplitudes of the appropriate channels. Another important factor was the postulate on the Regge behaviour of scattering amplitudes, which essentially reduces to a dynamical requirement that the bound states in the t - channel should define the asymptotic behaviour of the amplitude at high energies in the s -channel.

The use of the analyticity and of the assumption about the Regge behaviour of the amplitude leads to finite energy sum rules which establish an integral connection between the imaginary part of the scattering amplitude at low and medium energies and the Regge pole parameters defining the high-energy behaviour of the scattering amplitude[3-5].

The analysis of the finite-energy sum rules (FESR) has shown that in a number of cases the integrals of the imaginary parts of the scattering amplitudes are saturated with the contributions of the resonance states in the s-channel. This fact has led to the appearance of the notion of the so-called global duality: the integral of the sum of the resonances in the s-channel is equal to the integral of the sum of the Regge poles in the t-channel.

A next step was the introduction of the principle of local duality according to which the scattering amplitude is defined in an alternative manner either by the sum of the resonances in the s-channel, or by the sum of the Regge poles in the t-channels.

Veneziano [6] has suggested a model of the scattering amplitude having a Regge behaviour in all the channels and satisfying the requirement of crossing symmetry and the principle of local duality. However, we note that the amplitudes in the Venziano model and in its various generalizations [7-9] do not obey unitarity and have wrong analytic properties [10].

In this paper we study this problem using an auxiliary integral parametric representation for scattering amplitudes which obey unsubtracted dispersion relations [11].

We shall show on a simple example in perturbation theory that this representation might be useful in the study of asymptotic behaviour and the problem of expandability of scattering amplitudes in the Euler beta functions. Then, using FESR we give a possible formulation of the principle of "local duality" and show that dual amplitude defined in such a way can be expanded in general case in a discrete and a continuous series of Veneziano terms.

For the crossing symmetric amplitude and linear Regge trajectories the generalized Veneziano representation can be written in the following form :

$$F(s,t)=\int_0^\infty \rho(z)B(z-\alpha(s);z-\alpha(t)), \tag{1}$$

where a distribution function $\rho(z)$ consists of a discrete sum of the terms $\delta(z-n)$ and a continuous part at $z\geq z_0=\alpha(s_0)$, where s_0 is a threshold singularity.

Consider some invariant amplitude $T(s,t)$:

$$T(s,t)=F(s,t)\pm F(u,t), \tag{2}$$

where $F(s,t)$ is analytic in the complex s-plane with the cut at $s\geq s_0$ and obeys unsubtracted dispersion relations

$$F(s,t)=\frac{1}{\pi}\int_{s_0}^{\infty}\frac{\operatorname{Im}F(s',t)}{s'-s}\,ds'. \tag{3}$$

Using the evident identity

$$\frac{1}{a}=\int_0^1 dx\,x^{-1+a} \qquad \operatorname{Re}a>0 \tag{4}$$

we get the following integral parametric representation for the amplitude $F(s,t)$ x/:

$$F(s,t)=\int_0^1 dx\,x^{-1-s}f(x,t) \qquad s<s_0, \tag{5}$$

where

$$f(x,t)=\frac{1}{\pi}\int_{s_0}^{\infty}ds\,x^{s}\operatorname{Im}F(s,t). \tag{6}$$

One can see that the eq. (6) defines an analytic function of the variable x inside the circle $|x|<1$ with the cut at $-1<x\leq 0$.

x/ Obviously, the representation (5) determines the amplitude (3) in the half-plane $\operatorname{Re}(s-s_0)<0$. In the region $\operatorname{Re}(s-s_0)\geq 0$ the amplitude $F(s,t)$ can be determined by means of an analytic continuation.

For amplitudes $F(s,t)$ which decrease more slowly than $1/s$ the function $f(x,t)$ will have singularities at $x=1$ due to the divergence of the integral in eq. (6).

On the other hand it is easy to see that the asymptotic behaviour of $F(s,t)$ at $s\to-\infty$ is determined by the behaviour of the function $f(x,t)$ near the point $x=1$.

Now we make use of perturbation theory to get an information about the behaviour of the function $f(x,t)$ near $x=1$.

Consider the asymptotic behaviour of the ladder type diagrams [12,13]

$$F_n(s,t)\underset{s\to-\infty}{\to}-\frac{g^2}{s}\frac{1}{n!}[g^2\Delta(t)\ \ln(-s)]^n, \tag{7}$$

where

$$\Delta(t)=\frac{1}{8\pi^2}\int_{4m^2}^{\infty}\frac{dt'}{t'-t}\frac{1}{\sqrt{t'(t'-4m^2)}}. \tag{8}$$

As is well known the sum of the terms (7) gives the asymptotic Regge behaviour

$$F(s,t)\underset{s\to-\infty}{\to}g^2(-s)^{\alpha(t)}, \tag{9}$$

where

$$\alpha(t)=-1+g^2\Delta(t). \tag{10}$$

However, in doing so we lose the poles of the Regge residue at the points where $\alpha(t)$ = zero of positive integer which should appear in the thrue Regge-pole term.

We shall use the method of summation of perturbation theory expansion for spectral functions rather than for the Green functions itself. Such a method was suggested for the investigation of the photon Green function in quantum electrodynamics [14].

So we shall consider that the asymptotic of the ladder diagrams determines the main singularities of the function $f(x,t)$ at the point $x=1$.

Finding the asymptotic of $\operatorname{Im} F_n(s,t)$ from eq. (7) and using formula (6) we get near the point $x=1$.

$$f_n(x,t)=g^2\frac{1}{n!}[-g^2\Delta(t)\ln(1-x)]^n . \tag{11}$$

Taking the sum of the terms (11) we find at $x\approx 1$

$$f(x,t)=g^2(1-x)^{-1-\alpha(t)} . \tag{12}$$

Substituting eq. (12) in formula (5) we obtain just the expression for the Euler beta function

$$F(s,t)=g^2\int_0^1 dx\, x^{-1-s}(1-x)^{-1-\alpha(t)}=g^2\frac{\Gamma(-s)\Gamma[-\alpha(t)]}{\Gamma[-s-\alpha(t)]} . \tag{13}$$

The asymptotic form of eq. (13) is as follows

$$F(s,t)\underset{s\to\infty}{\to} g^2\Gamma[-\alpha(t)](-s)^{\alpha(t)} \tag{14}$$

which possesses the poles at the points where $\alpha(t)=$ zero or positive integer.

However, the expression (13) is meromorphic function of s whereas we have assumed the amplitude $F(s,t)$ to be an analytic function of s in the complex plane with the cut at $s\geq s_0$. The contradiction is due to the fact that eq. (12) gives the right behaviour of the function $f(x,t)$ only at the point $x=1$ and in particular is incorrect near the point $x=0$.

The behaviour of $f(x,t)$ near $x=0$ is determined by the structure of nearby singularities of the amplitude $F(s,t)$. Below we shall use FESR to study the properties of the function $f(s,t)$.

Using the Cauchy theorem for the function $\Psi(z) \equiv x^z F(z,t)$ one can get

$$f_R(x,t) = \frac{1}{\pi}\int_{s_0}^{R} ds\, x^s \operatorname{Im} F(s,t) = -\frac{1}{2\pi i}\int_{C_R} dz\, x^z F(z,t), \tag{15}$$

where C_R is a big circle of radius R .

Obviously

$$f(x,t) = \lim_{R\to\infty} f_R(x,t) . \tag{16}$$

As one can see from eq. (15) the function $f_R(x,t)$ is determined either by an imaginary part of $F(s,t)$ on the real axis or by an asymptotic behaviour of $F(s,t)$ on the big circle C_R in the complex s - plane.

Usually in studying FESR one considers the model in which the scattering amplitude is determined on the real axis by an asymptotic sum of the equidistant poles

$$F(s,t) = \sum_{n=0}^{\infty} g_n(t)\frac{1}{n+s_p-s} \qquad s - \text{real} \tag{17}$$

and has an asymptotic behaviour, which can be represented by an infinite sum of Regge poles[x/]

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x/ Obviously, one should consider the asymptotic behaviour (18) to be valid outside of some sector $|\arg s| < \epsilon$ inside of which on the real axis there appear the poles of eq. (17).

$$F(s,t) \underset{|s|\to\infty}{\to} -\sum_{n=0}^{\infty} \beta_n(t)\,\Gamma[n-\alpha(t)]\,(-s)^{\alpha(t)-n} \quad . \tag{18}$$

Substituting eqs. (17) and (18) in formula (15) and going to the limit $R \to \infty$ we find that $f(x,t)$ can be represented in alternative may be

$$f(x,t) = \sum_{n=0}^{\infty} g_n(t)\, x^{s_p+n} \tag{19}$$

or

$$f(x,t) = \sum_{n=0}^{\infty} \beta_n(t)(1-x)^{-1-\alpha(t)+n} \quad . \tag{20}$$

The expressions (19) and (20) allow one to give the following possible formulation of the principle of local duality: the function $f(x,t)$ is completely determined either by its expansion at the point $x=0$ or by its expansion at the point $x=1$.

One can see that for the case considered above the main singularities of $f(x,t)$ at the points $x=0$ and $x=1$ are factorized, i.e.

$$f(x,t) = x^{s_p}(1-x)^{-1-\alpha(t)}\,\Phi(x,t), \tag{21}$$

where $\Phi(x,t)$ is a regular function in the interval. Decompose $\Phi(x,t)$ on two parts by

$$\Phi(x,t) = A(x,t) + (1-2x)B(x,t), \tag{22}$$

where $A(s,t)$ and $B(s,t)$ are regular and symmetric under substitutions $x \to (1-x)$.

Using expansions of $A(x,t)$ and $B(x,t)$ in powers of $x(1-x)$:

$$A(x,t)=\sum_{n=0}^{\infty} a_n [x(1-x)]^n \tag{23a}$$

$$B(x,t)=\sum_{n=0}^{\infty} b_n [x(1-x)]^n \tag{23b}$$

we obtain the expansion of the amplitude $F(s,t)$ in the Euler beta functions:

$$F(s,t)=\sum_{n=0}^{\infty} a_n(t)\, B(n+s_p-s\,;\,n-\alpha(t))+ \\ +\sum_{n=0}^{\infty} b_n(t)[B[n+s_p-s\,;\,n+1-\alpha(t)]-B[n+s_p+1-s\,;\,n-\alpha(t)]]. \tag{24}$$

For the linear Regge trajectory with the unite slope i.e. $\alpha(t)=\alpha_0+t$ we can construct using eq. (24) an amplitude with a definite symmetry under crossing $s\to t$.

In particular case, when

$$j_{min}=\alpha(s_p)=\alpha_0+s_p=0. \tag{25}$$

and the coefficients $a_n(t)$ and $b_n(t)$ do not depend on t we get the following Veneziano type representations:

$$F(s,t)=\sum_{n=0}^{\infty} a_n\, B(n-\alpha(t)\,;\,n-\alpha(s)) \tag{26a}$$

and

$$F(s,t)=\sum_{n=0}^{\infty} b_n [\alpha(s)-\alpha(t)]\, \frac{\Gamma(n-\alpha(s))\,\Gamma(n-\alpha(t))}{\Gamma(2n+1-\alpha(s)-\alpha(t))} \tag{26b}$$

for crossing symmetric [15,16] and crossing antisymmetric amplitudes, respectively.

Suppose now, that $F(s,t)$ has a cut in the complex s -plane with an expansion on the threshold given by

$$\operatorname{Im}_{s\approx s_0} F(s,t) = \sum_{n=0}^{\infty} c_n(t)(s-s_0)^{1/2+n} . \qquad (27)$$

In this case a crossing symmetric amplitude can be constructed under the condition that in asymptotic there presents the contribution of a Regge cut:

$$F_{\substack{cut \\ |s|\to\infty}}(s,t) = (-s)^{\alpha'(t)} \sum_{n=0}^{\infty} d_n(t)[\ln(-s)]^{-3/2-n} . \qquad (28)$$

The corresponding behaviour of the function $f(s,t)$ near the points $x=0$ and $x=1$ is as follows:

$$f_{x\approx 0}(x,t) = x^{s_0} \sum \xi_n(t)[-\ln x(1-x)]^{-3/2-n} \qquad (29)$$

and

$$f_{x\approx 1}(x,t) = (1-x)^{-1-\alpha'(t)} \sum_{n=0}^{\infty} \eta_n(t)[-\ln x(1-x)]^{-3/2-n} . \qquad (30)$$

One can construct the function $f(x,t)$ of this type with factorizing main singularities in accordance with the principle of local duality in the form

$$f(x,t) = x^{s_0}(1-x)^{-1-\alpha'(t)} \sum_{n=0}^{\infty} \zeta_n(t)[-\ln x(1-x)]^{-3/2-n} . \qquad (31)$$

Substituting eq. (31) in the formula (5) and using the identity

$$[-\ell n\, x(1-x)]^{-3/2-n} = \frac{1}{\Gamma(3/2+n)} \int_0^\infty d\lambda\, \lambda^{1/2+n} [x(1-x)]^{\lambda} \tag{32}$$

we find the following continuous series in the Euler beta functions:

$$F(s,t) = \int_0^\infty d\lambda\, \Psi(\lambda, t)\, B(\lambda + s_0 - s\,;\, \lambda - \alpha'(t)), \tag{33}$$

where

$$\Psi(\lambda, t) = \sum_{n=0}^{\infty} \frac{\zeta_n(t)}{\Gamma(3/2+n)} \lambda^{1/2+n} . \tag{34}$$

In particular case when the position of the Regge cut $\alpha'(t)$ is a linear function of t with unite slope and $\alpha'(s_0) = 0$ and when the function $\rho(z) \equiv \Psi(z - s_0, t)$ does not depend on t we get the following generalized Veneziano representation for crossing symmetric amplitude:

$$F(s,t) = \int_{z_0}^\infty dz\, \rho(z)\, B(z - \alpha(s);\, z - \alpha(t)), \tag{35}$$

where $z_0 = \alpha(s_0)$.

The discrete sum of the beta functions (26a) can be included in the representation (35) if one adds to the distribution function $\rho(z)$ the sum $\Sigma\, a_n \delta(z-n)$.

Notice that the finite widths of resonances can be taken into account in eq. (35) by substitution of a discontinuity of the Breit-Wigner pole instead of the term $\delta(z-n)$ in the distribution function $\rho(z)$.

Thus the representations of type (33) and (35) might be useful in studying of the unitarity problem in the Veneziano model.

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